2 Problem 2

(a) The phrase "a particle of spin $s$" does not mean that if you were to measure the particle’s spin in an experiment, you’d find that it’s equal to $s$! This is simply not true. Why? In experiments, we can only measure observables - specifically, eigenvalues of Hermitian operators. So to measure the spin of a particle in an experiment means finding the magnitude of its spin vector.

$\hat{S}^2$ is a Hermitian operator which has eigenvalue given by $\hbar^2 s(s + 1)$. The magnitude of the particle’s spin (which is what you can actually measure in an experiment) is $\hbar \sqrt{s(s + 1)}$. For $s = 1$, this is $\sqrt{2}\hbar$.

So the lesson here is that $s$ is not an eigenvalue of a Hermitian operator - it’s just a quantum number, which you can use to label eigenstates (if the Hamiltonian commutes with the operator corresponding to that quantum number). Don’t confuse a quantum number with a measurable eigenvalue of a Hermitian operator!

(b) We are told that a particle is in a central potential with orbital angular momentum quantum number $l = 2$ and spin quantum number $s = 1$. Recall that from angular momentum addition, the possible values of the total angular momentum quantum number $j$ are

\[ j = |l - s|, |l - s| + 1, \ldots, l + s \]  

subject to the rule that $m_j = m_l + m_s$ where $m_j = -j, -j + 1, \ldots, j - 1, j$ for each possible value of $j$.

For the values of $l$ and $s$ in this problem, we find that the possible values of $j$ are $j = 1, 2, 3$. From equation (5), the eigenstates of $\hat{H}_{s-o}$ are given by $|j l s m_j\rangle$. The eigenstates are therefore,

\begin{align*}
\text{j = 1 eigenstates} & \quad \{ |121 - 1\rangle, |1210\rangle, |1211\rangle \} \\
\text{j = 2 eigenstates} & \quad \{ |221 - 2\rangle, |221 - 1\rangle, |2210\rangle, |2211\rangle, |2212\rangle \} \\
\text{j = 3 eigenstates} & \quad \{ |321 - 3\rangle, |321 - 2\rangle, |321 - 1\rangle, |3210\rangle, |3211\rangle, |3212\rangle, |3213\rangle \}
\end{align*}  

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Notice that for each value of $j$, there are $2j + 1$ eigenstates (because there are $2j + 1$ values of $m_j$).

The energies $E_{j s l}$ in equation (5) are independent of $m_j$. This implies the $j = 1$ eigenstates are $3$-fold degenerate, $j = 2$ eigenstates are $5$-fold degenerate, and $j = 3$ eigenstates are $7$-fold degenerate in energy. The energy eigenvalues are

\begin{align*}
\text{j = 1 eigenstates} & \quad E_{121} = \frac{\beta \hbar^2}{2} (1(1 + 1) - 2(2 + 1) - 1(1 + 1)) = -3\beta \hbar^2 \\
\text{j = 2 eigenstates} & \quad E_{221} = \frac{\beta \hbar^2}{2} (2(2 + 1) - 2(2 + 1) - 1(1 + 1)) = -\beta \hbar^2 \\
\text{j = 3 eigenstates} & \quad E_{321} = \frac{\beta \hbar^2}{2} (3(3 + 1) - 2(2 + 1) - 1(1 + 1)) = 2\beta \hbar^2
\end{align*}  

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