Phys 487 Discussion 4 – Spin-Orbit and Spin-Spin Interactions

Last week you found that when the Bohr Hamiltonian for a hydrogenic atom is supplemented with a spin-orbit term of the form $H_{\text{s-o}} = \beta\,\vec{l}\cdot\vec{s}$, these four commutators are obtained:

$$\left[H_{\text{Bohr + s-o}}, l^2\right] = 0 \qquad \left[H_{\text{Bohr + s-o}}, l_z\right] = \beta\left(-il_y\,s_x + i\,l_x\,s_y\right) \neq 0$$

$$\left[H_{\text{Bohr + s-o}}, s^2\right] = 0 \qquad \left[H_{\text{Bohr + s-o}}, s_z\right] = \beta\left(-i\,s_y\,l_x + i\,s_x\,l_y\right) \neq 0$$

As a result, l and s are good quantum numbers for an atomic electron when a spin-orbit force is present, but m_l and m_s are not good quantum numbers any more. You then turned to the electron's total angular momentum $\vec{j} \equiv \vec{l} + \vec{s}$. Using the supremely-useful technique

$$\vec{l}\cdot\vec{s} = (j^2 - l^2 - s^2)/2$$

(which you will need again ...) you found that

$$\left[H_{\text{Bohr + s-o}}, j^2\right] = 0 \qquad \left[H_{\text{Bohr + s-o}}, j_z\right] = 0\ .$$

Thus j and m_j are good quantum numbers, even in the presence of the spin-orbit force.

Problem 1 : Spin-Orbit Energy Correction (b) = *Qual Problem* [1]

A particle of spin $s = 1$ is in a central potential and has orbital angular momentum $l = 2$.

(a) "A particle of spin $s = 1$" is a bit of an odd phrase → what's being specified is the spin quantum number s, not the spin itself. What is the magnitude of the particle's *actual spin*? Your answer should have proper units of angular momentum (unlike the dimensionless quantum number "$s = 1$").

▶ IMPORTANT: This phrasing is EXTREMELY common. It is COMPLETELY NORMAL to say "a particle has spin 1" and mean that its spin *quantum number* is 1, not its *actual spin* (which is an angular momentum).

(b) Find the energy levels and degeneracies associated with a spin-orbit interaction of the form $H_{\text{s-o}} = \beta\vec{l}\cdot\vec{s}$, where β is a constant.

▶ HINT: Start by listing all possible states of the $s = 1$, $l = 2$ particle using our newly-discovered set of good quantum numbers.

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[1] **Q1 (a)** $\hbar\sqrt{s(s+1)} = \sqrt{2}\,\hbar$ for a spin-1 particle. **(b)** The n quantum number is irrelevant since you only want the contribution of the spin-orbit energy $\beta\vec{l}\cdot\vec{s}$, and that only depends on the magnitude and orientation of angular momenta →, list the possible values of j and m_j given $s = 1$, $l = 2$ … There are **three** possible values of j, each producing a different spin-orbit energy … You will need the **useful technique** from last week … The three energies are **not** { $2\beta, -\beta/2, -2\beta$ } … if that's what you got, you forgot part (a) (units!!) … **Answer**: spin-orbit energies = { $2\beta, -\beta, -3\beta$ } $\hbar^2$, with degeneracies { 7, 5, 3 }, for j = { 3, 2, 1 }

Problem 2 : Spin-Spin Interaction *Qual Problem* [2]

Two spin-½ particles are separated by a distance $\vec{a} = a\,\hat{z}$ and interact only through the magnetic dipole energy

$$H = \frac{\vec{\mu}_1 \cdot \vec{\mu}_2}{a^3} - 3\frac{(\vec{\mu}_1 \cdot \vec{a})(\vec{\mu}_2 \cdot \vec{a})}{a^5}$$

where μ_i is the magnetic moment of particle *i* due to its spin. The two particles have the same gyromagnetic ratio γ :

$$\vec{\mu}_i = \gamma\,\vec{s}_i \quad \text{for } i = 1, 2.$$

The energy eigenstates of the two-spin system are the eigenstates $|S\,M_S\rangle$ of the total spin (S^2) and total S_z.

(a) Write the Hamiltonian in terms of the spin operators $\vec{s}_i$ of the individual particles (and/or the magnitudes and/or components thereof).

(b) Write the Hamiltonian in terms of the operators S^2 and S_z where the capital *S* denotes TOTAL spin, as usual.

(c) Give the energy eigenvalues for all states $|S\,M_S\rangle$.

[2] **Q2** **(a)** $\hat{H} = \frac{\gamma^2}{a^3}\left(\hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 - 3\hat{s}_{1z}\,\hat{s}_{2z}\right)$ **(b)** $\hat{H} = \frac{\gamma^2}{2a^3}\left(\hat{S}^2 - \hat{s}_1^2 - \hat{s}_2^2 - 3\hat{S}_z^2 + 3\hat{s}_{1z}^2 + 3\hat{s}_{2z}^2\right)$ **(c)** $E_{00} = 0,\ E_{1\pm1} = -\frac{\gamma^2\hbar^2}{2a^3},\ E_{10} = \frac{\gamma^2\hbar^2}{2a^3}$