Last week you found that when the Bohr Hamiltonian for a hydrogenic atom is supplemented with a spin-orbit term of the form $H_{s-o} = \beta \mathbf{l} \cdot \mathbf{s}$, these four commutators are obtained:

$$
\begin{align*}
[H_{\text{Bohr}} + s-o, l^2] &= 0 \\
[H_{\text{Bohr}} + s-o, l_z] &= \beta (-i l_y s_x + i l_x s_y) \\
[H_{\text{Bohr}} + s-o, s^2] &= 0 \\
[H_{\text{Bohr}} + s-o, s_z] &= \beta (-i s_y l_x + i s_x l_y)
\end{align*}
$$

As a result, $l$ and $s$ are good quantum numbers for an atomic electron when a spin-orbit force is present, but $m_l$ and $m_s$ are not good quantum numbers any more. You then turned to the electron’s total angular momentum $\mathbf{j} \equiv \mathbf{l} + \mathbf{s}$. Using the supremely-useful technique

$$\mathbf{l} \cdot \mathbf{s} = (j^2 - l^2 - s^2) / 2$$

(which you will need again ...) you found that

$$
\begin{align*}
[H_{\text{Bohr}} + s-o, j^2] &= 0 \\
[H_{\text{Bohr}} + s-o, j_z] &= 0
\end{align*}
$$

Thus $j$ and $m_j$ are good quantum numbers, even in the presence of the spin-orbit force.

**Problem 1: Spin-Orbit Energy Correction**

A particle of spin $s = 1$ is in a central potential and has orbital angular momentum $l = 2$.

(a) “A particle of spin $s = 1$” is a bit of an odd phrase → what’s being specified is the spin quantum number $s$, not the spin itself. What is the magnitude of the particle’s actual spin? Your answer should have proper units of angular momentum (unlike the dimensionless quantum number “$s = 1$”).

▶ IMPORTANT: This phrasing is EXTREMELY common. It is COMPLETELY NORMAL to say “a particle has spin 1” and mean that its spin quantum number is 1, not its actual spin (which is an angular momentum).

(b) Find the energy levels and degeneracies associated with a spin-orbit interaction of the form $H_{s-o} = \beta \mathbf{l} \cdot \mathbf{s}$, where $\beta$ is a constant.

▶ HINT: Start by listing all possible states of the $s = 1, l = 2$ particle using our newly-discovered set of good quantum numbers.

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1 Q1 (a) $\hbar \sqrt{s(s+1)} = \sqrt{2} \hbar$ for a spin-1 particle. (b) The $n$ quantum number is irrelevant since you only want the contribution of the spin-orbit energy $\beta l \cdot s$, and that only depends on the magnitude and orientation of angular momenta →, list the possible values of $j$ and $m_j$ given $s = 1, l = 2$ … There are three possible values of $j$, each producing a different spin-orbit energy … You will need the useful technique from last week … The three energies are not {2$\beta$, $-2\beta$/2, $-2\beta$} … if that’s what you got, you forgot part (a) (units!!) … Answer: spin-orbit energies = {2$\beta$, $-\beta$, $-3\beta$} $\hbar^2$, with degeneracies {7, 5, 3}, for $j = \{3, 2, 1\}$
Problem 2 : Spin-Spin Interaction

Two spin-$\frac{1}{2}$ particles are separated by a distance $\bar{a} = a\hat{z}$ and interact only through the magnetic dipole energy

$$H = \frac{\hat{\mu}_1 \cdot \hat{\mu}_2}{a^3} - \frac{3(\hat{\mu}_1 \cdot \hat{a})(\hat{\mu}_2 \cdot \hat{a})}{a^5}$$

where $\mu_i$ is the magnetic moment of particle $i$ due to its spin. The two particles have the same gyromagnetic ratio $\gamma$

$$\hat{\mu}_i = \gamma \hat{s}_i \quad \text{for} \quad i = 1, 2.$$  

The energy eigenstates of the two-spin system are the eigenstates $|S M_S\rangle$ of the total spin ($S^2$) and total $S_z$.

(a) Write the Hamiltonian in terms of the spin operators $\hat{s}_i$ of the individual particles (and/or the magnitudes and/or components thereof).

(b) Write the Hamiltonian in terms of the operators $S^2$ and $S_z$ where the capital $S$ denotes TOTAL spin, as usual.

(c) Give the energy eigenvalues for all states $|S M_S\rangle$.

---

2 Q2 (a) $\hat{H} = \gamma^2 a^3 \left( \hat{s}_1 \cdot \hat{s}_2 - 3\hat{s}_{1z}\hat{s}_{2z} \right)$  
(b) $\hat{H} = \frac{\gamma^2}{2a^3} \left( S^2 - \hat{s}_1^2 - \hat{s}_2^2 - 3\hat{s}_{1z}^2 + 3\hat{s}_{2z}^2 \right)$  
(c) $E_{00} = 0$, $E_{1\pm1} = -\frac{\gamma^2\hbar^2}{2a^3}$, $E_{10} = \frac{\gamma^2\hbar^2}{2a^3}$