Phys 487 Discussion 3 – Magnetic Dipole Moment

Problem 1 : Zeeman Effect *adapted from Griffiths 4.33*

An electron is at rest in an oscillating external magnetic field that points in the z-direction:

$$\vec{B} = B_0 \cos(\omega t)\,\hat{z}$$

where B_0 and ω are constants. The magnetic field couples to the spin of the electron via its magnetic dipole moment, giving the following Hamiltonian:

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\gamma\,\hat{\vec{s}}\cdot\vec{B}$$

where γ is the gyromagnetic ratio and $\hat{\vec{s}}$ is the usual spin operator for a spin-½ particle :

$$\hat{\vec{s}} = \left(\hat{s}_x, \hat{s}_y, \hat{s}_z\right) = \frac{\hbar}{2}\left(\sigma_x, \sigma_y, \sigma_z\right)$$

(a) Write down the Hamiltonian in terms of the $\hat{s}_z$ operator.

(b) The electron's spin can be in the state

$$|s,m_s\rangle = \left|\frac{1}{2},+\frac{1}{2}\right\rangle = |\uparrow\rangle \quad \text{or}$$

$$|s,m_s\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle = |\downarrow\rangle \quad \text{or}$$

any linear combination of these two states. Write down the Hamiltonian as a 2 × 2 matrix in the ordered basis

$$\{\,|\uparrow\rangle,\ |\downarrow\rangle\,\}.$$

(c) Suppose that the electron is initially pointing in the $+x$-direction. That means that it starts int he $+\hbar/2$ eigenstate of $\hat{s}_x$, which is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(\,|\uparrow\rangle + |\downarrow\rangle\,\right)$$

Determine the time-evolved state $|\psi(t)\rangle$ under the given Hamiltonian.

▶ HINT: Since the Hamiltonian is **time-dependent**, you have to go back to basics: you have to write down the Schrödinger equation, write it down for this particular Hamiltonian, and solve it. Happily, the solution for this particular Hamiltonian is pretty simple to find.

Problem 2 : Spin-Orbit Coupling and Good Quantum Numbers *Checkpoints* [1]

Taking into account the effect of spin-orbit coupling, the Hamiltonian of the electron in a hydrogen atom is

$$H = H_0 + \beta\, \vec{l} \cdot \vec{s}$$

where β is a positive real number, H_0 is the Bohr Hamiltonian, and the second term is the spin-orbit interaction. Note that we are leaving off the ^ symbols on operators for brevity, as is commonly done. (At this point in your studies, it's clear in most situations what is an observable and and what isn't.)

The observables l^2, s^2, l_z, and s_z all commute with H_0, which means that their eigenvalues $\hbar^2\, l(l+1)$, $\hbar^2\, s(s+1)$, $\hbar\, m_l$, and $\hbar\, m_s$, are all *conserved* in the Bohr model, which in turn means that l, s, m_l, and m_s are all *good quantum numbers*. However, this is not necessarily true after the spin-orbit interaction is taken into account.

(a) Show that $[H, l_z]$ and $[H, s_z]$ are not zero. Hints are in the checkpoint.

(b) Show that $[H, l^2]$ and $[H, s^2]$ are zero.

(c) Given your findings in (a) and (b), indicate which of these quantum numbers are NOT good quantum numbers when the spin-orbit interaction is present:

$$l,\ s,\ m_l,\ m_s$$

(d) Show that these two observables ARE good quantum numbers even when the spin-orbit interaction is present:

$$j^2 = \left|\vec{l} + \vec{s}\right|^2 \quad \text{and} \quad j_z = l_z + s_z\ .$$

These correspond respectively to the magnitude² and z-component of the electron's TOTAL angular momentum.

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[1] **Q2 (a)** Hint 1: Expand the dot product component-by-component.
Hint 2: You need the canonical commutators of angular momentum: $[l_x, l_y] = i\hbar l_z$ & cyclic permutations
… which we copied over to spin: $[s_x, s_y] = i\hbar s_z$ & cyclic permutations
… which (just to point it out) therefore applies to *all* forms of angular momentum: $[j_x, j_y] = i\hbar j_z$ & perm.

(b) Hint: The angular momentum commutators lead to this OTHER important commutator: $[l^2, l_i] = 0$, and the same for s and j

(c) Answer: l and s are still good quantum numbers, but m_l and m_s are not.

(d) Hint: Perhaps you recall from 486 this SUPER-useful relation : $\vec{l} \cdot \vec{s} = \left(j^2 - l^2 - s^2\right)/2$