by Eli Chertkov

①

$$H = \hbar\omega \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix}$$

$$A = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) $$[H, A] = \hbar\omega\lambda \left\{ \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 2 \end{pmatrix} - \begin{pmatrix} & 1 & \\ 1 & & \\ & & 2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \right\}$$

$$= \hbar\omega\lambda \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \right\}$$

$$= \hbar\omega\lambda \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ A <u>is not</u> a symmetry.

$$[H, B] = \hbar\omega\mu \left\{ \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \begin{pmatrix} 2 & & \\ & & 1 \\ & 1 & \end{pmatrix} - \begin{pmatrix} 2 & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \right\}$$

$$= \hbar\omega\mu \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\}$$

$$= \hbar\omega\mu \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

↑ zero matrix

$\Rightarrow$ B <u>is</u> a symmetry.

1 con

(b) At $t=0$, the state is $|\psi(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

The expectation values under $|\psi(0)\rangle$ are

$$\langle\psi(0)|A|\psi(0)\rangle = (c_1^* \; c_2^* \; c_3^*) \begin{pmatrix} & \lambda & \\ \lambda & & \\ & & 2\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \lambda (c_1^* \; c_2^* \; c_3^*) \begin{pmatrix} c_2 \\ c_1 \\ 2c_3 \end{pmatrix}$$

$$= \lambda \left[c_1^* c_2 + c_2^* c_1 + 2|c_3|^2 \right]$$

$$\langle\psi(0)|B|\psi(0)\rangle = \mu (c_1^* \; c_2^* \; c_3^*) \begin{pmatrix} 2 & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \mu (c_1^* \; c_2^* \; c_3^*) \begin{pmatrix} 2c_1 \\ c_3 \\ c_2 \end{pmatrix}$$

$$= \mu \left(2|c_1|^2 + c_2^* c_3 + c_3^* c_2 \right)$$

(c) At $t>0$, the time-evolved state is

$$|\psi(t)\rangle = \begin{pmatrix} c_1 e^{-iE_1 t/\hbar} \\ c_2 e^{-iE_2 t/\hbar} \\ c_3 e^{-iE_3 t/\hbar} \end{pmatrix} = \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix}.$$

The expectation values under $|\psi(t)\rangle$ are

$$\langle\psi(t)|A|\psi(t)\rangle = \lambda \left(c_1^* e^{+i\omega t}, \; c_2^* e^{+2i\omega t}, \; c_3^* e^{+2i\omega t} \right)$$

$$\times \begin{pmatrix} & 1 & \\ 1 & & \\ & & 2 \end{pmatrix} \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix}$$

1con (Ccon) $\langle\psi(t)|A|\psi(t)\rangle = \lambda\Big[(c_1^* e^{+i\omega t})(c_2 e^{-2i\omega t})$

$+ (c_2^* e^{+2i\omega t})(c_1 e^{-i\omega t})$

$+ (c_3^* e^{+2i\omega t})(2c_3 e^{-2i\omega t})\Big]$

$= \lambda\left[c_1^* c_2\, e^{-i\omega t} + c_2^* c_1\, e^{+i\omega t} + 2|c_3|^2\right]$

$\langle\psi(t)|B|\psi(t)\rangle = \mu\Big[(c_1^* e^{+i\omega t})(2c_1 e^{-i\omega t})$

$+ (c_2^* e^{+2i\omega t})(c_3 e^{-2i\omega t})$

$+ (c_3^* e^{+2i\omega t})(c_2 e^{-2i\omega t})\Big]$

$= \mu\left[2|c_1|^2 + c_2^* c_3 + c_3^* c_2\right]$

$\langle A\rangle$ varies with time. It is not constant.

$\langle B\rangle$ is constant.

(d) If $|\psi(0)\rangle = |\mu\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}$, then $c_1 = 0,\ c_2 = c_3 = 1/\sqrt{2}$

and $|\psi(t)\rangle = \dfrac{e^{-2i\omega t}}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1\end{pmatrix} \propto \dfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1\end{pmatrix} = |\mu\rangle.$

$|\psi(t)\rangle$ stays an eigenstate of B with eigenvalue μ.