Phys 487 Discussion 2 – Essential Skills

Problem 1 : Time Evolution *adapted from Griffiths 2.38* [1]

The energy eigenstates of an infinite square well that extends from $x = 0$ to $x = a$ are :

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \quad \text{for } n = 1, 2, 3, \ldots \quad \text{with energies } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}.$$

At times $t < 0$, a particle of mass m sits in the ground state of an ∞ well with the very convenient width $a = 1$.

At time $t = 0$, the well suddenly expands to twice its original size: the right wall moves from a to $2a$ so quickly that it leaves the particle's wavefunction undisturbed at time $t = 0+$. This is called the **sudden approximation**: a system's Hamiltonian changes so quickly that, for a brief moment, the system's wavefunction doesn't have time to react significantly. (We will revisit this approximation in more detail later on in the course.)

At time $t = 0+$, the energy of the particle is measured.

(a) What is the most probable value of the measured energy, and what is the probability of getting that result?

▶ HINT 1: Project! Onto! Eigenstates! ▶ HINT 2 & checkpoints: see footnotes.

(b) What is the next most probably result, and what is its probability?

(c) What is the expectation value of the energy?

(d) OK enough with the energy measurements. Restart the same experiment. At time $t = 0+$, the particle is in initial state $\psi(x,t=0+) \equiv \psi_0(0) = \sqrt{2}\sin(\pi x)$ and is sitting in a double-sized well that runs to $x = 2a = 2$. How does the wavefunction evolve with time, i.e. what is $\Psi(x,t)$ for times $t > 0$?

▶ HINT 1: Your answer should be a big integral that you should IN NO WAY attempt to compute. (Computers enjoy doing that part. ☺)

▶ HINT 2: If you're attempting to solve the Schrödinger equation from scratch … well, re-deriving the procedure you need is fabulous … here's a hint: "separation of variables" … but perhaps you can recall our procedure for calculating the **time-evolution of a wavefunction** when the Hamiltonian is independent of time (which is the only case we covered in 486).

… hint 2′ : Project! Onto! Eigenstates! ☺

… hint 2″ : Project *what*? → the initial wavefunction $\psi(x, t{=}0+)$

… hint 2‴ : Project *onto which eigenstates*? → onto the eigenstates of ______ because YOU KNOW THEIR TIME DEPENDENCE.

… hint 2⁗ : The Schrödinger equation controls the time-evolution of wavefunctions. What observable appears in the Schrödinger equation? That one observable is what goes in the blank above. You absolutely know the time-dependence of the eigenstates of that very important observable.

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[1] **Q1 (a)** HINT 2: From your memory / the 486 formula sheet, the **master probability formula** : $\text{Prob}(q) = \left|\langle e_q|\psi\rangle\right|^2$.
Answer: The most probably energy is $E'_2 = \pi^2\hbar^2/2m$, with probability 50%. **(b)** The next most probable value is $E'_1 = \pi^2\hbar^2/8m$, with probability ≈ 36%. **(c)** $\langle H\rangle = \pi^2\,\hbar^2/2m$. **(d)** HINT 2‴ blank is **the Hamiltonian**. The energy eigenstates $|E\rangle$ of a time-independent Hamiltonian (which is all we treated in PHYS 486) have a very simple time-dependence: $|E(t)\rangle = |E\rangle\, e^{-i\omega t}$ where $\omega = E/\hbar$. Soooo, the time-dependent wavefunction is always $|\Psi(t)\rangle = \sum_E e^{-iEt/\hbar}|E\rangle\langle E|\psi_0\rangle$.
For this particular situation, it is $\Psi(t) = \sum_{n=1}^{\infty} e^{-i\frac{n^2\pi^2\hbar}{8m}t}\int_0^1 \sin\left(\frac{n\pi x}{2}\right)\sin(\pi x)\,dx$.

Problem 2 : Commutators, Symmetries, and Conserved Quantities *adapted from Griffiths 3.38*

The Hamiltonian for a certain 3-level system is represented by this matrix:

$$\mathbf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

where ω is a positive real number. Two other observables, A and B, are represented in the same basis as the matrices

$$\mathbf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \mathbf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where λ and μ are positive real numbers.

(a) Compute the commutators $[H, A]$ and $[H, B]$.

Thing #1 to recall: If an operator Q **commutes with the Hamiltonian** of a system (i.e. if $[Q, H] = 0$), then Q is called a **symmetry** of the system. Is either A or B a symmetry of our 3-level system?

(b) The system starts out at time $t = 0$ in a generic normalized state $|\psi(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ where $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Compute the expectation values of A and B at $t = 0$.

(c) Find the time-evolved state $|\psi(t)\rangle$. Compute the expectation values of A and B at time $t > 0$.

Thing #2 to recall: If an observable Q **commutes with the Hamiltonian** of a system then Q is **conserved** i.e.:

- Its expectation value $\langle Q \rangle$ is a constant of motion
- If you put the system in an eigenstate of Q, it will <u>remain</u> in that eigenstate.

(d) The observable B has an eigenvector $|\mu\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ with eigenvalue μ.

Show that if $|\psi(0)\rangle = |\mu\rangle$ at time $t = 0$, then the time-evolved state $|\psi(t)\rangle \sim |\mu\rangle$ for all times $t > 0$, where "~" means "proportional to".

▶ WHAT THIS DEMONSTRATES: You found earlier that the operator B is a symmetry of our Hamiltonian. That means B is a conserved quantity, and so its eigenvalues thus provide **good quantum numbers** for the system. If a system is in an eigenstate of a symmetry operator, it will *remain* in that eigenstate over time and will always have the same quantum number → that's what we mean by a good quantum number.