

1a) No symmetry. Exchanging two different particles doesn't take you back to where you started.

b) To put two fermions in two states  $\psi_1, \psi_2$ , we build the

$$\text{wf } \psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)$$

If we try to put them in the same wavefunction, then

$$\psi(x_1, x_2) = \psi_1(x_1)\psi_1(x_2) - \psi_1(x_2)\psi_1(x_1) = 0$$

So we don't get a normalizable wavefunction.

2a) We know that for a two-particle system,  $\langle \hat{O} \rangle = \iint \psi^*(x_1, x_2) \hat{O} \psi(x_1, x_2)$ .

Using this, we can just write out each bracket.

$$\langle x_1^2 \rangle = \int dx_1 \int dx_2 \psi_a^*(x_1) \psi_b^*(x_2) x_1^2 \psi_a(x_1) \psi_b(x_2)$$

$$= \underbrace{\int dx_1 |\psi_a(x_1)|^2 x_1^2}_{\langle x^2 \rangle_a} \underbrace{\int dx_2 |\psi_b(x_2)|^2}_1 \quad \text{b/c } \psi_b \text{ is normalized}$$

$$= \langle x^2 \rangle_a$$

Similarly,

$$\langle x_2^2 \rangle = \int |\psi_a(x_1)|^2 dx_1 \int |\psi_b(x_2)|^2 x_2^2 dx_2 = \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \int x_1 |\psi_a(x_1)|^2 dx_1 \int x_2 |\psi_b(x_2)|^2 dx_2 = \langle x \rangle_a \langle x \rangle_b$$

Thus in total,  $\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b$

b) Our wavefunction now is  $\frac{1}{\sqrt{2}} [\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)]$ , w/  $+$   $\rightarrow$  Symmetric  
 $-$   $\rightarrow$  Antisymmetric

We now just do all the brackets again.

$$\begin{aligned}
 \langle x_1^2 \rangle &= \frac{1}{2} \int dx_1 \int dx_2 (\psi_a^*(x_1)\psi_b^*(x_2) \pm \psi_b^*(x_1)\psi_a^*(x_2)) x_1^2 (\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)) \\
 &= \frac{1}{2} \int dx_1 \int dx_2 \left[ \psi_a^*(x_1)\psi_b^*(x_2)\psi_a(x_1)\psi_b(x_2) x_1^2 + \psi_b^*(x_1)\psi_a^*(x_2)\psi_b(x_1)\psi_a(x_2) x_1^2 \right. \\
 &\quad \left. \pm \psi_a^*(x_1)\psi_b^*(x_2)\psi_a(x_2)\psi_b(x_1) x_1^2 \pm \psi_a^*(x_2)\psi_b^*(x_1)\psi_a(x_1)\psi_b(x_2) x_1^2 \right] \\
 &= \frac{1}{2} \left[ \underbrace{\int dx_1 |\psi_a(x_1)|^2 x_1^2}_{\langle x^2 \rangle_a} \underbrace{\int dx_2 |\psi_b(x_2)|^2}_{1} + \underbrace{\int dx_1 |\psi_b(x_1)|^2 x_1^2}_{\langle x^2 \rangle_b} \underbrace{\int dx_2 |\psi_a(x_2)|^2}_{1} \right. \\
 &\quad \left. \pm \int \psi_a^*(x_1)\psi_b(x_1) x_1^2 dx_1 \underbrace{\int \psi_b^*(x_2)\psi_a(x_2) dx_2}_0 \pm \int \psi_b^*(x_1)\psi_a(x_1) x_1^2 dx_1 \underbrace{\int \psi_a^*(x_2)\psi_b(x_2) dx_2}_0 \right] \\
 &= \frac{1}{2} \langle x^2 \rangle_a + \frac{1}{2} \langle x^2 \rangle_b
 \end{aligned}$$

Similarly,

$$\langle x_2^2 \rangle = \frac{1}{2} \langle x^2 \rangle_a + \frac{1}{2} \langle x^2 \rangle_b$$

Finally,

$$\begin{aligned}
 \langle x_1 x_2 \rangle &= \frac{1}{2} \int dx_1 \int dx_2 \left[ \psi_a^*(x_1)\psi_b^*(x_2)\psi_a(x_1)\psi_b(x_2) x_1 x_2 + \psi_b^*(x_1)\psi_a^*(x_2)\psi_b(x_1)\psi_a(x_2) x_1 x_2 \right. \\
 &\quad \left. \pm \psi_a^*(x_1)\psi_b^*(x_2)\psi_a(x_2)\psi_b(x_1) x_1 x_2 \pm \psi_b^*(x_1)\psi_a^*(x_2)\psi_b(x_2)\psi_a(x_1) x_1 x_2 \right] \\
 &= \frac{1}{2} \left[ \underbrace{\int |\psi_a(x)|^2 x_1 dx_1}_{\langle x \rangle_a} \underbrace{\int |\psi_b(x_2)|^2 x_2 dx_2}_{\langle x \rangle_b} + \underbrace{\int |\psi_b(x_1)|^2 x_1 dx_1}_{\langle x \rangle_b} \underbrace{\int |\psi_a(x_2)|^2 x_2 dx_2}_{\langle x \rangle_a} \right. \\
 &\quad \left. \pm \underbrace{\int \psi_a^*(x_1) x_1 \psi_b(x_1) dx_1}_{\langle \psi_a | \hat{x} | \psi_b \rangle} \underbrace{\int \psi_b^*(x_2) x_2 \psi_a(x_2) dx_2}_{\langle \psi_b | \hat{x} | \psi_a \rangle} \pm \underbrace{\int \psi_b^*(x_1) x_1 \psi_a(x_1) dx_1}_{\langle \psi_b | \hat{x} | \psi_a \rangle} \underbrace{\int \psi_a^*(x_2) x_2 \psi_b(x_2) dx_2}_{\langle \psi_a | \hat{x} | \psi_b \rangle} \right]
 \end{aligned}$$

$$= \frac{1}{2} \left[ \langle x \rangle_a \langle x \rangle_b + \langle x_a \rangle \langle x_b \rangle \pm 2 \langle \psi_a | \hat{x} | \psi_b \rangle \langle \psi_b | \hat{x} | \psi_a \rangle \right]$$

$$= \langle x \rangle_a \langle x \rangle_b \pm |\langle \psi_a | \hat{x} | \psi_b \rangle|^2$$

Thus for symmetric (+) state, we have

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b - 2 |\langle \psi_a | \hat{x} | \psi_b \rangle|^2$$

For antisymmetric (-) state,

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b + 2 |\langle \psi_a | \hat{x} | \psi_b \rangle|^2$$

c) Punchline: A symmetric state is closer ( $(x_1 - x_2)^2$  is smaller), while an antisymmetric state is further ( $(x_1 - x_2)^2$  is larger).

The distinguishable case is exactly between.

d) In general, we need

$$\begin{aligned} 1 = \langle \Psi | \Psi \rangle &= C^2 \iint (\psi_a^*(x_1) \psi_b^*(x_2) \pm \psi_a^*(x_2) \psi_b^*(x_1)) (\psi_a(x_1) \psi_b(x_2) \pm \psi_a(x_2) \psi_b(x_1)) dx_1 dx_2 \\ &= C^2 \left[ \underbrace{\int \psi_a^*(x_1) \psi_a(x_1) dx_1}_1 \underbrace{\int \psi_b^*(x_2) \psi_b(x_2) dx_2}_1 + \underbrace{\int \psi_b^*(x_1) \psi_b(x_1) dx_1}_1 \underbrace{\int \psi_a^*(x_2) \psi_a(x_2) dx_2}_1 \right. \\ &\quad \left. \pm \underbrace{\int \psi_a^*(x_1) \psi_b(x_1) dx_1}_{\langle \psi_a | \psi_b \rangle} \underbrace{\int \psi_b^*(x_2) \psi_a(x_2) dx_2}_{\langle \psi_b | \psi_a \rangle} \pm \underbrace{\int \psi_b^*(x_1) \psi_a(x_1) dx_1}_{\langle \psi_b | \psi_a \rangle} \underbrace{\int \psi_a^*(x_2) \psi_b(x_2) dx_2}_{\langle \psi_a | \psi_b \rangle} \right] \\ &= C^2 \left[ 2 \pm 2 |\langle \psi_a | \psi_b \rangle|^2 \right] \end{aligned}$$

Thus,

$$C = \frac{1}{\sqrt{2} \sqrt{1 \pm |\langle \psi_a | \psi_b \rangle|^2}}$$

Thus, if  $a+b$  are orthogonal,  $\langle \psi_a | \psi_b \rangle = 0$  and

$$C = \frac{1}{\sqrt{2}} \text{ for both symmetric + antisymmetric states.}$$

If  $\psi_a = \psi_b$ , then  $\langle \psi_a | \psi_b \rangle = 1$ , and

$$C = \frac{1}{\sqrt{2} \sqrt{1 \pm 1}}. \text{ In the case of a symmetric state, we}$$

find  $C = \frac{1}{2}$ , and for an antisymmetric state, we find  $C = \frac{1}{0}$ , which makes sense, since in part (1b) we showed an antisymmetric state in the same wavefunction ~~could~~ couldn't exist.

$$3 a \bar{z}) E_{n_1, n_2} = E_{n_1} + E_{n_2} = (n_1^2 + n_2^2) K$$

i) Ground state is the lowest allowed energy,  $n_1 = n_2 = 1$ . Thus,  $E_{\text{ground}} = E_{11} = 2K$ . There is only one state w/ energy  $2K$ , thus ground state has no degeneracy.

ii) For distinguishable ptles, there are two ways to get the first excited state. Either  $(n_1=1, n_2=2)$ , or  $(n_1=2, n_2=1)$ . So the excited energy is  $5K$ , w/ degeneracy 2.

$$b \bar{z}) \psi_{n_1, n_2}^{\pm}(x_1, x_2) = \psi_{n_1}^{\pm}(x_1) \psi_{n_2}^{\pm}(x_2) + \psi_{n_2}^{\pm}(x_1) \psi_{n_1}^{\pm}(x_2)$$

ii) Ground state is still given by  $n_1 = n_2 = 1$ , has energy  $2K$ .

iii) Here,  $(n_1=1, n_2=2)$  and  $(n_1=2, n_2=1)$  are the same state. So 1st excited state has energy  $5K$ , degeneracy 1.

~~$$\psi_{n_1, n_2}^{\pm}(x_1, x_2) = \psi_{n_1}^{\pm}(x_1) \psi_{n_2}^{\pm}(x_2) + \psi_{n_2}^{\pm}(x_1) \psi_{n_1}^{\pm}(x_2)$$~~

$$(i) \Psi_{n_1, n_2}(x_1, x_2) = \Psi_{n_1}(x_1) \Psi_{n_2}(x_2) - \Psi_{n_2}(x_1) \Psi_{n_1}(x_2)$$

(ii) If we set  $n_1 = n_2 = 1$ , we get  $\Psi_{11}(x_1, x_2) = 0 \rightarrow$  not a valid state.

Thus, we must have  $n_1 \neq n_2$ . We still have that

$(n_1=1, n_2=2)$  and  $(n_1=2, n_2=1)$  are the same state, so our ground

state has energy  $5K$  and no degeneracy.

(iii) The first excited state has  ~~$\{1, 2\}$~~   $\{n_1, n_2\} = \{1, 3\}$ . It still has degeneracy 1, energy  $10K$ .