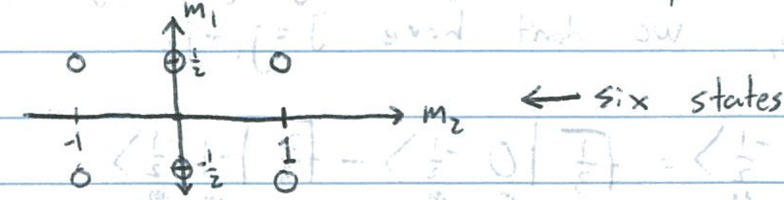
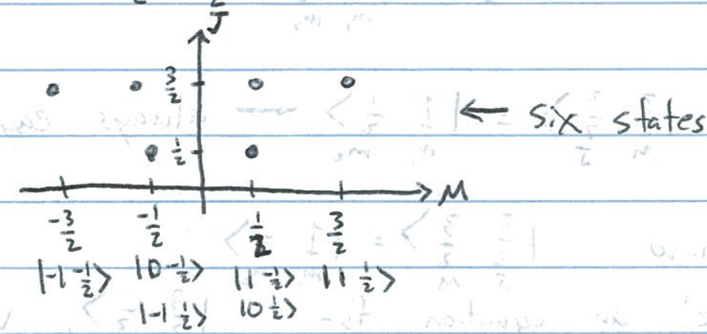
1 a) The allowed values of m_j are $j, j-1, \ldots, -j$. So $m_1 = 1, 0, -1$, and $m_2 = \frac{1}{2}, \frac{-1}{2}$. These values are uncorrelated; we can have any pair (m_1, m_2)



b) We know J can be $\frac{3}{2}, \frac{1}{2}$. If $J = \frac{3}{2}$, $M = \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}$. If $J = \frac{1}{2}$, $M = \frac{1}{2}, -\frac{1}{2}$



c) If we want to form a $|J M\rangle$ state out of $|m_1 m_2\rangle$ states, we must have $m_1 + m_2 = M$. Thus, for the $|\underset{J}{\tfrac{3}{2}}\ \underset{M}{\tfrac{3}{2}}\rangle$, it can only be made of $|\underset{m_1}{1}\ \underset{m_2}{\tfrac{1}{2}}\rangle$.

Similarly,
$|\frac{3}{2} \frac{1}{2}\rangle$ can be made of $|0 \frac{1}{2}\rangle + |1 -\frac{1}{2}\rangle$
$|\frac{3}{2} -\frac{1}{2}\rangle$ can be made of $|0 -\frac{1}{2}\rangle + |-1, \frac{1}{2}\rangle$
$|\frac{3}{2} -\frac{3}{2}\rangle$ can be made of $|-1 -\frac{1}{2}\rangle$
$|\frac{1}{2} \frac{1}{2}\rangle$ can be made of $|0 \frac{1}{2}\rangle + |1 -\frac{1}{2}\rangle$
$|\frac{1}{2} -\frac{1}{2}\rangle$ can be made of $|0 -\frac{1}{2}\rangle + |-1 \frac{1}{2}\rangle$

d) ~~We~~ Say we have a state $|j_1 m_1; j_2 m_2\rangle$. Then

$$(J_1 + J_2)^2 |j_1 m_1; j_2 m_2\rangle = J_1^2 |j_1 m_1; j_2 m_2\rangle + J_2^2 |j_1 m_1; j_2 m_2\rangle + 2\vec{J}_1 \cdot \vec{J}_2 |j_1 m_1; j_2 m_2\rangle$$

$$= \left(\hbar^2 j_1(j_1+1) + \hbar^2 j_2(j_2+1) + 2\vec{J}_1\cdot J_2\right)|j_1 m_1; j_2 m_2\rangle$$

Which is <u>not</u> the same as

$$\hbar^2(j_1+j_2)(j_1+j_2+1)|j_1 m_1; j_2 m_2\rangle.$$

Thus, we don't have $J = j_1 + j_2$

e) $|\underset{J}{\tfrac{1}{2}}, \underset{M}{-\tfrac{1}{2}}\rangle = \sqrt{\tfrac{1}{3}}\,|\underset{m_1}{0}\ \underset{m_2}{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{2}{3}}\,|\underset{m_1}{-1}\ \underset{m_2}{\tfrac{1}{2}}\rangle$

f) In (c) we said that the $|\underset{J}{\tfrac{3}{2}}, \underset{M}{-\tfrac{3}{2}}\rangle$ state could <u>only</u> be written as the $|\underset{m_1}{-1}\ \underset{m_2}{-\tfrac{1}{2}}\rangle$ state. So $|\underset{J}{\tfrac{3}{2}}, \underset{M}{-\tfrac{3}{2}}\rangle = |\underset{m_1}{-1}, \underset{m_2}{-\tfrac{1}{2}}\rangle$

g) Also $|\underset{M}{\tfrac{3}{2}}\ \underset{J}{\tfrac{3}{2}}\rangle = |\underset{m_1}{1}\ \underset{m_2}{\tfrac{1}{2}}\rangle \longleftarrow$ always easy to find top/bottom states!

h) A) We know $|\underset{J}{\tfrac{3}{2}}\ \underset{M}{\tfrac{3}{2}}\rangle = |\underset{m_1}{1}\ \underset{m_2}{\tfrac{1}{2}}\rangle$

B) To get an equation for $|\tfrac{3}{2}\ \tfrac{1}{2}\rangle$, we apply the J^- operator to both sides of the equation in (A).

$$J^-|\underset{M}{\tfrac{3}{2}}\ \underset{J}{\tfrac{3}{2}}\rangle = J^-|\underset{m_1}{1}\ \underset{m_2}{\tfrac{1}{2}}\rangle$$

The left hand side (LHS) of this equation we can simplify using our formula for J^-:

$$LHS = J^-|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}\left(\tfrac{3}{2}+1\right) - \tfrac{3}{2}\left(\tfrac{3}{2}-1\right)}\ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

The RHS we simplify using $J^- = J_1^- + J_2^-$, and the formula for J_i^-. Note J_1^- only looks at j_1, m_1, and J_2^- only at j_2, m_2.

$$RHS = (J_1^- + J_2^-)|1\ \tfrac{1}{2}\rangle = J_1^-|1\ \tfrac{1}{2}\rangle + J_2^-|1, \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{\underset{j_1}{1}(1+1) - \underset{m_1}{1}(1-1)}\ |0\ \tfrac{1}{2}\rangle + \hbar\sqrt{\underset{j_2}{\tfrac{1}{2}}\left(\tfrac{1}{2}+1\right) - \underset{m_2}{\tfrac{1}{2}}\left(\tfrac{1}{2}-1\right)}\ |1\ -\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2}\ |0\ \tfrac{1}{2}\rangle + \hbar\,|1\ -\tfrac{1}{2}\rangle$$

Thus, $\hbar\sqrt{3}\,|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \hbar\sqrt{2}\,|0\ \tfrac{1}{2}\rangle + \hbar\,|1\ -\tfrac{1}{2}\rangle$, or

$$\underset{J\ \ M}{|\tfrac{3}{2}\ \tfrac{1}{2}\rangle} = \sqrt{\tfrac{2}{3}}\,\underset{m_1\ m_2}{|0\ \tfrac{1}{2}\rangle} + \sqrt{\tfrac{1}{3}}\,\underset{m_1\ m_2}{|1\ -\tfrac{1}{2}\rangle}$$

To find $|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$ we repeat:

$$J^-|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}(J_1^- + J_2^-)|0\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}(J_1^- + J_2^-)|1\ -\tfrac{1}{2}\rangle$$

zero b/c $-\tfrac{1}{2}$ can't be lowered

$$\hbar\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\,|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}J_1^-|0\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}J_2^-|0\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}J_1^-|1\ -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}J_2^-|1\ -\tfrac{1}{2}\rangle$$

$$\cancel{2\hbar|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\hbar\sqrt{0(0+1) - 0(0-1)}\,|0\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\hbar\sqrt{1(1+1)}}$$

$$2\hbar|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\hbar\sqrt{1(1+1) - 0(0-1)}\,|-1\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\,|0\ -\tfrac{1}{2}\rangle$$
$$+ \sqrt{\tfrac{1}{3}}\hbar\sqrt{1(1+1) - 1(1-1)}\,|0\ -\tfrac{1}{2}\rangle$$

$$2\hbar|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \tfrac{2}{\sqrt{3}}\hbar|-1\ \tfrac{1}{2}\rangle + 2\sqrt{\tfrac{2}{3}}\hbar|0\ -\tfrac{1}{2}\rangle, \text{ or}$$

$$\underset{M\ \ J}{|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle} = \sqrt{\tfrac{1}{3}}\,\underset{m_1\ m_2}{|-1\ \tfrac{1}{2}\rangle} + \sqrt{\tfrac{2}{3}}\,\underset{m_1\ m_2}{|0\ -\tfrac{1}{2}\rangle}$$

Finally, we already know $|\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |-1\ -\tfrac{1}{2}\rangle$. But if we wanted, we could get this result by again applying J^- to both sides.

C) Now, we want to find $\underset{J\ \ M}{|\tfrac{1}{2}, \tfrac{1}{2}\rangle}$. From our work above, we know $|\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|0\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1\ -\tfrac{1}{2}\rangle$. From (c), we also know that

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle = A|0\ \tfrac{1}{2}\rangle + B|1, -\tfrac{1}{2}\rangle \quad \text{for some } A \text{ and } B.$$

Finally, we know $|\tfrac{3}{2}, \tfrac{1}{2}\rangle$ + $|\tfrac{1}{2}, \tfrac{1}{2}\rangle$ must be orthogonal, b/c they are eigenstates of J^2 w/ different eigenvalues. Thus,

$$0 = \langle \underset{J\ M}{\tfrac{3}{2}\ \tfrac{1}{2}} | \underset{J\ M}{\tfrac{1}{2}\ \tfrac{1}{2}} \rangle = \left(\sqrt{\tfrac{2}{3}}\langle 0\ \tfrac{1}{2}| + \sqrt{\tfrac{1}{3}}\langle 1\ -\tfrac{1}{2}|\right)\left(A|0\ \tfrac{1}{2}\rangle + B|1\ -\tfrac{1}{2}\rangle\right)$$
$$= \sqrt{\tfrac{2}{3}}A + \sqrt{\tfrac{1}{3}}B.$$

We then find $B = -\sqrt{2}A$. Since $A^2 + B^2 = 1$ to normalize, we then have

$$A^2 + 2A^2 = 1, \text{ or } A = \tfrac{1}{\sqrt{3}},\quad B = -\sqrt{\tfrac{2}{3}}.$$

We thus have

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|0\ \tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|1\ -\tfrac{1}{2}\rangle$$

We figured this out using only orthonormality.

Finally, we can find $|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$ by applying J^- to both sides of our $|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$ equation

$$J^-|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\left(J_1^- + J_2^-\right)|0\ \tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\left(J_1^- + J_2^-\right)|1\ -\tfrac{1}{2}\rangle$$

$$\hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,J_1^-|0\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,J_2^-|0\ \tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,J_1^-|1\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,J_2^-|1\ -\tfrac{1}{2}\rangle$$

$$\hbar|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,\hbar\sqrt{1(1+1) - 0(0-1)}\,|-1, \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,\hbar\sqrt{(\tfrac{1}{2})(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\,|0\ -\tfrac{1}{2}\rangle$$
$$- \sqrt{\tfrac{2}{3}}\,\hbar\sqrt{1(1+1) - 1(1-1)}\,|0\ -\tfrac{1}{2}\rangle + 0$$

$$\hbar|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,\hbar|-1, \tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,\hbar|0\ -\tfrac{1}{2}\rangle$$

So,

$$|\underset{J}{\tfrac{1}{2}}\ \underset{M}{-\tfrac{1}{2}}\rangle = \sqrt{\tfrac{2}{3}}\,|\underset{m_1}{-1}\ \underset{m_2}{\tfrac{1}{2}}\rangle - \sqrt{\tfrac{1}{3}}\,|\underset{m_1}{0}\ \underset{m_2}{-\tfrac{1}{2}}\rangle$$

And we're done!

Problem 2 → Homework