Phys 486 Discussion 14 – Angular Momentum Addition : Clebsch-Gordan Coefficients

We are learning how to add two angular momenta, j_1 and j_2. We can add them in two ways: we can

- list all the eigenstates $|m_1 m_2\rangle$ of the operators j_{1z} & j_{2z} of the INDIVIDUAL ang. momenta, OR
- list all the eigenstates $|JM\rangle$ of the TOTAL ang. mom. operators $J^2 \equiv |\vec{j}_1 + \vec{j}_2|^2$ & $J_z \equiv (j_{1z} + j_{2z})$.

The eigenstates $|m_1 m_2\rangle$ and $|JM\rangle$ provide different bases with which to describe the sum of one angular momentum j_1 with another one j_2. In class, we learned how to read tables of **Clebsch-Gordan coefficients** to express a $|JM\rangle$ as a linear combination of $|m_1 m_2\rangle$'s. These tables are now on our website, filename Formulae-CGtables.pdf. In this discussion, we will learn how to calculate the CG coefficients so that we understand exactly what they are, and where they come from, and that the scary-looking CG tables are not in the least bit mysterious. ☺

Problem 1 : Deuterium Atom

We'll continue to explore our example from class, which was to add these two angular momenta:

- $j_1 = 1$ is the spin s_d of a deuterium nucleus (a.k.a. a "deuteron")
- $j_2 = ½$ is the spin s_e of an electron.

If we bring the deuteron and electron together, we get a deuterium atom with total spin $j_1 \oplus j_2$. The "$\oplus$" symbol means "an addition that's more complex than 2+3 = 5". As we discussed, "$j_1 \otimes j_2$" is also used.

(a) Make a plot with the m_1 axis pointing upward and the m_2 axis pointing sideways, and mark with solid circles all the points (m_1, m_2) where $|m_1 m_2\rangle$ is a physically-possible state. How many states do you have?

(b) Make a plot with the J axis pointing upward and the M axis pointing sideways, and mark with solid circles all the points (J, M) where $|JM\rangle$ is a physically possible state. **Leave lots of space between your tick marks on the horizontal (M) axis** … like an inch of space … we're going to write things under these tick marks. To make your plot, you need the first of the two angular momentum addition rules we learned today:

(1) The total J quantum number runs from $|j_1 - j_2|$ to $|j_1 + j_2|$ in steps of 1.
(2) The total M quantum number is additive: $M = m_1 + m_2$.

You also need the rule restricting the m quantum number for any angular momentum:

- The m (or M, or m_2, …) quantum number runs from $-j$ to $+j$ (or $-J$ to $+J$, or $-j_2$ to $+j_2$,…) in steps of 1.

How many states do you have? It should be the same as in part (a)!

(c) **Underneath** each of the M tick marks that has at least one solid circle above it — i.e. under each M value that has at least one valid $|JM\rangle$ eigenstate — write a list of all the $|m_1 m_2\rangle$ states that might contribute to said M value. HINT: You need the second of our two angular momentum addition rules.

(d) As we discussed, $m_1 + m_2 = M$, but $j_1 + j_2 \neq J$. The m quantum number is additive, but the j quantum number is not. Do you understand why this is? If not, check the hint[1] ... if it's still unclear, ask!!!

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[1] j and m are **quantum numbers**. They **label the eigenvalues** of certain operators, but they **are not eigenvalues**. (For a more familiar example, recall the energy quantum number n : it's an integer that labels energy eigenvalues but it is not an energy itself; we derived formulae like $E_n = -13.6$ eV / n^2 to relate n to the corresponding eigenvalue E_n. The relations between j & m and the e-values they label are: $\hat{j}_z |\, j\, m\rangle = \hbar m\, |\, j\, m\rangle$ and $\hat{j}\, |\, j\, m\rangle = \hbar\sqrt{j(j+1)}\, |\, j\, m\rangle$. The m eigenvalue is linear in m, therefore m's are additive.

(e) Consult the Clebsch-Gordan tables on the web and write the eigenstate $\left|J,M\right\rangle = \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_{J,M}$ as a linear combination of $\left|m_1,m_2\right\rangle$ eigenstates.

(f) Consult nothing and write the $\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle_{J,M}$ eigenstate as a linear combination of $\left|m_1,m_2\right\rangle$ eigenstates.

(g) You just realized that the $\left|J,M\right\rangle$ states $\left|\tfrac{3}{2},\pm\tfrac{3}{2}\right\rangle_{J,M}$ each match a SINGLE $\left|m_1,m_2\right\rangle$ state: $\left|\pm 1,\pm\tfrac{1}{2}\right\rangle_{m_1,m_2}$.
These are called **stretched states** because all j vectors are maximally aligned along or against the z axis.
The stretched states should be found at the top-right and top-left corners of the (M, J) plot you've been making. Circle the stretched state with positive M. This will be the **starting point** for our Clebsch-Gordan calculations because it is always uniquely determined : $\left|J_{\max}, M_{\max}\right\rangle = \left|m_{1\max}, m_{2\max}\right\rangle = \left|+j_1,+j_2\right\rangle$. Simple starting point!

(h) THE WHOLE THING: Here is the **procedure** for building all the $\left|J, M\right\rangle$ states from $\left|m_1, m_2\right\rangle$ states, and thereby calculating all the Clebsch-Gordan (CG) coefficients for the given j_1 and j_2 values:

A. Build the **stretched state** $\left|J_{\max}, M_{\max}\right\rangle$ = top-right point on your plot.

B. Build the states with the same J but lower M — step to the LEFT on your plot — by applying the **step-down** operator $J_- = j_{1-} + j_{2-}$. This step has two parts:

1. use $\hat{J}_{\pm}\left|j, m\right\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\ \left|j, m\pm 1\right\rangle$ $\rightarrow$ apply J_- to the $\left|J, M\right\rangle$ form
2. use the concept $\hat{Q}_{1+2} = \hat{Q}_{1\text{only}} + \hat{Q}_{2\text{only}}$ $\rightarrow$ apply $J_- = j_{1-} + j_{2-}$ to the $\left|m_1, m_2\right\rangle$ form

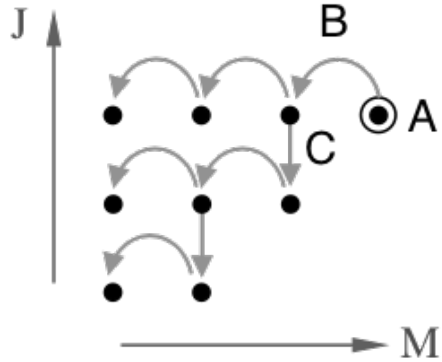
Step-down until you have filled in the whole line with the J you are currently working on.
Check the CG tables on the course website $\rightarrow$ are your answers correct?

C. Go back to the right-hand-side of the J line you've just completed, move one step to the left, to $\left|J, J-1\right\rangle_{J,M}$, then move DOWN one step and build $\left|J-1, J-1\right\rangle_{J,M}$ using **orthonormality**.

How does that work exactly? $\rightarrow$ Look at the list of $|m_1\ m_2\rangle$ states you wrote below this M column, and you'll see that there are exactly the same number of them as there are $|J\,M\rangle$ states in the column. It must be so. This whole exercise is a change of basis: we are building an orthonormal basis $|J\,M\rangle$ from a different orthonormal basis $|m_1\ m_2\rangle$. The states with a particular M value — call it M_0 — are a subset of both bases, satisfying $M = M_0$ on the one hand and $m_1+m_2 = M_0$ on the other hand. The number of $|m_1\ m_2\rangle$ states in this subspace must be the same as the number of $|J\,M\rangle$ states since both sets span the same subspace. Now realize that you have already built all but one of the $|J\,M\rangle$ states in this column. Since you only have one left, and you know all the $|m_1\ m_2\rangle$ states you can use to build it, you can build the final $|J\,M\rangle$ state in the column by making it orthonormal to the other $|J\,M\rangle$ states in the column. Get the idea? You will when you try it!

Repeat steps B and C until all the states $|J\,M\rangle$ are written as linear combinations of the states $|m_1\ m_2\rangle$. The figure at right may help you to visualize the procedure:

A. START at **stretched state** with maximum M
B. move LEFT via **step-down** operator $J_- = j_{1-} + j_{2-}$
C. move DOWN at bottom of a column via **orthonormality**



Problem 2 : Repeat for 3/2 x 1 $\rightarrow$ will be on homework 13

Add a spin-3/2 particle and a spin-1 particle to build the state $|J, M\rangle = |\tfrac{3}{2}, +\tfrac{1}{2}\rangle_{J,M}$ from $|m_1, m_2\rangle$ states.

Calculate the result using the A, B, C procedure above, doing only the steps you need (!!!) to reach the state you want. Check your answer against the CG table on the website, of course.