1 a) $P(S_z = -\frac{\hbar}{2}) = |\langle e_{-\frac{\hbar}{2}} | \chi \rangle|^2$, where $|\chi\rangle$ is our current state, and $|e_{-\frac{\hbar}{2}}\rangle$ is the eigenstate of S_z w/ eigenvalue $-\frac{\hbar}{2}$.

In our case, we know $|e_{-\frac{\hbar}{2}}\rangle = |\chi_-\rangle$, so

$$P(S_z = -\tfrac{\hbar}{2}) = |\langle \chi_- | \chi \rangle|^2 = \left| \tfrac{3}{5} \langle \chi_- | \chi_+ \rangle + \tfrac{4}{5} \langle \chi_- | \chi_- \rangle \right|^2 = \left(\tfrac{4}{5}\right)^2 = \tfrac{16}{25}$$

b) We have that $\chi = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$ in the S_z-eigenbasis. We need to find ~~where there the~~ the eigenstate of the $\underline{S_x}$ operator w/ eigenvalue $+\frac{\hbar}{2}$. I'll call this state $|e_+\rangle$. We need $\hat{S}_x |e_+\rangle = \frac{\hbar}{2} |e_+\rangle$, or writing $e_+ = \begin{pmatrix} a \\ b \end{pmatrix}$ in the S_z-eigenbasis,

$$\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}.$$ This is solved by $a = b$, so

$e_+ = \begin{pmatrix} a \\ a \end{pmatrix}$ for some a. We find a by normalization:

$$1 = \langle e_+ | e_+ \rangle = (a^*, a^*) \begin{pmatrix} a \\ a \end{pmatrix} = 2|a|^2,$$ so we take $a = \frac{1}{\sqrt{2}}$.

Now, $$P(S_x = \tfrac{\hbar}{2}) = |\langle e_+ | \chi \rangle|^2 = \left| \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \right|^2 = \left(\frac{7}{5\sqrt{2}} \right)^2 = \frac{49}{50}$$

c) We need to find the state $|e_-\rangle$, which satisfies $\hat{S}_x |e_-\rangle = -\frac{\hbar}{2} |e_-\rangle$. Again writing this in the S_z-eigenbasis, we find

$$\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix},$$ which is solved w/ $a = -b$. We find a by normalizing,

$$1 = \langle e_- | e_- \rangle = (a^*, -a^*) \begin{pmatrix} a \\ -a \end{pmatrix} = 2|a|^2,$$ so again $a = \frac{1}{\sqrt{2}}$. Then

$$P(S_x = -\tfrac{\hbar}{2}) = |\langle e_- | \chi \rangle|^2 = \left| \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right) \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \right|^2 = \left| \frac{1}{5\sqrt{2}} \right|^2 = \frac{1}{50}$$

These probabilities add to one, as required!

3) The allowed values for m_s are -1, 0, and 1, so we have three states, $|1\rangle$, $|0\rangle$, $|-1\rangle$.

These states are by definition the $\hat{S}_z$-eigenstates, so

$$\hat{S}_z|1\rangle = \hbar|1\rangle \qquad \hat{S}_z|0\rangle = 0 \qquad \hat{S}_z|-1\rangle = -\hbar|-1\rangle.$$

In matrix form, ordering the basis elements from largest m_s to smallest,

$$\hat{S}_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

Now, we find $\hat{S}_+$.

$$\hat{S}_+|-1\rangle = \hbar\sqrt{1(1+1)-(-1)(-1+1)}\,|0\rangle = \hbar\sqrt{2}\,|0\rangle$$

$$S_+|0\rangle = \hbar\sqrt{1(1+1)-0(0+1)}\,|1\rangle = \hbar\sqrt{2}\,|0\rangle$$

$$S_+|1\rangle = \hbar\sqrt{1(1+1)-1(1+1)}\,|\text{"}2\text{"}\rangle = 0$$

So in matrix form, $S_+ = \begin{pmatrix} 0 & \hbar\sqrt{2} & 0 \\ 0 & 0 & \hbar\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$

We similarly do $\hat{S}_-$:

$$\hat{S}_-|-1) = \hbar\sqrt{1(1+1)-(-1)(-1-1)}\,|\text{"}-2\text{"}\rangle = 0$$

$$\hat{S}_-|0\rangle = \hbar\sqrt{1(1+1)-0(0-1)}\,|-1\rangle = \hbar\sqrt{2}\,|-1\rangle$$

$$\hat{S}_-|1\rangle = \hbar\sqrt{1(1+1)-1(1-1)}\,|0\rangle = \hbar\sqrt{2}\,|0\rangle$$

So,

$$S_- = \begin{pmatrix} 0 & 0 & 0 \\ \hbar\sqrt{2} & 0 & 0 \\ 0 & \hbar\sqrt{2} & 0 \end{pmatrix}$$

Thus,

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{2i}(S_+ - S_-) = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

4a) $1 = \langle \chi | \chi \rangle = A^* (-3i, 4)\begin{pmatrix} 3i \\ 4 \end{pmatrix} A = |A|^2 [9+16] = 25|A|^2$, so

$A = \frac{1}{5}$.

b) $\langle S_x \rangle = \langle \chi | S_x | \chi \rangle = \frac{1}{25} (-3i \quad 4) \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{1}{25} (-3i \quad 4) \begin{pmatrix} 2\hbar \\ 3\hbar i/2 \end{pmatrix}$

$= \frac{1}{25} [-6\hbar i + 6\hbar i] = 0$

$\langle S_y \rangle = \langle \chi | S_y | \chi \rangle = \frac{1}{25} (-3i \quad 4) \begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{1}{25} (-3i \quad 4) \begin{pmatrix} -2i\hbar \\ -3\hbar/2 \end{pmatrix}$

$= \frac{1}{25} [-6\hbar - 6\hbar] = -\frac{12}{25}\hbar$

$\langle S_z \rangle = \langle \chi | S_z | \chi \rangle = \frac{1}{25} (-3i \quad 4) \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{1}{25} (-3i \quad 4) \begin{pmatrix} \frac{3i\hbar}{2} \\ -2\hbar \end{pmatrix}$

$= \frac{1}{25} [9\hbar/2 - 8\hbar] = \frac{7\hbar}{50}$

c) $\sigma_{S_x} = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} \qquad \sigma_{S_y} = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} \qquad \sigma_{S_z} = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2}$.

Note that $S_x^2 = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and similarly

$S_y^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad S_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So,

$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}$.

Thus, $\sigma_{S_x} = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2} \qquad \sigma_{S_y} = \sqrt{\frac{\hbar^2}{4} - \frac{144}{625}\hbar^2} = \frac{7\hbar}{50}, \quad \sigma_{S_z} = \sqrt{\frac{\hbar^2}{4} - \frac{49\hbar^2}{2500}} = \frac{12\hbar}{25}$

d) $\sigma_{S_x}\sigma_{S_y} = \frac{7\hbar^2}{100}$, while $\frac{\hbar}{2}|\langle S_z \rangle| = \frac{\hbar}{2} \times \frac{7\hbar}{50} = \frac{7\hbar^2}{100}$, so $\sigma_x \sigma_y \geq \frac{\hbar}{2}|\langle S_z \rangle|$ saturates

$\sigma_{S_y}\sigma_{S_z} = \frac{42\hbar^2}{625}$, while $\frac{\hbar}{2}|\langle S_x \rangle| = 0$, so $\sigma_y \sigma_z \geq \frac{\hbar}{2}|\langle S_x \rangle|$ trivially.

$\sigma_{S_z}\sigma_{S_x} = \frac{6\hbar^2}{25}$, while $\frac{\hbar}{2}|\langle S_y \rangle| = \frac{\hbar}{2} \times \frac{12\hbar}{25} = \frac{6\hbar^2}{25}$, so $\sigma_z \sigma_x \geq \frac{\hbar}{2}|\langle S_y \rangle|$

saturates.