1a) We know $\hat{L}^2 Y_\ell^m = \hbar^2 \ell(\ell+1)$. Our guess for Ψ is $\Psi = R(r)Y(\theta,\phi)$

We thus have

$$E\Psi = \hat{H}\Psi, \quad \text{or} \quad ERY = \hat{H}RY, \text{ or}$$

$$ERY = \frac{1}{r^2}\left[-\frac{\hbar^2}{2m}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)Y + R\frac{\hat{L}^2 Y}{2m}\right] + V(r)RY. \text{ Using } \hat{L}^2 Y = \hbar^2\ell(\ell+1)Y,$$

$$= \frac{1}{r^2}\left[-\frac{\hbar^2}{2m}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)Y + \frac{\hbar^2}{2m}\ell(\ell+1)RY\right] + V(r)RY. \text{ Dividing by } Y,$$

$$ER = \frac{1}{r^2}\left[-\frac{\hbar^2}{2m}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\hbar^2}{2m}\ell(\ell+1)R\right] + V(r)R$$

b) We want to make the substitution $u(r) = rR(r)$, or $R = \frac{u}{r}$.

~~Then~~ $\cancel{\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{R}{r}\right)\right) = \frac{\partial}{\partial r}\left(r^2\left(\frac{R'}{r} - \frac{R}{r^2}\right)\right) = \frac{\partial}{\partial r}\left(rR' - R\right) = rR''}$

Then $\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right) = \frac{\partial}{\partial r}\left(r^2\left(\frac{u'}{r} - \frac{u}{r^2}\right)\right) = \frac{\partial}{\partial r}\left(ru' - u\right) = ru''$

Thus, plugging this in to our equation (a),

$$E\frac{u}{r} = \frac{1}{r^2}\left[-\frac{\hbar^2}{2m}ru'' + \frac{\hbar^2}{2m}\ell(\ell+1)\frac{u}{r}\right] + V(r)\frac{u}{r}, \text{ or}$$

$$Eu = -\frac{\hbar^2}{2m}u'' + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u$$

Multiplying by $-\frac{2m}{\hbar^2}$ gives

$$-\frac{2mE}{\hbar^2}u = u'' - \frac{2m}{\hbar^2}\left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u$$

The units of $\frac{2mE}{\hbar^2}$ are $\frac{[m][E]}{[\hbar]^2} = \frac{(kg)\left(kg\frac{m^2}{s^2}\right)}{\left(kg\frac{m^2}{s}\right)^2} = \frac{1}{m^2}$

c) To make everything unitless, we multiply both sides by r^{2}.

$$-\frac{2mE}{\hbar^2}r^2u = u''r^2 - \frac{2m}{\hbar^2}\left[r^2V(r) + \frac{\hbar^2}{2m}\ell(\ell+1)\right]u, \text{ or}$$

$$r^2u'' = \frac{2m}{\hbar^2}\left(V(r)r^2 - Er^2\right)u + \ell(\ell+1)u$$

d) $\frac{e^2}{4\pi\varepsilon_0} = \frac{e^2}{4\pi\varepsilon_0 r^2} r^2$. The first part is force, while the second is m^2.

So $\left[\frac{e^2}{4\pi\varepsilon_0}\right] = \left[\frac{e^2}{4\pi\varepsilon_0 r^2}\right][r^2] = \left(\frac{kg\ m}{s^2}\right)(m^2) = \frac{kg\ m^3}{s^2}$

e) $[\hbar c] = \left(\frac{kg\ m^2}{s}\right)\left(\frac{m}{s}\right) = \frac{kg\,m^3}{s^2}$

f) $\hbar c = (6.582 \times 10^{-16}\ eV\cdot sec)(2.998\times 10^{17}\ nm/sec) = 197\ eV\cdot nm$

g) $\alpha = \frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{\hbar c}$. So $\frac{1}{\alpha} = \frac{\hbar c}{e^2/4\pi\varepsilon_0} = 137.036$

h) We'll first replace all the E's w/ $K = \frac{\sqrt{-2mE}}{\hbar}$. The eq'n from (c) becomes

$$r^2 u'' = \left[K^2 r^2 + \ell(\ell+1) - \frac{Ze^2}{4\pi\varepsilon_0}\frac{2mr}{\hbar^2}\right]u$$

Now, replace r w/ ρ/K. Then $\frac{\partial}{\partial r} = K\frac{\partial}{\partial \rho}$, and

$$K^2\left(\frac{\rho}{K}\right)^2 u'' = \left[K^2\left(\frac{\rho}{K}\right)^2 + \ell(\ell+1) - \frac{Ze^2}{4\pi\varepsilon_0}\frac{2m}{\hbar^2}\left(\frac{\rho}{K}\right)\right]u, \text{ or}$$

$$\rho^2 u'' = \left[\rho^2 + \ell(\ell+1) - \frac{Ze^2}{4\pi\varepsilon_0}\frac{2m}{\hbar^2 K}\rho\right]u$$

i) Call $\frac{Ze^2}{4\pi\varepsilon_0}\frac{2m}{\hbar^2 K} = \lambda$, where λ is clearly dimensionless.

Dividing by ρ^2, we get $u'' = \left[1 - \frac{\lambda}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u$

2a) In the limit $\rho\to\infty$, the dominant terms are $u'' = u$, or $u(\rho) = Ae^{\rho} + Be^{-\rho}$. Since the wavefunction must $\to 0$ as $\rho\to\infty$, we have $A = 0$, so $u_\infty(\rho) \sim e^{-\rho}$.

b) In the limit $\rho\to 0$, the dominant terms are $u'' = \frac{\ell(\ell+1)}{\rho^2}u$.

This equation has solutions $A\rho^{\ell+1} + \frac{B}{\rho^\ell}$. Since the wavefunction ~~probably~~ can't $\to\infty$ as $\rho\to 0$, we have $B=0$. Thus,

$u_0(\rho) \sim \rho^{\ell+1}$

3a) Plugging in the guess $u(\rho) = e^{-\rho}\rho^{\ell+1}h(\rho)$; and doing algebra, we get

$$\rho h'' + 2(\ell+1-\rho)h' + (\lambda - 2(\ell+1))h = 0.$$

As our solution to $h(\rho)$, we guess $h(\rho) = \sum_{j=0}^{\infty} a_j \rho^j$. Then

$h'(\rho) = \sum_{j=0}^{\infty} j a_j \rho^{j-1}$, $h''(\rho) = \sum_{j=0}^{\infty} j(j-1)a_j \rho^{j-2}$. Plugging this in, we get

$$\rho\left(\sum_{j=0}^{\infty} a_j j(j-1)\rho^{j-2}\right) + 2(\ell+1-\rho)\left(\sum_{j=0}^{\infty} j a_j \rho^{j-1}\right) + (\lambda - 2(\ell+1))\left(\sum_{j=0}^{\infty} a_j \rho^j\right) = 0, \text{ or}$$

$$\sum_{j=0}^{\infty} j(j-1)a_j\rho^{j-1} + 2(\ell+1)\sum_{j=0}^{\infty} j a_j \rho^{j-1} - 2\sum_{j=0}^{\infty} j a_j \rho^j + (\lambda-2(\ell+1))\sum_{j=0}^{\infty} a_j \rho^j = 0, \text{ o.}$$

$$\sum_{j=0}^{\infty}\left[j(j-1) + 2(\ell+1)j\right]a_j\rho^{j-1} + \sum_{j=0}^{\infty}\left[\lambda - 2(\ell+1) - 2j\right]a_j\rho^j = 0$$

We want to gather all terms with the <u>same</u> power of ρ^j. To do this, we re-index the first summation. Letting $\bar{j} = j-1$, this becomes

$$\sum_{\bar{j}=-1}^{\infty}\left[(\bar{j}+1)\bar{j} + 2(\ell+1)(\bar{j}+1)\right]a_{\bar{j}+1}\rho^{\bar{j}}, \quad \text{or} \quad (\text{renaming } \bar{j}\ j)$$

$$\sum_{j=0}^{\infty}\left[(j+1)j + 2(\ell+1)(j+1)\right]a_{j+1}\rho^j$$

Here, I dropped the $j=-1$ term because it is zero. Thus in total,

$$\sum_{j=0}^{\infty}\left\{\left[(j+1)j + 2(\ell+1)(j+1)\right]a_{j+1} + \left[\lambda - 2(\ell+1) - 2j\right]a_j\right\}\rho^j = 0$$

For this series to be zero, each term must be zero. So,

$$\left[(j+1)j + 2(\ell+1)(j+1)\right]a_{j+1} + \left[\lambda - 2(\ell+1) - 2j\right]a_j = 0, \text{ or}$$

$$a_{j+1} = \frac{2(\ell+1) + 2j - \lambda}{(j+1)j + 2(\ell+1)(j+1)}$$

We can go further. For this series to terminate, we need

$$2(\ell+1) + 2j_{max} - \lambda = 0 \quad \text{for some } j_{max}, \text{ or}$$

$$\lambda = 2(\ell + 1 + j_{max}), \text{ or}$$

$$\frac{Ze^2}{4\pi\varepsilon_0}\frac{2m}{\hbar^2 K} = 2(\ell+1+j_{max}), \quad \text{or}$$

$$\frac{Ze^2}{4\pi\varepsilon_0}\frac{2m}{\hbar^2}\left(\frac{\hbar^2}{-2mE}\right)^{1/2} = 2(\ell+1+j_{max}), \quad \text{or}$$

$$\frac{Ze^2}{4\pi\varepsilon_0}\frac{m}{\hbar^2}\frac{1}{\ell+1+j_{max}} = \left(\frac{-2mE}{\hbar^2}\right)^{1/2}, \quad \text{or}$$

$$-\frac{\hbar^2}{2m}\left(\frac{Ze^2}{4\pi\varepsilon_0}\frac{m}{\hbar^2}\right)^2\frac{1}{(\ell+1+j_{max})^2} = E, \quad \text{or}$$

$$E_N = \frac{-Z^2 e^4 m}{2(4\pi\varepsilon_0)^2 \hbar^2}\frac{1}{N^2}, \qquad \text{where } N = \ell + 1 + j_{max}$$

$$\text{So} \quad E_N = \frac{(\text{Something})}{N^2}$$