1a) • $[\hat{A},\hat{A}] = \hat{A}\hat{A} - \hat{A}\hat{A} = 0$

• $[\hat{A},\hat{B}+\hat{C}] = \hat{A}(\hat{B}+\hat{C}) - (\hat{B}+\hat{C})\hat{A} = \hat{A}\hat{B} + \hat{A}\hat{C} - \hat{B}\hat{A} - \hat{C}\hat{A}$

$= (\hat{A}\hat{B} - \hat{B}\hat{A}) + (\hat{A}\hat{C} - \hat{C}\hat{A}) = [\hat{A},\hat{B}] + [\hat{A},\hat{C}]$

$[\hat{A}+\hat{B},\hat{C}] = (\hat{A}+\hat{B})\hat{C} - \hat{C}(\hat{A}+\hat{B}) = (\hat{A}\hat{C} - \hat{C}\hat{A}) + (\hat{B}\hat{C} - \hat{C}\hat{B}) = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]$

• $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = -(\hat{B}\hat{A} - \hat{A}\hat{B}) = -[\hat{B},\hat{A}]$

b) $[x,y]\Psi = (xy - yx)\Psi = xy\Psi - yx\Psi = 0$

$[x,p_x]\Psi = x\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi - \left(-i\hbar\frac{\partial}{\partial x}\right)(x\Psi) = -i\hbar x\frac{\partial\Psi}{\partial x} + i\hbar\Psi + i\hbar x\frac{\partial\Psi}{\partial x} = i\hbar\Psi$, so

$[x,p_x] = i\hbar$

$[x,p_y]\Psi = x\left(-i\hbar\frac{\partial}{\partial y}\right)\Psi - \left(-i\hbar\frac{\partial}{\partial y}\right)(x\Psi) = -i\hbar x\frac{\partial\Psi}{\partial y} + i\hbar x\frac{\partial\Psi}{\partial y} = 0$

$[p_x,p_y] = \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial y}\right)\Psi - \left(-i\hbar\frac{\partial}{\partial y}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi = \hbar^2\frac{\partial^2\Psi}{\partial x\partial y} - \hbar^2\frac{\partial^2\Psi}{\partial y\partial x} = 0$

c) Clearly, p_i's always commute w/ each other and x_i's always commute w/ each other. So,

$[x_i,x_j] = 0, \quad [p_i,p_j] = 0.$

We also see that $[x_i,p_j]$ is <u>zero</u> when $i \neq j$, and $i\hbar$ when $i=j$. So, compactly:

$$[x_i,p_j] = i\hbar\delta_{ij}$$

d) $[AB,CD] = ABCD - CDAB$ because B <u>commutes</u> w/ everything

$= BACD - BCDA$

$= B(ACD - CDA)$

$= B[A,CD]$

Similarly, $[AB,CD] = ABCD - CDAB = (ACD - CDA)B = [A,CD]B$

2a) $L_x = yp_z - zp_y$; $L_y = zp_x - xp_z$.

$$
\begin{aligned}
[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\
&= [yp_z - zp_y, zp_x] - [yp_z - zp_y, xp_z] \\
&= [yp_z, zp_x] - [zp_y, zp_x] - [yp_z, xp_z] + [zp_y, xp_z]
\end{aligned}
$$

use 1(a), second bullet point.

use 1(d)

$$
\begin{aligned}
&= yp_x \underbrace{[p_z, z]}_{-i\hbar} - p_y p_x \underbrace{[z, z]}_{0} - xy \underbrace{[p_z, p_z]}_{0} + xp_y \underbrace{[z, p_z]}_{i\hbar} \\
&= i\hbar(xp_y - yp_x) \\
&= i\hbar L_z
\end{aligned}
$$

b) $[AB, C] = ABC - CAB$

$A[B,C] + [A,C]B = A(BC - CB) + (AC - CA)B = ABC - CAB$

So the two are equal.

c) $[L^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x]$

using 2(b)

$$
\begin{aligned}
&= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\
&= 0 + L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z
\end{aligned}
$$

Now, $[L_x, L_y] = i\hbar L_z$, so $[L_y, L_x] = -i\hbar L_z$.

Using the cyclic property of $x \to y \to z \to x$, $[L_z, L_x] = i\hbar L_y$

So in total,

$$
\begin{aligned}
[L^2, L_x] &= -i\hbar L_y L_z - i\hbar L_z L_y + i\hbar L_z L_y + i\hbar L_y L_z \\
&= 0
\end{aligned}
$$

Similarly, $[L^2, L_y] = [L^2, L_z] = 0$.

3a) $[L_z, L_\pm] = [L_z, L_x \pm iL_y] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x)$

$$= i\hbar L_y \pm \hbar L_x = \hbar(L_x \pm iL_y) = \hbar L_\pm$$

Thus, $c_\pm = \pm\hbar$

b) We know $[L_z, L_\pm] = \pm\hbar L_\pm$, or $L_z L_\pm = L_\pm L_z \pm \hbar L_\pm$.

Using this, we can see

$$L_z(L_\pm Y_{\ell m}) = L_\pm L_z Y_{\ell m} \pm \hbar L_\pm Y_{\ell m}$$

$$= L_\pm \hbar m Y_{\ell m} \pm \hbar L_\pm Y_{\ell m}$$

$$= \hbar(m \pm 1) L_\pm Y_{\ell m}$$

So, $(L_\pm Y_{\ell m})$ is an eigenfcn of L_z w/ eigenvalue $(m \pm 1)\hbar$.

c) We note that $L_\pm Y_{\ell m}$ is an eigenfunction of L_z w/ eigval $(m+1)\hbar$. We can <u>also</u> ask what $\hat{L}^2$ does to $L_\pm Y_{\ell m}$.

Note that $[\hat{L}^2, L_\pm] = [L^2, L_x] \pm i[L^2, L_y] = 0$, so $L^2 L_\pm = L_\pm L^2$.

We then find

$$L^2 L_\pm Y_{\ell m} = L_\pm L^2 Y_{\ell m} = L_\pm \hbar^2 \ell(\ell+1) Y_{\ell m} = \hbar^2 \ell(\ell+1) L_\pm Y_{\ell m}$$

<u>So</u>: $L_\pm Y_{\ell m}$ has eigenvalue $\hbar(m \pm 1)$ w/ respect to L_z, and eigval $\hbar^2 \ell(\ell+1)$ w/r/t L^2. There is only one function that has these properties: $Y_{\ell(m\pm1)}$ We thus see that

$$L_\pm Y_{\ell m} = Y_{\ell(m\pm1)}$$

In other words, L_+ <u>raises</u> the m-number by 1, leaving ℓ unchanged, while L_- <u>lowers</u> m by one.

This allows us to generate <u>all</u> $Y_{\ell m}$ for a given ℓ, if we know just <u>one</u> of them!