Physics 486 Discussion 11 – Angular Momentum : Commutators and Ladder Operators

Problem 1 : Commutator Warmup

Lots of commutators to do today, so let's start with a warmup of things you've seen before, and make a couple of important observations.

(a) First, make sure these relations are <u>obvious</u> to you. If not, do some work until they *are* obvious:

- $\left[\hat{A},\hat{A}\right]=0$ i.e. anything **commutes with itself**
- $\left[\hat{A},\hat{B}+\hat{C}\right]=\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right]$ and $\left[\hat{A}+\hat{B},\hat{C}\right]=\left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right]$ i.e. commutators are **distributive**
- $\left[\hat{A},\hat{B}\right]=-\left[\hat{B},\hat{A}\right]$

(b) Explicitly calculate the four commutators below. For expediency, I have stopped putting hats on everything because everything is an operator ($\hat{x}=x$, $\hat{p}_x=-i\hbar\partial_x$, etc … where ∂_x is useful shorthand for $\partial/\partial x$).

$$[x,y]\,, \qquad [x,p_x]\,, \qquad [x,p_y]\,, \qquad [p_x,p_y]$$

▶ Did you find that three of them are zero and one of them is $i\hbar$? If you did NOT find that, I am 99% sure of what went wrong: These commutators are commutators of OPERATORS, so the commutators themselves are OPERATORS, and WHENEVER you are doing calculations with operators you should provide them with a FUNCTION TO WORK ON. (Do you recall this advice from homework and lecture?) Please try again using this technique, and you will for sure succeed! ☺

(c) After working through the above, make sure that these **canonical commutation relations** are 100% clear :

$$\boxed{[x_i,x_j]=0 \qquad [p_i,p_j]=0 \qquad [x_i,p_j]=i\hbar\delta_{ij}} \qquad \text{where } i,j\in\{x,y,z\}$$

(d) After working through part (b), make sure the following statement is also 100% clear :

$$\left[\hat{A}\hat{B},\hat{C}\hat{D}\right] = \hat{B}\left[\hat{A},\hat{C}\hat{D}\right] = \left[\hat{A},\hat{C}\hat{D}\right]\hat{B} \text{ if } \hat{B} \text{ commutes with all the other operators } (\hat{A}\,,\ \hat{C}\,,\ \&\ \hat{D})$$

To see the obviousness of the above, i.e. that <u>an operator $\hat{B}$ that commutes with all the others</u> can be <u>freely yanked out of a commutator and put anywhere</u>, think of $\hat{B}$ as the number 8. 8 definitely commutes with everything. ☺

Problem 2 : Angular Momentum

Checkpoints [1]

The angular momentum operator is $\vec{L}=\vec{r}\times\vec{p}$. (I've left off the hats again since everything is an operator.)

$$\hat{\vec{L}}=\hat{\vec{r}}\times\hat{\vec{p}}=\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ x & y & z \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix} \quad \text{(there, all the hats are back ☺).}$$

(a) Calculate this commutator: $[L_x, L_y]$. First, you have to expand L_x and L_y into what you get from the cross product, e.g. into terms like $x\,p_y - y\,p_x$. This is a bigger job that in looks like, with many terms. To complete it efficiently, use the <u>distributive</u> property we mentioned in problem 1(a) above to expand the commutator into four commutators, then use the properties 1(c) and 1(d) to spot the fact that two of them are zero. Go for it!

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[1] **Q2** (a) turn the page ☺ (c) 0!

The result is so important that I will put it in a box as well : $\boxed{[L_x, L_y] = i\hbar L_z, \quad \text{etc}}$ where "etc" means cycle the indices to get the others, like $[L_z, L_x] = i\hbar L_y$

(b) Here is a useful theorem: $\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$

Just expand it and two terms will cancel. This is not something you should memorize (!), rather know that it exists and that you can re-derive it in 20 seconds when needed. For example …

(c) Now we come to another very important commutator : $[L^2, L_x]$ where $L^2 = L_x^2 + L_y^2 + L_z^2$.
Calculate that one using symmetry as much as you can as well as the part (a) and (b) relations.
Once you get the result, the results for $[L^2, L_y]$ and $[L^2, L_z]$ will be obvious, no extra work required!

▶ That result needs a box. It is because of $\boxed{[L^2, L_i] = 0}$, where $i \in \{x, y, z\}$, that the spherical harmonics $Y_l^m(\theta, \phi)$ can be simultaneous eigenfunctions of the operators L^2 and L_z. Similarly, because $[L_x, L_z]$ and $[L_y, L_z]$ are NON-zero, the spherical harmonics are NOT eigenfunctions of L_x and L_y. Whenever you write down the spherical harmonics, $Y_l^m(\theta, \phi)$, you are implicitly choosing one axis to be the "**quantization axis**" with respect to which the ***m*** quantum number is defined. By default this is the *z* axis, meaning $L_z Y_{lm} = (m\hbar)\, Y_{lm}$.

Problem 3 : Welcome to Ladder Operators *Checkpoints* [2]

Check out these two operators : $\boxed{L_\pm \equiv L_x \pm i\, L_y}$. Strange, these L_+ and L_- operators … what good are they?

(a) To find out, first prove the following : $\left[L_z, L_\pm\right] = c_\pm L_\pm$ where $c_\pm$ is a scalar for you to determine.

(b) Now apply the L_z operator to the wavefunction $(L_\pm Y_{lm})$. Show that $(L_\pm Y_{lm})$ is an eigenfunction of L_z and find out what its eigenvalue is. (In math: show that $L_z (L_\pm Y_{lm}) = \lambda_\pm (L_\pm Y_{lm})$ and find out what $\lambda_\pm$ is.)

(c) Once you find your result, does it make sense that $L_\pm$ are called the **raising & lowering operators for L_z** ? What IS $L_\pm Y_{lm}$? i.e. what do these operators DO to the spherical harmonics?

▶ Raising & lowering operators are also called **ladder operators**, or step-up & step-down operators, because they "step" eigenfunctions up or down the eigenvalue "ladder". Say you have an operator *Q* and you would like to find its { eigen-things }. Well, if you have one e-function of *Q* and *Q*'s ladder operators, $Q\pm$, you can generate the whole e-spectrum by successively applying $Q\pm$ to your one e-function. (You can even get that one e-function from the ladder ops: insist that the { e-things } must have a "bottom rung" and solve $Q_- \psi_{\text{bottom}} = 0$.) Wow, this sounds like the *ideal* way to generate eigen-spectra! Why don't we use it all the time?? Reason: it is not at all trivial to find the ladder operators of *Q* (!!) and it's not guaranteed that they exist at all. There are exactly two operators whose ladder ops are easier to find than calculating the { e-spectrum } by another method: (1) L_z and (2) the Hamiltonian of the 1D harmonic oscillator. Seriously, that's it.

There's one more note on the next page, wouldn't fit here.

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[2] **Q3 (a)** $a_\pm = \pm\hbar$ **(b)** $\lambda_\pm = \hbar(m \pm 1)$ **(c)** $L_\pm Y_{l\,m} \sim Y_{l\,(m+1)}$. **NOTE** that's a proportional-to sign "~", it is *not* an equals sign. Why? The part(b) result is $L_z\,(L_\pm Y_{l\,m}) = \hbar(m \pm 1)\,(L_\pm Y_{l\,m})$. In words: $(L_\pm Y_{l\,m})$ is an eigenfunction of L_z with eigenvalue $\hbar(m \pm 1)$. Eigenfunctions are only *ever* determined up to a multiplicative constant: $\hat{Q} f_q = q\, f_q$ means f_q is an eigenfunction of operator $\hat{Q}$ with eigenvalue *q* … but $5\, f_q$ works just as well, or πf_q, or $-2i\, f_q$. Since f_q is on *both sides* of the defining equation $\hat{Q} f_q = q\, f_q$ for an eigenfunction of $\hat{Q}$, *any multiple of* f_q works just as well as f_q. Some other aspect of your physical system is required to determine the overall normalization of your eigenfunctions. For QM wavefunctions, the normalization is determined by total probability; for classical coupled-oscillator modes, the normalization is determined by initial conditions; etc.

▶ In class, I advertised ladder operators as the fourth and final utility of commutators in QM.
Reason: that part (a) commutator $[Q, Q_\pm] = c_\pm\, Q_\pm$ is the <u>essential property</u> that $Q_\pm$ must possess to be the ladder op of Q. I dearly wish we could find a ladder op for L^2 : together with the ladder op for L_z that you just studied, we could use it to generate all the spherical harmonics! That would be very nice … but L^2 commutes with everything! (well, almost everything) I've never seen a ladder operator for L^2 … but I also haven't seen a proof that one doesn't exist. ¯_(ツ)_/¯

Here is a recap of the vital **roles that commutators play in QM**:

- Generalized Uncertainty Principle : $\sigma_A \sigma_B \geq \left| \frac{1}{2i} \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right|$
- Compatible Operators : ops with common { e-spectra } = ops that commute with each other
- Generalized Ehrenfest Theorem : $\frac{d\langle \hat{Q} \rangle}{dt} = \frac{i}{\hbar} \left\langle \left[\hat{H}, \hat{Q} \right] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle \rightarrow$ conserved quantities
- Ladder Operators : $\hat{Q}_\pm$ of $\hat{Q}$ must satisfy $\left[\hat{Q}, \hat{Q}_\pm \right] = c_\pm \hat{Q}_\pm$ where $c_\pm$ are constant scalars