1 a) Plugging $\Psi(\vec{r}) = R(r)Y(\theta, \varphi)$ into the equation

$$\left(\hat{p}_r^2 + \frac{\hat{L}^2}{r^2} + 2mV\right)\Psi = 2mE\Psi,$$

and noting that $\hat{p}_r$ only acts on R, while $\hat{L}$ only acts on Y, we get

$$\left(\hat{p}_r^2 + \frac{\hat{L}^2}{r^2} + 2mV\right)RY = 2mERY, \quad \text{or}$$

$$Y\hat{p}_r^2 R + R\frac{\hat{L}^2 Y}{r^2} = 2m(E-V)RY, \quad \text{or dividing by } RY$$

$$\frac{\hat{p}_r^2 R}{R} + \frac{\hat{L}^2 Y}{r^2 Y} = 2m(E-V).$$

Multiplying by r^2 gives

$$r^2\frac{\hat{p}_r^2 R}{R} + \frac{\hat{L}^2 Y}{Y} = 2m(E-V)r^2, \quad \text{or}$$

$$\underbrace{\left[r^2\frac{p_r^2 R}{R} - 2m(E-V)r^2\right]}_{\text{only depends on } r} + \underbrace{\frac{\hat{L}^2 Y}{Y}}_{\text{Only depends on } \theta,\, \phi} = 0$$

We can then set each term equal to a constant. I'll call the constant for the Y-equation $\hbar^2\lambda^2$. Then

$$\frac{\hat{L}^2 Y}{Y} = \hbar\lambda^2, \quad \text{or} \quad \frac{\hat{L}^2 Y}{\hbar^2} = \lambda^2 Y$$

The constant for the R-equation must then be $-\hbar^2\lambda^2$, so the R-equation is

$$r^2\frac{\hat{p}_r^2 R}{R} - 2m(E-V)r^2 = -\hbar^2\lambda^2$$

b) The units of $\hbar$ are $kg \frac{m^2}{s}$. The units of L are also $kg \frac{m^2}{s}$. Thus, $L^2/\hbar^2$ is unitless, and so λ is unitless.

c) This equation becomes

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial TF}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 TF}{\partial\phi^2}\right]=\lambda^2 TF \quad \text{, or}$$

$$-\frac{F}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)-\frac{T}{\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}=\lambda^2 TF. \quad \text{Dividing by } TF,$$

$$-\frac{1}{T\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)-\frac{1}{F\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}=\lambda^2. \quad \text{Multiplying by } \sin^2\theta,$$

$$-\frac{\sin\theta}{T}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)-\frac{1}{F}\frac{\partial^2 F}{\partial\phi^2}=\lambda^2\sin^2\theta, \quad \text{or}$$

$$0=\underbrace{\left[\frac{\sin\theta}{T}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\lambda^2\sin^2\theta\right]}_{\text{only depends on } \theta}+\underbrace{\left[\frac{1}{F}\frac{\partial^2 F}{\partial\phi^2}\right]}_{\text{only depends on } \phi}$$

We can then separate these into two equations:

$$\frac{\sin\theta}{T}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\lambda^2\sin^2\theta=-\mu$$

$$\frac{1}{F}\frac{\partial^2 F}{\partial\phi^2}=\mu$$

d) This equation is $F'' = \mu F$. We have two possibilities. If $\mu < 0$, then we have solutions

$$F(\phi) = e^{ik\phi}, \text{ where } k = \pm\sqrt{-\mu}$$

If $\mu > 0$, we have solutions

$$F(\phi) = e^{k\phi}, \quad k = \pm\sqrt{\mu}$$

e) If $\mu > 0$, our most general solution is

$$F(\phi) = Ae^{\sqrt{\mu}\phi} + Be^{-\sqrt{\mu}\phi}$$

We need $F(\phi + 2\pi n) = F(\phi)$. This clearly can't happen. As $n \to \infty$, the first term $\to \infty$, while the second $\to 0$. So we can't come back to $F(\phi)$.

If $\mu < 0$, we have $F(\phi) = Ae^{ik\phi} + Be^{-ik\phi}$. In this case, $F(\phi + 2\pi n) = F(\phi)$ provided $e^{i2\pi nk}$ and $e^{-i2\pi nk}$ are both 1. This happens iff k is an integer, call it m.

Thus, we see that μ is negative and $\sqrt{-\mu} = m$, or $\mu = -m^2$.

f) We now get $F(\phi) = e^{im\phi}$, $m = 0, \pm 1, \pm 2, \ldots$ Note the $\pm$ means we're getting both the $e^{ik\phi}$ and the $e^{-ik\phi}$ solutions.

g) Plugging in $\mu = -m^2$ to our T-equation from (c),

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial T}{\partial\theta}\right) = (m^2 - \lambda^2 \sin^2\theta)T$$

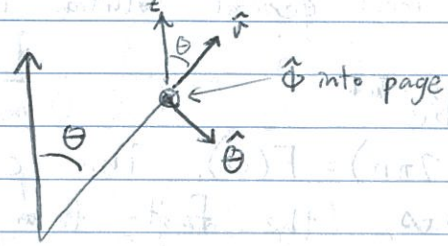
The original equation for Y was

$$\frac{\hat{L}^2 Y}{\hbar^2} = \lambda^2 Y, \quad \text{or} \quad \hat{L}^2 Y = \hbar^2 \lambda^2 Y.$$

Now we know $\lambda^2 = \ell(\ell+1)$, so

$\hat{L}^2 Y = \hbar^2 \ell(\ell+1) Y$, so Y is an eigenfcn of $\hat{L}^2$ w/ eigenvalue $\hbar^2 \ell(\ell+1)$.

h)



$$\hat{L}_z = \hat{z} \cdot \vec{L}$$

$$= \frac{\hbar}{i}\left(\hat{z} \cdot \hat{\Phi} \frac{\partial}{\partial \theta} - \frac{\hat{z} \cdot \hat{\theta}}{\sin\theta} \frac{\partial}{\partial \phi}\right)$$

Note $\hat{z} \cdot \hat{\Phi} = 0$, b/c these are always perpendicular. Meanwhile, $\hat{z} \cdot \hat{\theta} = -\sin\theta$. Thus,

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

i) $\hat{L}_z P_\ell^m(\cos\theta) e^{im\phi} = \frac{\hbar}{i}\frac{\partial}{\partial\phi} P_\ell^m(\cos\theta)\, e^{im\phi}$

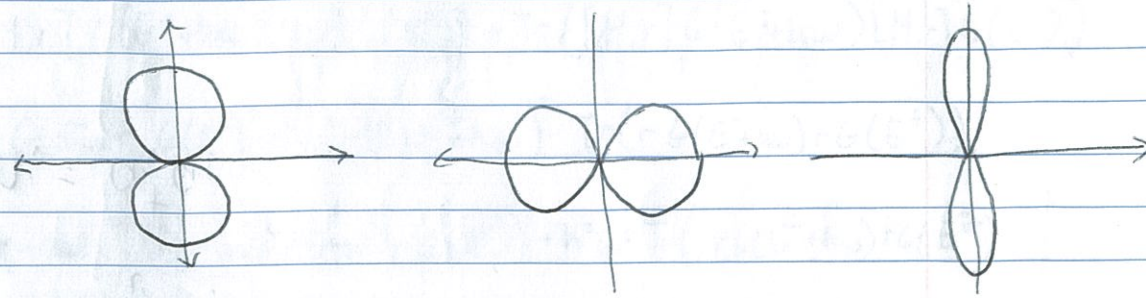
$$= P_\ell^m(\cos\theta) \frac{\hbar}{i}\frac{\partial}{\partial\phi} e^{im\phi}$$

$$= P_\ell^m(\cos\theta)\, \hbar m\, e^{im\phi}$$

so $\hat{L}_z Y_\ell^m = \hbar m Y_\ell^m$, and Y_ℓ^m is an eigenfcn of $\hat{L}_z$ w/ eigenvalue $\hbar m$.

$Y_1^{1*} Y_1^1 = \sin^2\theta \qquad Y_1^{0*} Y_1^0 = \cos^2\theta \qquad Y_2^{2*} Y_2^2 = \sin^4\theta$

j)



k) $\vec{j}_1^{\,0} = \text{Re}\left[\cos\theta \dfrac{\vec{\nabla}}{im}\cos\theta\right] = \text{Re}\left[\cos\theta \dfrac{\hat{\theta}}{irm}\dfrac{\partial}{\partial\theta}\cos\theta\right] = 0$

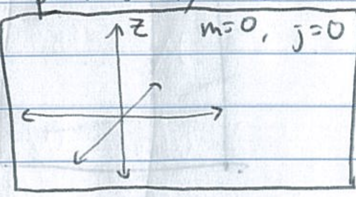
$$\vec{j}_1^{\,1} = \text{Re}\left[\sin\theta\, e^{-i\phi}\frac{\vec{\nabla}}{im}\sin\theta\, e^{i\phi}\right]$$

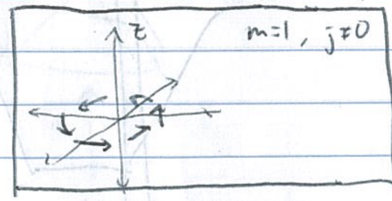
$$= \text{Re}\left[\sin\theta\, e^{-i\phi}\frac{1}{im}\left(\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\sin\theta\, e^{i\phi}\right]$$

$$= \text{Re}\left[\sin\theta\, e^{-i\phi}\frac{1}{im}\left(\hat{\theta}\frac{\cos\theta\, e^{i\phi}}{r} + \hat{\phi}\frac{i}{r}e^{i\phi}\right)\right]$$

$$= \text{Re}\left[\hat{\theta}\frac{1}{imr}\cos\theta\sin\theta + \hat{\phi}\frac{\sin\theta}{mr}\right] = \frac{\sin\theta}{mr}\hat{\phi}$$

So when $L_z = 0$, no $\vec{j}$. But when $L_z = \hbar$, we have a probability current in the $\hat{\phi}$ direction, around $\hat{z}$





l) $$\hat{L}^2\hat{L}_z\Psi = \frac{1}{r^2}\left[-\frac{\hbar^2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\hbar}{i}\frac{\partial\Psi}{\partial\phi}\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\hbar\,\partial^3\Psi}{i\,\partial\phi^3}\right]$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial\phi}\left(\frac{1}{r^2}\left[-\frac{\hbar^2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right]$$

$$= \hat{L}_z\hat{L}^2\Psi, \quad \text{so} \quad [\hat{L}_z, \hat{L}^2] = 0$$