1a) Let $|f\rangle$ be an eigenvector of $\hat{Q}$ w/ eigenvalue q_f. Then $\hat{Q}|f\rangle = q|f\rangle$. ~~But we also have~~ We then have:

$$\langle f|\hat{Q}f\rangle = q_f\langle f|f\rangle$$

We also have

$$\underset{\substack{\uparrow \\ \text{property of} \\ \langle\,|\,\rangle: \\ \langle f|g\rangle = \langle g|f\rangle^*}}{\langle f|\hat{Q}f\rangle =} \langle \hat{Q}f|f\rangle^* \underset{\substack{\uparrow \\ Q \text{ Hermitian}}}{=} \langle f|\hat{Q}f\rangle^* = [q_f\langle f|f\rangle]^* = q_f^*\langle f|f\rangle$$

Thus, $q_f^* = q_f$, and q_f is real.

b) Again, let $|f\rangle$ be an eigenvector w/ eigenvalue q. Then

$\langle\hat{Q}\rangle = \langle f|\hat{Q}f\rangle = q_f\langle f|f\rangle = q_f$, since $\langle f|f\rangle = 1$ for $|f\rangle$ normalized.

$\langle\hat{Q}^2\rangle = \langle f|\hat{Q}^2 f\rangle = \langle f|\hat{Q}q_f f\rangle = q_f\langle f|\hat{Q}f\rangle = q_f^2\langle f|f\rangle = q^2$

Thus, $\sigma_q = q_f^2 - q_f^2 = 0$, and we only measure q_f.

c) Say $|f_1\rangle$ + $|f_2\rangle$ are eigenvectors w/ eigenvalues $q_1 + q_2$, with $q_1 \neq q_2$. Then $\langle f_1|\hat{Q}f_2\rangle = q_2\langle f_1|f_2\rangle$. But we also know

$\langle f_1|\hat{Q}f_2\rangle = \langle\hat{Q}f_1|f_2\rangle = q_1\langle f_1|f_2\rangle$. So,

$q_1\langle f_1|f_2\rangle = q_2\langle f_1|f_2\rangle$. Since $q_1 \neq q_2$, this means $\langle f_1|f_2\rangle = 0$

d) $\hat{P}|\Psi_p\rangle = |-i\hbar\frac{\partial}{\partial x}A_p e^{i\frac{p}{\hbar}x}\rangle = p|\Psi_p\rangle$, so $|\Psi_p\rangle$ is an eigenstate.

We want to choose A_p so that $\langle\Psi_{p_1}|\Psi_{p_2}\rangle = \delta(p_1 - p_2)$. We have

$$\langle\Psi_{p_1}|\Psi_{p_2}\rangle = A_{p_1}^* A_{p_2}\int e^{i\frac{p_2-p_1}{\hbar}x}\,dx$$

Fourier transform of δ-fcn

$$= A_{p_1}^* A_{p_2}\,2\pi\,\delta\left(\frac{p_2-p_1}{\hbar}\right)$$

$$= A_{p_1}^* A_{p_2}\,2\pi\hbar\,\delta(p_2-p_1)$$

$= \delta(p_2 - p_1)$ if we choose $A_p = \frac{1}{\sqrt{2\pi\hbar}}$

How do we know $\delta(\frac{x}{\hbar}) = \hbar\,\delta(x)$? We know $\delta(\frac{x}{\hbar})$ should be proportional to a δ-fcn. We also know

$$\int_{-\infty}^{\infty}\delta\left(\tfrac{x}{\hbar}\right)dx = \int_{-\infty}^{\infty}\delta(u)\,\hbar\,du \leftarrow \text{let } u = \tfrac{x}{\hbar}$$

$$= \hbar\int_{-\infty}^{\infty}\delta(u)\,du = \hbar$$

Thus, $\delta(\frac{x}{\hbar}) = \hbar\,\delta(x)$.

e) Consider $\hat{X} B_0 \delta(x-x_0) = x B_0 \delta(x-x_0)$. Since $\delta(x-x_0)$ is zero when $x \neq$ we can replace x w/ x_0, so $\hat{X} B_0 \delta(x-x_0) = x_0 B \delta(x-x_0)$. So $B_0 \delta(x-x_0)$ is an eigenfcn w/ eigenvalue x_0.

We want $\langle \Psi_{x_1} | \Psi_{x_2} \rangle = \delta(x_1 - x_2)$, and we pick B_0 to make this true.

We have
$$\langle \Psi_{x_1} | \Psi_{x_2} \rangle = B_{x_1}^* B_{x_2} \int_{-\infty}^{\infty} \delta(x-x_1)\,\delta(x-x_2)\,dx = B_{x_1}^* B_{x_2}\,\delta(x_2 - x_1)$$
$$= \delta(x_2 - x_1) \quad \text{if} \quad B_x = 1$$

f) We have $\langle f | (\hat{A} \pm \hat{B}) f \rangle = \langle f | \hat{A} f \rangle \pm \langle f | \hat{B} f \rangle = \langle \hat{A} f | f \rangle \pm \langle \hat{B} f | f \rangle = \langle (\hat{A} \pm \hat{B}) f | f \rangle$, so $\hat{A} \pm \hat{B}$ is Hermitian.

g) We have
$$\langle f | \hat{A}\hat{B} f \rangle \underset{\text{b/c } \hat{A} \text{ Hermitian}}{=} \langle \hat{A} f | \hat{B} f \rangle \underset{\text{b/c } \hat{B} \text{ Hermitian}}{=} \langle \hat{B}\hat{A} f | f \rangle \overset{\text{only if } \hat{B}\hat{A} = \hat{A}\hat{B}}{=} \langle \hat{A}\hat{B} f | f \rangle$$

So $\hat{A}\hat{B}$ is Hermitian if $\hat{A}\hat{B} = \hat{B}\hat{A}$, or $\hat{A}\hat{B} - \hat{B}\hat{A} = 0$, or $[\hat{A}, \hat{B}] = 0$.

2 Equivalent Definitions of a Hermitian Operator

Suppose $\langle f|\hat{Q}f\rangle = \langle\hat{Q}f|f\rangle\ \forall f(x)$ in Hilbert space. For any two functions $g(x)$ and $h(x)$ denote $f_1(x) = g(x) + h(x)$ and $f_2(x) = g(x) + ih(x)$, then since the operators $\hat{Q}$ is linear,

$$\begin{aligned}
\langle f_1|\hat{Q}f_1\rangle &= \langle(g+h)|\hat{Q}(g+h)\rangle \\
&= \langle g|\hat{Q}g\rangle + \langle g|\hat{Q}h\rangle + \langle h|\hat{Q}g\rangle + \langle h|\hat{Q}h\rangle\,, && (1) \\
\langle f_1\hat{Q}|f_1\rangle &= \langle(g+h)\hat{Q}|(g+h)\rangle \\
&= \langle g\hat{Q}|g\rangle + \langle g\hat{Q}|h\rangle + \langle h\hat{Q}|g\rangle + \langle h\hat{Q}|h\rangle\,, && (2) \\
\langle f_2|\hat{Q}f_2\rangle &= \langle(g-ih)|\hat{Q}(g+ih)\rangle \\
&= \langle g|\hat{Q}g\rangle + i\langle g|\hat{Q}h\rangle - i\langle h|\hat{Q}g\rangle - \langle h|\hat{Q}h\rangle\,, && (3) \\
\langle f_2\hat{Q}|f_2\rangle &= \langle(g-ih)\hat{Q}|(g+ih)\rangle \\
&= \langle g\hat{Q}|g\rangle + i\langle g\hat{Q}|h\rangle - i\langle h\hat{Q}|g\rangle - \langle h\hat{Q}|h\rangle\,. && (4)
\end{aligned}$$

Considering that $\langle g|\hat{Q}g\rangle = \langle\hat{Q}g|g\rangle$ and $\langle h|\hat{Q}h\rangle = \langle\hat{Q}h|h\rangle$, from (1) and (2) we have

$$\langle g\hat{Q}|h\rangle + \langle h\hat{Q}|g\rangle = \langle g|\hat{Q}h\rangle + \langle h|\hat{Q}g\rangle\,, \tag{5}$$

and from (3) and (4) we have

$$\langle g\hat{Q}|h\rangle - \langle h\hat{Q}|g\rangle = \langle g|\hat{Q}h\rangle - \langle h|\hat{Q}g\rangle\,. \tag{6}$$

Adding (5) and (6) together,

$$\langle h\hat{Q}|g\rangle = \langle g|\hat{Q}h\rangle\,.$$