Q1

(a) $\langle \Psi_2 | \Psi_2 \rangle = \left(\frac{4}{5}\right)^2 \underset{\substack{\uparrow \\ \text{One b/c } \phi_1, \phi_2 \\ \text{are normalized}}}{\langle \phi_1 | \phi_1 \rangle} - \frac{12}{25} \underset{\substack{\uparrow \\ \text{Zero b/c eigenstates are orthogonal}}}{\langle \phi_1 | \phi_2 \rangle} - \frac{12}{25} \langle \phi_2 | \phi_1 \rangle + \left(\frac{3}{5}\right)^2 \langle \phi_2 | \phi_2 \rangle = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = 1$

$\langle \Psi_1 | \Psi_2 \rangle = \left(\frac{3}{5}\langle \phi_1 | + \frac{4}{5}\langle \phi_2 |\right)\left(\frac{4}{5}|\phi_1\rangle - \frac{3}{5}|\phi_2\rangle\right) = \frac{12}{25}\langle \phi_1 | \phi_1 \rangle + \frac{16}{25}\langle \phi_2 | \phi_1 \rangle - \frac{9}{25}\langle \phi_1 | \phi_2 \rangle - \frac{12}{25}\langle \phi_2 | \phi_2 \rangle$

$= \frac{12}{25} - \frac{12}{25} = 0$

(b) From the 1st equation, we have $|\phi_1\rangle = \frac{5}{3}|\Psi_1\rangle - \frac{4}{3}|\phi_2\rangle$. Plugging this in to the 2nd equation gives $|\Psi_2\rangle = \frac{4}{5}\left(\frac{5}{3}|\Psi_1\rangle - \frac{4}{3}|\phi_2\rangle\right) - \frac{3}{5}|\phi_2\rangle$,

or $|\phi_2\rangle = \dfrac{4|\Psi_1\rangle - 3|\Psi_2\rangle}{5}$. Plugging this in to our earlier eq,

$|\phi_1\rangle = \frac{5}{3}|\Psi_1\rangle - \frac{4}{3}\left(\frac{4}{5}|\Psi_1\rangle - \frac{3}{5}|\Psi_2\rangle\right) = \dfrac{3|\Psi_1\rangle + 4|\Psi_2\rangle}{5}$

(c) After measuring a_1, we're in the eigenstate of $\hat{A}$ w/ eigenvalue a_1, so our new state is $|\Psi_1\rangle$

(d) We measure b_1 w/ probability $|\langle \phi_1 | \Psi_1 \rangle|^2 = \left|\frac{3}{5}\right|^2 = \frac{9}{25}$

We measure b_2 w/ probability $|\langle \phi_2 | \Psi_1 \rangle|^2 = \frac{16}{25}$

Eigenstate of our measurement operator $\hat{B}$.   our current state

(e) If we measured b_1: $P(a_1) = |\langle \Psi_1 | \phi_1 \rangle|^2 = \frac{9}{25}$

eigenstate of measurement op   Current state

If we measured b_2: $P(a_1) = |\langle \Psi_1 | \phi_2 \rangle|^2 = \frac{16}{25}$

So the total probability of getting a_1 is

$P(b_1)P(a_1|b_1) + P(b_2)P(a_1|b_2)$

$= \frac{9}{25} \times \frac{9}{25} + \frac{16}{25} \times \frac{16}{25}$

$= \frac{337}{625} \approx .5392$

So it is <u>not</u> 1!

(f) In this case, $P(a_1) = \frac{16}{25}$, as shown above.

(g)

First, calculate $\hat{A}\hat{B}\Psi_1$: $\hat{A}\hat{B}|\Psi_1\rangle = \hat{A}\hat{B}\left[\frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_2\rangle\right]$

$$= \hat{A}\left(\frac{3b_1}{5}|\phi_1\rangle + \frac{4b_2}{5}|\phi_2\rangle\right)$$

$$= \hat{A}\left(\frac{3b_1}{5}\left(\frac{3}{5}|\Psi_1\rangle + \frac{4}{5}|\Psi_2\rangle\right) + \frac{4b_2}{5}\left(\frac{4}{5}|\Psi_1\rangle - \frac{3}{5}|\Psi_2\rangle\right)\right)$$

$$= \hat{A}\left(\left(\frac{9}{25}b_1 + \frac{16}{25}b_2\right)|\Psi_1\rangle + \left(\frac{12}{25}b_1 - \frac{12}{25}b_2\right)|\Psi_2\rangle\right)$$

$$= \left(\frac{9}{25}b_1 + \frac{16}{25}b_2\right)a_1|\Psi_1\rangle + \left(\frac{12}{25}b_1 - \frac{12}{25}b_2\right)a_2|\Psi_2\rangle$$

Now: $\hat{B}\hat{A}|\Psi_1\rangle = \hat{B}a_1|\Psi_1\rangle$

$$= \hat{B}\left(\frac{3}{5}a_1|\phi_1\rangle + \frac{4}{5}a_1|\phi_2\rangle\right)$$

$$= \frac{3}{5}a_1b_1|\phi_1\rangle + \frac{4}{5}a_1b_2|\phi_2\rangle$$

$$= \frac{3}{5}a_1b_1\left(\frac{3}{5}|\Psi_1\rangle + \frac{4}{5}|\Psi_2\rangle\right) + \frac{4}{5}a_1b_2\left(\frac{4}{5}|\Psi_1\rangle - \frac{3}{5}|\Psi_2\rangle\right)$$

$$= \left(\frac{9}{25}a_1b_1 + \frac{16}{25}a_1b_2\right)|\Psi_1\rangle + \left(\frac{12}{25}a_1b_1 - \frac{12}{25}a_1b_2\right)|\Psi_2\rangle$$

So in total,

$$[\hat{A},\hat{B}]|\Psi_1\rangle = \left(\frac{9}{25}a_1b_1 + \frac{16}{25}a_1b_2 - \frac{9}{25}a_1b_1 - \frac{16}{25}a_1b_2\right)|\Psi_1\rangle$$

$$+ \left(\frac{12}{25}a_2b_1 - \frac{12}{25}a_2b_2 - \frac{12}{25}a_1b_1 + \frac{12}{25}a_1b_2\right)|\Psi_2\rangle$$

$$= \frac{12}{25}\left(a_2b_1 - a_2b_2 - a_1b_1 + a_1b_2\right)|\Psi_2\rangle$$

$$= \frac{12}{25}(a_1 - a_2)(-b_1 + b_2)|\Psi_2\rangle$$

We thus see $[\hat{A},\hat{B}] \neq 0$ unless $a_1 = a_2$ or $b_1 = b_2$.

a) If this Hamiltonian is written in the basis $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$, then in coords

Q2 we have $|\psi_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $|\psi_2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $|\psi_3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Meanwhile,

$$H|\psi_1\rangle = \begin{pmatrix}a & 0 & b\\0 & c & 0\\b & 0 & a\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}a\\0\\b\end{pmatrix} \neq K\begin{pmatrix}1\\0\\0\end{pmatrix},$$ thus $|\psi_1\rangle$ is not an eig of H.

Similarly, you can show $|\psi_3\rangle$ is not an eig of H.

b) Note that $H|\mathcal{S}(0)\rangle = c|\mathcal{S}(0)\rangle$, so $|\mathcal{S}(0)\rangle$ is an eigenstate of H. Thus, its time-dependence is given by multiplying by $e^{-iEt/\hbar}$. Here, $E=c$, so

$$|\mathcal{S}(t)\rangle = \begin{pmatrix}0\\e^{-ict/\hbar}\\0\end{pmatrix}$$

c) Note that $H|\mathcal{S}(0)\rangle = \begin{pmatrix}b\\0\\a\end{pmatrix} \neq K\begin{pmatrix}0\\0\\1\end{pmatrix}$, so $|\mathcal{S}(0)\rangle$ is not an eigenstate of H. We thus need to expand $|\mathcal{S}(0)\rangle$ in eigenstates of H; each eigenstate we will then be able to find the time-dependence of.

First: Find eigenstates of H.

$$\det(H - E\mathbb{1}) = 0, \quad \text{or} \quad \begin{vmatrix}a-E & 0 & b\\0 & c-E & 0\\b & 0 & a-E\end{vmatrix} = 0, \quad \text{or}$$

$$0 = (c-E)\left[(a-E)^2 - b^2\right] = (c-E)\left(a^2 - 2aE + E^2 - b^2\right)$$
$$= (c-E)(E-a+b)(E-a-b)$$

Thus, $E=c$ or $E=a-b$ or $E=a+b$

$\boxed{\text{If } E=c}$ we have

$$\begin{pmatrix}a & 0 & b\\0 & c & 0\\b & 0 & a\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix} = c\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}, \text{ or } \begin{pmatrix}ae_1 + be_3\\ce_2\\be_1 + ae_3\end{pmatrix} = \begin{pmatrix}ce_1\\ce_2\\ce_3\end{pmatrix}$$

The ~~Rep~~ equations are then

$$\begin{cases}ae_1 + be_3 = ce_1\\ be_1 + ae_3 = ce_3\end{cases} \qquad ce_2 = ce_2$$

always true

solve for e_1, get

$$e_1 = \frac{b}{c-a}e_3, \text{ plug in, get}$$

$$\left(\frac{b^2}{c-a} + a\right)e_3 = ce_3, \quad \text{or} \quad 0 = \left(\frac{b^2}{c-a} + a - c\right)e_3, \text{ or}$$

$$0 = \frac{b^2-(a-c)^2}{a-c} e_3.$$ Since $b^2-(a-c)^2 \neq 0$, we have $e_3=0$.

Plugging $e_3=0$ to $e_1 = \frac{b}{c-a} e_3$, we find $e_1=0$. To normalize, we pick $e_2=1$, so our eigenstate is $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

$\boxed{\text{If } E=a+b}$

$$\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix} = \begin{pmatrix}(a+b)e_1\\(a+b)e_2\\(a+b)e_3\end{pmatrix}, \text{ or } \begin{pmatrix}ae_1+be_3\\ce_2\\be_1+ae_3\end{pmatrix} = \begin{pmatrix}(a+b)e_1\\(a+b)e_2\\(a+b)e_3\end{pmatrix}$$

Middle eq'n gives $(a+b-c)e_2=0$, or $e_2=0$.

Top eq'n gives $ae_1+be_3 = ae_1+be_1$, or $e_3=e_1$

For normalization, pick $e_1=e_3=\frac{1}{\sqrt{2}}$, so our eigstate is $\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

$\boxed{\text{If } E=a-b}$

$$\begin{pmatrix}ae_1+be_3\\ce_2\\ae_3+be_1\end{pmatrix} = \begin{pmatrix}ae_1-be_1\\ae_2-be_2\\ae_3-be_3\end{pmatrix}$$

Middle eqn gives $(a-c-b)e_2=0$, or $e_2=0$.

Top eqn gives $ae_1+be_3=ae_1-be_1$, or $e_3=-e_1$.

For normalization, pick $e_1=\frac{1}{\sqrt{2}}$, so our eigenstate is $\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}$

Second: Expand $\begin{pmatrix}0\\0\\1\end{pmatrix}$ in eigenstates. We want to write

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = A\begin{pmatrix}0\\1\\0\end{pmatrix} + B\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} + C\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}.$$ We note that

$$A = \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = 0, \quad B = \begin{pmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}, \quad C = \begin{pmatrix}1/\sqrt{2} & 0 & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = -\frac{1}{\sqrt{2}}.$$

Thus, $$\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}$$

We know the first eigenstate has time-dependence $e^{-i(a+b)t/\hbar}$, and the second has $e^{-i(a-b)t/\hbar}$, so

$$|\mathcal{S}(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} e^{\frac{-i(a+b)t}{\hbar}} - \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix} e^{-i(a-b)t/\hbar} = \begin{pmatrix}\frac{1}{2}e^{\frac{-i(a+b)t}{\hbar}} - \frac{1}{2}e^{\frac{-i(a-b)t}{\hbar}}\\ 0 \\ \frac{1}{2}e^{-i(a+b)t/\hbar} + \frac{1}{2}e^{-i(a-b)t/\hbar}\end{pmatrix}$$