Physics 486 Discussion 7 – Formalism

Problem 1: Sequential Measurements

adapted from Griffiths 3.27; Checkpoints

In this problem, we will perform some quantum calculations using only the generalized mathematical language of inner product spaces: kets for states, bra-kets for inner products, letters with hats for operators. We will not use any particular representation of the states and operators at all: no wavefunctions or differential operators, no column vectors or matrices, just kets and \( \hat{Q} \)s. This language is perfectly suited to the axioms of QM! ☝️

Consider operators \( \hat{A} \) and \( \hat{B} \). The eigenstates of operator \( \hat{A} \) are \( |\psi_1\rangle \) and \( |\psi_2\rangle \). These are the ONLY two eigenstates \( \Rightarrow \) we are considering a small, two-dimensional space spanned by exactly two eigenstates as it makes a good sandbox in which to play and learn. ☺️ The eigenstates of operator \( \hat{B} \) are \( |\phi_1\rangle \) and \( |\phi_2\rangle \) and are orthonormal, i.e. the \( |\phi\rangle \)'s are each normalized to 1 and \( |\phi_1\rangle \) is orthogonal to \( |\phi_2\rangle \). As have learned, in the generalized notation of inner product spaces, this orthonormality condition is written:

\[
\langle \phi_i | \phi_j \rangle = \delta_{ij}.
\]

The corresponding eigenvalues are \( a_1, a_2 \) for operator \( \hat{A} \) and \( b_1, b_2 \) for operator \( \hat{B} \). Finally, the eigenstates of the two different operators are related to each other in the following way:

\[
|\psi_1\rangle = \frac{3|\phi_1\rangle + 4|\phi_2\rangle}{5} \quad \text{and} \quad |\psi_2\rangle = \frac{4|\phi_1\rangle - 3|\phi_2\rangle}{5}.
\]

(a) Using the fact that \( |\phi_{1,2}\rangle \) are orthonormal, show that \( |\psi_2\rangle \) is also normalized, and that \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are orthogonal to each other.

(b) Write the eigenstates \( |\phi_{1,2}\rangle \) in terms of the eigenstates \( |\psi_{1,2}\rangle \).

(c) An experiment is performed to measure the observable \( A \). The measurement yields the value \( a_1 \). What is the state of the system immediately after this measurement?

Something new: We now introduce a very important formula that we didn’t quite get to in class yet, which is the master formula for probability:

\[
\text{Prob}(q) = |\langle e_q | \psi \rangle|^2
\]

In words: If a system is in state \( |\psi\rangle \) and we measure an observable \( \hat{Q} \), the probability of obtaining the eigenvalue \( q \) is calculated by projecting the state \( |\psi\rangle \) onto the eigenstate \( |e_q\rangle \) and taking the norm-squared.

(d) The observable \( B \) is now measured (i.e. after \( A \) was measured to be \( a_1 \)). What are the possible results and what are their probabilities?

(e) Right after the measurement of \( B \), \( A \) is measured again. What is the probability of getting the same value \( a_1 \) as in part (c)?

(f) What if, in part (d), the result of measurement \( B \) was found to be \( b_2 \); what is the possibility of getting \( a_1 \) in this case?

(g) Recall the definition of a commutator: \( [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \). Calculate \( [\hat{A}, \hat{B}]|\psi_1\rangle \). Do \( A \) and \( B \) commute?

► Looking ahead: Commutators appear again! (The first time was in last week’s discussion.) As we will shortly learn, operators that don’t commute with each other – as in this exercise – are called incompatible

\[
\text{Q1 (b)} \quad |\phi_1\rangle = \left( \frac{3}{5}|\psi_1\rangle + \frac{4}{5}|\psi_2\rangle \right) \quad |\phi_2\rangle = \left( \frac{4}{5}|\psi_1\rangle - \frac{3}{5}|\psi_2\rangle \right) \\
\text{(c)} \quad |\psi_1\rangle \\
\text{(d)} \quad \text{P}(b_1) = 9/25 = 36\% \quad \text{and} \quad \text{P}(b_2) = 16/25 = 64\% \\
\text{(e)} \quad \text{P}(a_1) = 337/625 = 53.9\% \\
\text{(f)} \quad \text{P}(a_1) = 16/25 \\
\text{(g)} \quad 12/25 \:\left( a_1 - a_2 (b_2 - b_1) \right) |\psi_2\rangle
\]
observables because the measurement of one affects the value of the other. The best-known pair of incompatible observables is \( \hat{x} \) & \( \hat{p}_x \). Just like for position and momentum, every pair of incompatible observables has an associated uncertainty principle.

**Problem 2 : Matrix Representation of States and Operators**  
Griffiths 3.37; Hints & Checkpoints

In today’s lecture, we learned about the **matrix representation** of QM states and operators, which is an alternative to the **wavefunction representation** that we have been using to date. A quick formula summary:

- **QM states** are represented as **column vectors** with components \( \psi_i = \langle e_i | \psi \rangle \)

- **QM operators** are represented as **matrices** with components \( Q_{ij} = \langle e_i | \hat{O} | e_j \rangle \)

- The **inner product** between two states is calculated as follows: \( \langle f | g \rangle = \hat{f}^*^T \hat{g} = f_i^* g_i \)

Now for some practice!

The Hamiltonian for a certain 3-level system is represented by the following matrix :

\[
H = \begin{pmatrix}
    a & 0 & b \\
    0 & c & 0 \\
    b & 0 & a
\end{pmatrix}
\]

where \( a, b, \) and \( c \) are real numbers. (Assume \( a - c \neq \pm b \)).

Find the system’s state at time \( t > 0 \) if the system starts in state

(a) \[
\begin{pmatrix}
    0 \\
    1 \\
    0
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
    0 \\
    0 \\
    1
\end{pmatrix}
\]

at \( t = 0 \).

➤ **GUIDANCE:** The matrix representation is new, so please do consult the step-by-step hints in the footnote if you are stuck or uncertain!

\[2\] Q2 Hints: You know how to do time-evolution: express the starting state as a linear combination of the system’s **energy eigenstates** … so you must first find the eigenvectors and eigenvalues of the given Hamiltonian matrix … the fastest way to do that, if possible, is to **guess** the form of the eigenvectors and **try them out** to find the eigenvalues → look at the form of \( H \) and guess; and once you have one or two of the eigenvectors, remember that a Hermitian matrix like our \( H \) has **orthogonal** eigenvectors, very helpful!

(a) \[
\Psi(t) = e^{-ict/h} \begin{pmatrix}
    0 \\
    1 \\
    0
\end{pmatrix}
\]

(b) \[
\Psi(t) = e^{-iat/h} \begin{pmatrix}
    -i \sin(bt/h) \\
    0 \\
    \cos(bt/h)
\end{pmatrix}
\]