Physics 486 Discussion 7 – Formalism

Problem 1 : Sequential Measurements *adapted from Griffiths 3.27; Checkpoints* [1]

In this problem, we will perform some quantum calculations using only the generalized mathematical language of **inner product spaces**: kets for states, bra-kets for inner products, letters with hats for operators. We will not use any particular representation of the states and operators at all: no wavefunctions or differential operators, no column vectors or matrices, just kets and $\hat{Q}s$. This language is perfectly suited to the axioms of QM! ☺

Consider operators $\hat{A}$ and $\hat{B}$. The eigenstates of operator $\hat{A}$ are $|\psi_1\rangle$ and $|\psi_2\rangle$. These are the ONLY two eigenstates → we are considering a small, two-dimensional space spanned by exactly two eigenstates as it makes a good sandbox in which to play and learn. ☺ The eigenstates of operator $\hat{B}$ are $|\phi_1\rangle$ and $|\phi_2\rangle$and are **orthonormal**, i.e. the $|\phi\rangle$'s are each normalized to 1 and $|\phi_1\rangle$ is orthogonal to $|\phi_2\rangle$. As have learned, in the generalized notation of inner product spaces, this orthonormality condition is written :

$$\langle\phi_i|\phi_j\rangle = \delta_{ij}.$$

The corresponding eigenvalues are a_1, a_2 for operator $\hat{A}$ and b_1, b_2 for operator $\hat{B}$. Finally, the eigenstates of the two different operators are related to each other in the following way:

$$|\psi_1\rangle = \frac{3|\phi_1\rangle + 4|\phi_2\rangle}{5} \quad \text{and} \quad |\psi_2\rangle = \frac{4|\phi_1\rangle - 3|\phi_2\rangle}{5}.$$

(a) Using the fact that $|\phi_{1,2}\rangle$ are orthonormal, show that $|\psi_2\rangle$ is also normalized, and that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal to each other.

(b) Write the eigenstates $|\phi_{1,2}\rangle$ in terms of the eigenstates $|\psi_{1,2}\rangle$.

(c) An experiment is performed to measure the observable A. The measurement yields the value a_1. What is the state of the system immediately after this measurement?

Something new: We now introduce a very important formula that we didn't quite get to in class yet, which is the **master formula for probability** :

$$\boxed{\text{Prob}(q) = \left|\langle e_q|\psi\rangle\right|^2}$$

In words: If a system is in state $|\psi\rangle$ and we measure an observable $\hat{Q}$, the probability of obtaining the eigenvalue q is calculated by **projecting** the state $|\psi\rangle$ onto the eigenstate $|e_q\rangle$ and taking the norm-squared.

(d) The observable B is now measured (i.e. after A was measured to be a_1). What are the possible results and what are their probabilities?

(e) Right after the measurement of B, A is measured again. What is the probability of getting the same value a_1 as in part (c)?

(f) What if, in part (d), the result of measurement B was found to be b_2; what is the possibility of getting a1 in this case?

(g) Recall the definition of a commutator : $\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$. Calculate $\left[\hat{A}, \hat{B}\right]\psi_1$. Do A and B commute?

▶ **Looking ahead**: Commutators appear again! (The first time was in last week's discussion.) As we will shortly learn, operators that don't commute with each other – as in this exercise – are called **incompatible**

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[1] **Q1 (b)** $|\phi_1\rangle = (\ 3|\psi_1\rangle + 4|\psi_2\rangle\ )/5$, $|\phi_2\rangle = (\ 4|\psi_1\rangle - 3|\psi_2\rangle\ )/5$ **(c)** $|\psi_1\rangle$
(d) P(b_1) = 9/25 = 36% and P(b_2) = 16/25 =64% **(e)** P(a_1) = 337/625 = 53.9% **(f)** P(a_1) = 16/25 **(g)** 12/25 $(a_1 - a_2)(b_2 - b_1)\,|\psi_2\rangle$

observables because the measurement of one affects the value of the other. The best-known pair of incompatible observables is $\hat{x}$ & $\hat{p}_x$. Just like for position and momentum, every pair of incompatible observables has an associated uncertainty principle.

Problem 2 : Matrix Representation of States and Operators *Griffiths 3.37; Hints & Checkpoints* [2]

In today's lecture, we learned about the **matrix representation** of QM states and operators, which is an alternative to the **wavefunction representation** that we have been using to date. A quick formula summary:

- QM <u>states</u> are represented as <u>column vectors</u> with components $\boxed{\psi_i = \langle e_i | \psi \rangle}$
- QM <u>operators</u> are represented as <u>matrices</u> with components $\boxed{Q_{ij} = \langle e_i | \hat{Q}\, e_j \rangle}$
- The <u>inner product</u> between two states is calculated as follows: $\boxed{\langle f | g \rangle = \vec{f}^{*\mathrm{T}} \vec{g} = f_i^* g_i}$

Now for some practice!
The Hamiltonian for a certain 3-level system is represented by the following matrix :

$$\mathbf{H} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix} \quad \text{where } a, b, \text{ and } c \text{ are real numbers. (Assume } a - c \neq \pm b\text{).}$$

Find the system's state at time $t > 0$ if the system starts in state (a) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ at $t = 0$.

▶ **GUIDANCE**: The matrix representation is new, so please do consult the step-by-step hints in the footnote if you are stuck or uncertain!

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[2] **Q2** Hints: You know how to do time-evolution: express the starting state as a linear combination of the system's <u>energy eigenstates</u>
… so you must first find the eigenvectors and eigenvalues of the given Hamiltonian matrix
… the fastest way to do that, if possible, is to <u>guess</u> the form of the eigenvectors and <u>try them out</u> to find the eigenvalues → look at the form of **H** and guess; and once you have one or two of the eigenvectors, remember that a Hermitian matrix like our **H** has <u>orthogonal</u> eigenvectors, very helpful!

(a) $|\Psi(t)\rangle = e^{-ict/\hbar} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ **(b)** $|\Psi(t)\rangle = e^{-iat/\hbar} \begin{pmatrix} -i\sin(bt/\hbar) \\ 0 \\ \cos(bt/\hbar) \end{pmatrix}$