1a) $i\hbar \frac{\partial}{\partial t}(\Psi^* \Psi) = i\hbar\left(\dot{\Psi}^* \Psi + \Psi^* \dot{\Psi}\right)$

Now, Schrodinger's Eqn says

$$i\hbar \dot{\Psi} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

Taking the conjugate of both sides gives

$$-i\hbar \dot{\Psi}^* = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V^* \Psi^*$$

Plugging these in gives

$$i\hbar \frac{\partial}{\partial t}(\Psi^* \Psi) = -\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V^* \Psi^*\right)\Psi + \Psi^*\left(\frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi\right)$$

$$= \frac{-\hbar^2}{2m}\left[\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right] + (V - V^*)\Psi^* \Psi$$

$$= -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right] + (V - V^*)\Psi^* \Psi$$

b) $\frac{d\langle x\rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \Psi^* x \Psi \, dx$

$$= \int_{-\infty}^{\infty} x \frac{d}{dt}(\Psi^* \Psi)\, dx$$

$$= \int_{-\infty}^{\infty} x\left[\frac{-\hbar^2}{2m}\frac{\partial}{\partial x}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\frac{1}{i\hbar}\right] dx$$

$$= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x \frac{\partial}{\partial x}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) dx$$

Integrate by parts

$$= \frac{-i\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) dx$$

c) We have

Integrate 2nd term by parts

$$\frac{d\langle x\rangle}{dt} = \frac{-i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x} + \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\Psi$$

$$= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x} - \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}$$

$$= \frac{-i\hbar}{m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}$$

$$= \frac{1}{m}\int_{-\infty}^{\infty}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi$$

$$= \frac{\langle p\rangle}{m}$$

d) We know $\langle p \rangle = \int \Psi^* (-i\hbar \frac{\partial}{\partial x}) \Psi \, dx$. We want to take the derivative of this.

$$\frac{d\langle p \rangle}{dt} = \underbrace{\int \dot{\Psi}^* (-i\hbar \frac{\partial}{\partial x}) \Psi}_{1} + \underbrace{\int \Psi^* (-i\hbar \frac{\partial}{\partial x}) \dot{\Psi}}_{2}$$

Now, Schrodinger's equation says that

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi, \text{ or}$$

$$\dot{\Psi} = \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \frac{V}{i\hbar} \Psi$$

Thus, we can write

$$2 = \int \Psi^* (-i\hbar \frac{\partial}{\partial x}) \left[\frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \frac{V}{i\hbar} \Psi \right]$$

$$= \int \Psi^* \frac{\partial^3 \Psi}{\partial x^3} \times \frac{\hbar^2}{2m} - \int \Psi^* \frac{\partial}{\partial x} (V\Psi)$$

$$= \frac{\hbar^2}{2m} \int \Psi^* \frac{\partial^3 \Psi}{\partial x^3} - \int \Psi^* V \frac{\partial \Psi}{\partial x} - \int \Psi^* \frac{\partial V}{\partial x} \Psi$$

Now, since we know the equation for $\dot{\Psi}$, we can find the equation for $\dot{\Psi}^*$ by taking the conjugate of both sides

$$\dot{\Psi}^* = -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} - \frac{V}{i\hbar} \Psi^*$$

Thus, we can write

$$1 = \int \left(-\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} - \frac{V}{i\hbar} \Psi^* \right) (-i\hbar \frac{\partial}{\partial x}) \Psi$$

$$= -\frac{\hbar^2}{2m} \int \frac{\partial^2 \Psi^*}{\partial x^2} \frac{\partial \Psi}{\partial x} + \int \Psi^* V \frac{\partial \Psi}{\partial x}$$

$$= -\frac{\hbar^2}{2m} \int \Psi^* \frac{\partial^3 \Psi}{\partial x^3} + \int \Psi^* V \frac{\partial \Psi}{\partial x}$$

where in the last line, I used integration by parts twice (see postscript)

Thus, $\frac{\partial\langle p\rangle}{\partial t} = ① + ② = -\int \Psi^* \frac{\partial V}{\partial x}\Psi = \int \Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi = \left\langle -\frac{\partial V}{\partial x}\right\rangle$

This is just the classical $F = ma$, or $\underbrace{\frac{\partial p}{\partial t}}_{ma} = \underbrace{-\frac{\partial V}{\partial x}}_{F}$

but w/ expectations on both sides.

$\boxed{\text{Postscript}}$: How do we know $\int \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x} = \int \Psi^* \frac{\partial^3\Psi}{\partial x^3}$?

Integration by parts! Say we have two functions $f + g$, which both go to zero at $\pm\infty$. Integration by parts says

$$\int_{-\infty}^{\infty} f \frac{\partial g}{\partial x}\,dx = fg\Big|_{-\infty}^{\infty} - \int \frac{\partial f}{\partial x} g\,dx$$

$$= -\int \frac{\partial f}{\partial x} g\,dx, \quad \text{since } f(\pm\infty) = 0$$

So in the case where the fcns vanish at $\pm\infty$, integration by parts lets us move derivatives from one function to another. Each time we do, we get a $(-)$ sign.

Now, wavefunctions always vanish at $\pm\infty$, otherwise they wouldn't be normalizable. So in our case, we can integrate by parts twice to get, e.g.,

$$\int \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x} = -\int \frac{\partial\Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2} = \int \Psi^* \frac{\partial^3\Psi}{\partial x^3}$$

Here, $f = \frac{\partial\Psi}{\partial x}$, $g = \frac{\partial^2\Psi^*}{\partial x}$

Here, $f = \frac{\partial^2\Psi}{\partial x^2}$, $g = \Psi^*$

3) Say $\Psi(x,t)$ is a solution to Schrödingers equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi.$$

Let's add a constant term to our potential energy:

$$V_{new} = V + V_0$$

We want to show that $\Psi_{new} = e^{-\frac{iV_0 t}{\hbar}}\Psi$ is a solution to the new Schrödinger Eqn,

$$i\hbar \frac{\partial \Psi_{new}}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_{new}}{\partial x^2} + V_{new}\Psi_{new}$$

We'll just evaluate each side, and show they're equal.

$$i\hbar \frac{\partial \Psi_{new}}{\partial t} = i\hbar \frac{\partial}{\partial t}\left(e^{-\frac{iV_0 t}{\hbar}}\Psi\right) = i\hbar\left(\frac{-iV_0}{\hbar}\right)e^{-\frac{iV_0 t}{\hbar}}\Psi + i\hbar e^{-\frac{iV_0 t}{\hbar}}\frac{\partial \Psi}{\partial t}$$

$$= V_0\Psi_{new} + e^{-iV_0 t/\hbar}\left(i\hbar\frac{\partial \Psi}{\partial t}\right)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_{new}}{\partial x^2} + V_{new}\Psi_{new} = e^{-iV_0 t/\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi\right) + V_0\Psi_{new}$$

We want to show that these are equal. We know

$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$. Multiplying both sides by $e^{-\frac{iV_0 t}{\hbar}}$,

$e^{-iV_0 t/\hbar}\left(i\hbar\frac{\partial \Psi}{\partial t}\right) = e^{-iV_0 t/\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi\right)$. Adding $V_0\Psi_{new}$ to both,

$$V_0\Psi_{new} + e^{-iV_0 t/\hbar}\left(i\hbar\frac{\partial \Psi}{\partial t}\right) = e^{-\frac{iV_0 t}{\hbar}}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi\right) + V_0\Psi_{new}$$

Thus, $i\hbar \frac{\partial \Psi_{new}}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_{new}}{\partial x^2} + V_{new}\Psi_{new}$, and Ψ_{new} does indeed solve the schrödinger eqn. This phase has <u>no</u> <u>effect</u> on dynamical variables!