DENSITY OF STATES $n(E_f)$ for SPONTANEOUS EMISSION

- Considering spontaneous transition $i \to f$ where i & f are specific states

- e.g. decay $2p \to 1s$ in H : $E_f = E_i \pm \hbar\omega$
  E_f specific, E_i specific

continuum of photon states due to vacuum fluctuations

$$\Downarrow$$

$$n(E_\gamma)$$

METHOD:

① PERIODIC BOUNDARY CONDITIONS

Consider plane wave states $\psi_\gamma \sim e^{i\vec{k}\cdot\vec{r}}$ Can't normalize over all space $\therefore$ can't COUNT # states

...

Introduce the BOX OF CONVENIENCE : size $L \times L \times L$

Do all our physics w/in Box; run $L \to \infty$ @ end

Normalize plane waves w/in box

$$\psi(x,y,z) = \frac{e^{ik_x x}}{\sqrt{L}} \frac{e^{ik_y y}}{\sqrt{L}} \frac{e^{ik_z z}}{\sqrt{L}}$$

Plane waves are e-states of $\vec{p}$ with e-value $\hbar\vec{k}$

... must ensure e-values $\vec{p} = \hbar\vec{k}$ are ONLY REAL

i.e. insist $\left[\hat{p} \text{ is HERMITIAN}\right]$: $\langle f | \hat{p} g\rangle = \langle \hat{p} f | g\rangle$

$\forall$ two allowed wavefn's f, g

$$\left\langle f \,\middle|\, \frac{\hbar}{i}\frac{\partial}{\partial x} g\right\rangle = \left\langle \frac{\hbar}{i}\frac{\partial}{\partial x} f \,\middle|\, g\right\rangle$$

$$\int_0^L dx\, f^* \left[\frac{\hbar}{i}\frac{dg}{dx}\right] = \int_0^L dx \left[\frac{\hbar}{i}\frac{df}{dx}\right]^* g$$

(NOTE: case $f=g$,
$\langle f | \hat{p} f\rangle = \langle \hat{p} f | f\rangle$
$\quad\quad = \quad\quad\quad\quad =$
$\langle \hat{p}\rangle = \langle \hat{p}\rangle^*$ i.e. $\langle \hat{p}\rangle$ REAL)

$\Downarrow$ I.B.P. $\quad\quad\quad\quad \Downarrow$

$$\frac{\hbar}{i}\left[f^* g\Big|_0^L - \int_0^L \frac{df^*}{dx} g\right] = -\frac{\hbar}{i}\int_0^L dx\, \frac{df^*}{dx} g \quad \Rightarrow f^* g\Big|_0^L = \emptyset \quad \forall \text{ TWO allowed } f, g$$

~~same~~ $\quad\quad\quad$ ~~same~~

$$\therefore f^*(L) g(L) = f^*(0) g(0)$$

$$\therefore \frac{g(L)}{g(0)} = \left[\frac{f(L)}{f(0)}\right]^{-1\,*} = e^{i\alpha}$$

CONSTANT indep of f, g

take $f = g = \psi$:

$$\psi^*\psi\Big|_{x=0} = \psi^*\psi\Big|_{x=L}$$

$$\frac{\psi(L)}{\psi(0)} = \text{CONST} = \left[\frac{\psi(L)}{\psi(0)}\right]^{*\,-1} = e^{i\alpha}$$

FOR DEFINITENESS,

take $\alpha = 0$ $\Rightarrow$

CONDITION ON WAVEFN's:

$$\boxed{\psi(L) = \psi(0)}$$

(will ensure later this choice doesn't matter!)

Return to plane wave basis: $\psi(x) = \dfrac{e^{ik_x x}}{\sqrt{L}}$

Impose $\psi(L) = \psi(0)$:

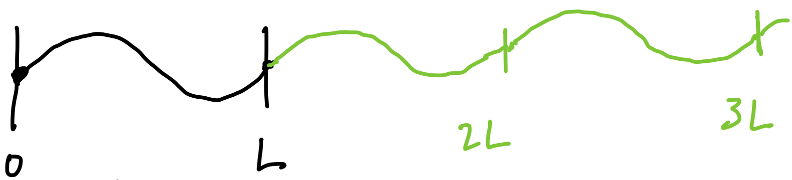
periodic boundary cond. *

$$e^{ik_x L} = 1 \Rightarrow k_x = \frac{2\pi n}{L} \Rightarrow \hbar k_x = \boxed{p_x = n_x \cdot \frac{2\pi\hbar}{L}}$$

$= \dfrac{2\pi\hbar}{L}, \dfrac{4\pi\hbar}{L}, \dfrac{6\pi\hbar}{L}, \ldots$

* "called PERIODIC B.C."

$\therefore$ plane waves restricted to



"wall paper" boundary conditions

Want $n(E_\gamma)$...

For PHOTONS, $E_\gamma = p_\gamma c$ ... drop γ subscript for brevity:

$$n(E) \equiv \frac{dn}{dE} \underset{3D}{=} \frac{dn_x\, dn_y\, dn_z}{dE} = \frac{d^3 n}{dE} \quad \ldots \text{ now relate } dn \text{ to } dp:$$

*

$$\hookrightarrow \quad d^3 n = \left(dp \cdot \frac{L}{2\pi\hbar}\right)^3 = \left(\frac{L}{2\pi\hbar}\right)^3 \cdot d^3 p$$

$$d^3 p = p^2\, dp\, d\Omega_p$$

$$p = \frac{E}{c} \quad \therefore dp = \frac{dE}{c}$$

$$\therefore d^3 p = \frac{E^2\, dE}{c^3} \cdot d\Omega_p$$

$$d^3 n = 2\left(\frac{L}{2\pi\hbar}\right)^3 \cdot d\Omega_p \frac{E^2\, dE}{c^3}$$

* $\times 2$ independent POLARIZATION STATES of LIGHT

(Electric field amplitude $\vec{E}_0 e^{i\vec{k}\cdot\vec{r}}$ ⇢ from Maxwell's equ's in vacuum, $\vec{E}_0 \perp \vec{k}$ ∴ in a plane, spanned by two indep directions

e.g. $\vec{E}_0 e^{ikz}$ with $\vec{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y}$
$\vec{k} \parallel z$)

$$n(E) = \frac{d^3n}{dE} = 2 \cdot \frac{L^3}{8\pi^3\hbar^3} \cdot d\Omega_p \cdot \frac{E^2}{c}$$

integrate over angles of $\vec{p}$: $\int d\Omega_p = 4\pi$

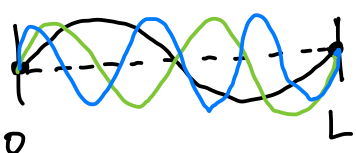
$$\boxed{n_\gamma(E_\gamma) = \frac{L^3}{\pi^2\hbar^3}\frac{E_\gamma^2}{c^3}}$$

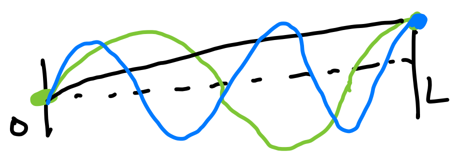
number density of photon states in vacuum of $L\times L\times L$

bound. cond. was:
$\psi(L) = \psi(0)\, e^{i\alpha}$ CONST ... suppose we chose instead $\alpha = \pi/2$ (rather than $\alpha=0$)

$\alpha = \pi/2$: $\psi(L) = \psi(0)\cdot e^{i\pi/2}$

$\alpha = 0$





$kL = \alpha + n2\pi$

$\therefore p = \hbar k = \hbar/L[\alpha + 2\pi n]$

$dp = dn\cdot(2\pi\hbar/L)$

SPACING BETWEEN allowed momenta IS THE SAME

Collect everything into F.G.R:

transition rate $W_{i\to f} = \frac{2\pi}{\hbar} |V_{fi}|^2 n(E_f)\Big|_{E_i \pm \hbar\omega}$ **

now we have this!

from ① ② for E1 radiation: $|\vec{V}_{fi}|^2 = \frac{E_0^2}{3} \underbrace{|\langle f | q\vec{r} | i \rangle|^2}_{\vec{p}_{fi}}$

$\vec{p}_{fi}$: transition electric dipole moment

$$\Rightarrow W_{i\to f} = \frac{2\pi}{\hbar} \frac{\vec{E}_0^2}{3} |\vec{p}_{fi}|^2 \cdot \frac{L^3}{\pi^2 \hbar^3} \frac{E_\gamma^2}{c^3}$$

** $E_\gamma = E_f - E_i$ on resonance

recall: from E&M $\underbrace{\varepsilon_0 E^2}_{\text{field}} = \frac{\text{energy of EM wave}}{\text{volume}} = \frac{E_\gamma}{L^3} = \frac{\hbar\omega_{fi}}{L^3}$

$u_{\substack{EM\\ \text{wave}}} = u_E + u_B = 2u_E$

$\underbrace{\varepsilon_0 E^2}_{\text{field}} \cdot L^3 = E_\gamma$

combination can be replaced with something physical/non-artificial $\rightarrow E_\gamma$

Our $H'(\vec{r},t) = V(\vec{r})[e^{i\omega t} + e^{-i\omega t}]$

... arises from $\vec{E}(\vec{r},t) = (E_0 e^{i\vec{k}\cdot\vec{r}}) \underbrace{[e^{i\omega t} + e^{-i\omega t}]}_{2\cos\omega t}$

$\therefore \langle u_{EM} \rangle_{\substack{\text{cycle-}\\ \text{average}}} = E_0^2 \cdot \langle (2\cos\omega t)^2 \rangle$

$= E_0^2 \cdot 4 \cdot \frac{1}{2} = 2E_0$

$$\therefore \boxed{W_{i\to f} = \frac{1}{3\pi\varepsilon_0} \frac{|\vec{p}_{fi}|^2}{\hbar c^3} \omega_{fi}^3 \equiv A} = A_{i\to f} \text{ Einstein's coefficient}$$

$\frac{\text{prob}}{\text{time}}$ = RATE of spontaneous photon emission via E1 radiation

crucial: artificial L is gone as it must be

USUAL formula is A = above $\times 4\pi\varepsilon_0$

difference: Griffiths/us/experimentalists = MKS units
theorists/≈all other QM books = Gaussian units

SURVIVAL GUIDE to MKS vs G units

(1) $F_E = \left(\frac{q_1 q_2}{r^2}\right)_G = \left(\frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\right)_{MKS}$: $q_G \leftrightarrow \frac{q_{MKS}}{\sqrt{4\pi\varepsilon_0}}$

(2) $F_E = (qE)_{both}$: $E_G \leftrightarrow E_{MKS}\sqrt{4\pi\varepsilon_0}$

(3) $F_B = \left(q\frac{v}{c}\times B\right)_G = (qv\times B)_{MKS}$: $B_G \leftrightarrow B_{MKS}\cdot c\sqrt{4\pi\varepsilon_0}$
$= B_{MKS}\cdot\sqrt{\frac{4\pi}{\mu_0}}$

(4) $\vec{B} = (\vec{\nabla}\times\vec{A})_{both}$

$\vec{E} = \left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)_G = \left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right)_{MKS}$