

EM RADIATION as Perturbation

jargon: E1 = elec. dipole
 E2 = elec. quadr.
 ...
 M1 = mag. dipole
 ...

Electric Dipole (E1) Radiation

E1 radiation is dominant when

- ① $F_B \ll F_E$ on charges in your system
 - ② $\vec{k} \cdot \vec{r} \ll 1$
- } very common in atomic physics

① $F_E = qE$... compare with $F_B = |q\vec{v} \times \vec{B}| \leq qvB$

- compare $E \neq B$: for EM waves in vacuum, $B = \frac{E}{c}$
- how big is v for atomic electrons?

Virial Theorem for p'cles bound in $-\frac{1}{r}$ potential:

$$\langle E \rangle = \langle V \rangle + \langle T \rangle \quad \dots \text{Virial Thm for } 1/r$$

NEG NEG POS

$$= -\langle T \rangle = -\frac{mv^2}{2}$$

$|\langle V \rangle| = 2\langle T \rangle$

Bohr $E_n = -\frac{(Z\alpha)^2}{2n^2} (m_e c^2) = -\frac{m_e c^2}{2} \cdot \frac{v^2}{c^2}$

$\therefore \frac{v}{c} = \frac{Z\alpha}{n} \leq Z\alpha = \frac{Z}{137} \approx 1\% \text{ for light atoms}$
 -10%

$$\Rightarrow \frac{F_B}{F_E} \text{ on } e\text{'s in } \sim \text{light atoms} \leq \frac{qvB}{qE} = \frac{v}{c} = \mathcal{O}(1\%) \ll 1 \quad //$$

in presence of EM waves

② $\vec{k} \cdot \vec{r} \leq \frac{2\pi r}{\lambda} \rightarrow r \approx 1 \text{ \AA} = 0.1 \text{ nm}$

EM wave
 (with $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$)
 PLANE WAVE

eg. visual range $\lambda = 400 - 700 \text{ nm}$
 cf. RF waves have MUCH larger λ

$$\Rightarrow \text{in } \underline{\text{huge}} \text{ range of cases, } \vec{k} \cdot \vec{r} \ll 1 //$$

With approx ① $F_B \ll F_E$ ② $\vec{k} \cdot \vec{r} \ll 1$

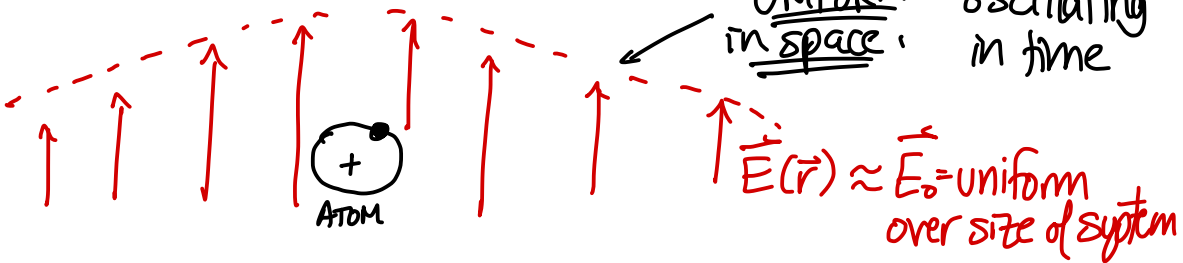
part of EM wave to consider:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_0 e^{i\vec{k} \cdot \vec{r}} e^{-i\omega t}$$

where $e^{i\vec{k} \cdot \vec{r}} = 1 + i\vec{k} \cdot \vec{r} + \frac{(i\vec{k} \cdot \vec{r})^2}{2!} + \dots$

$$\approx 1 \text{ for } \vec{k} \cdot \vec{r} \ll 1 \text{ i.e. } \vec{E}(\vec{r}, t) \approx \vec{E}_0 e^{-i\omega t}$$

UNIFORM in space, oscillating in time



Goal: Evaluate FGR $W_{i \rightarrow f}$ for E1 radiation

need ① average $\overline{V_{fi}}$ over polarizⁿ states of $\vec{E}_0 = \hat{\epsilon} E_0$
(assuming INCOHERENT source of radiation)

② $|\overline{V_{fi}}|$ transition matrix elem \rightarrow selection rules

③ $n(E_f)$ density of states

Potential of a charge q in an E1 radⁿ field:

$$V(\vec{r}) = -q \vec{E}_0 \cdot \vec{r}$$

just electrostatic potential energy $= \int_{\vec{r}_0, \text{REF}}^{\vec{r}} \vec{F}_E \cdot d\vec{l}$
 $=$ WORK integral

⊕ spatial part \uparrow is what goes into F.G.R. (as part of V_{fi})

① Assuming INCOHERENT source of radiation (e.g. flashlight, blackbody source)

$$V_{fi} = \langle f | -q \vec{E}_0 \cdot \vec{r} | i \rangle \dots \text{define polarizⁿ vector } \hat{\epsilon} : \\ \vec{E}_0 = E_0 \hat{\epsilon}$$

$$= -q E_0 \langle f | \hat{\epsilon} \cdot \vec{r} | i \rangle \quad (\text{classical})$$

$\hat{\epsilon}$ is \perp to \vec{k} of photon but

randomly oriented: $\vec{r} \cdot \hat{z} = r \cos\Theta$

\Rightarrow average over random Θ [Gr p-353]

$$|\overline{\langle \vec{r} \cdot \hat{z} \rangle}_{if}|^2 = \left(\frac{1}{3}\right) |\langle \vec{r} \rangle_{if}|^2 \sim |\overline{V_{fi}}|^2$$

NOT an expectation value!

$$\langle \hat{Q} \rangle_{if} \equiv \langle i | \hat{Q} | f \rangle$$

= TRANSITION matx. elem

2) When is V_{fi} between atomic states non-zero?

SELECTION RULES for E1 transitions:

Take $|i\rangle, |f\rangle = |n, l, m\rangle_{i,f}$ single-particle states

Transition

Matx elem is $V_{fi} = -E_0 \langle n_f, l_f, m_f | q \vec{r} | n_i, l_i, m_i \rangle$

* (or any central potential)

When is V_{fi} non-zero?

Gr. p.360-361: when

$$\bullet l_i - l_f = \pm 1$$

$$\bullet m_i - m_f = 0, \pm 1$$

E1 selection rules

$\equiv \vec{p}_{fi} :: q \vec{r}$ is electric dipole moment of a single charge q relative to the origin

... BUT \vec{p}_{fi} is "TRANSITION electric dipole moment",

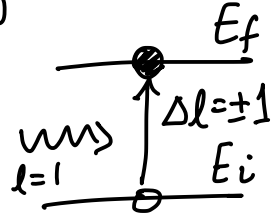
What does this tell us about an E1 photon?

NOT actual EDM of any single charge distribution

⇒ E1 photon must carry angular momentum of 1

QFT: E1 photon is a "SPIN*1 BOSON"

*ie. orbital ang. momentum of 1, but intrinsic to an E1 photon



⊕ E1 transitions are so dominant

that transitions mediated by E2 (electric quadrupole) are called FORBIDDEN transition

M1 (magnetic dipole)

∴ terms in expansion of EM wave

③ Finally, need density of states $n(E_f)$

...

DENSITY OF STATES $n(E_f)$ for SPONTANEOUS EMISSION

- Considering spontaneous transition $i \rightarrow f$
where $i \neq f$ are specific states

• eg. decay

$$2p \rightarrow 1s \text{ in H} : E_f = E_i \pm \hbar\omega$$

continuum of photon states due to vacuum fluctuations

$$\Downarrow \\ n(E_f)$$

METHOD:

PERIODIC BOUNDARY CONDITIONS

Consider plane wave states

$$\psi_{\mathbf{k}} \sim e^{i\mathbf{k} \cdot \mathbf{r}}$$

Can't normalize over all space
 \therefore can't COUNT # states

Introduce the BOX OF CONVENIENCE: size $L \times L \times L$

Do all our physics w/in Box; run $L \rightarrow \infty$ @ end

Normalize plane waves w/in box

$$\psi(x, y, z) = \frac{e^{ik_x x}}{\sqrt{L}} \frac{e^{ik_y y}}{\sqrt{L}} \frac{e^{ik_z z}}{\sqrt{L}}$$

Plane waves are e-states of \hat{p} with value $\hbar k$
 ... must ensure e-values $\hat{p} = \hbar k$ are ONLY REAL

i.e. insist $[\hat{p} \text{ is HERMITIAN}] : \langle f | \hat{p} g \rangle = \langle \hat{p} f | g \rangle$

forall two allowed wavefn's f, g

(NOTE: case $f=g$,
 $\langle f | \hat{p} f \rangle = \langle \hat{p} f | f \rangle$)

$\langle \hat{p} \rangle = \langle \hat{p} \rangle^*$ i.e. $\langle \hat{p} \rangle$ REAL

$$\langle f | \frac{\hbar}{i} \frac{\partial}{\partial x} g \rangle = \langle \frac{\hbar}{i} \frac{\partial}{\partial x} f | g \rangle$$

$$\int_0^L dx f^* \left[\frac{\hbar}{i} \frac{dg}{dx} \right] = \int_0^L dx \left[\frac{\hbar}{i} \frac{df}{dx} \right]^* g$$

↓ I.B.P.

$$\frac{\hbar}{i} \left[f^* g \right]_0^L - \int_0^L \frac{df^*}{dx} g = - \frac{\hbar}{i} \int_0^L dx \frac{df}{dx} g$$

~~same~~ ~~same~~

$$\Rightarrow f^* g \Big|_0^L = 0 \quad \forall \text{ TWO allowed } f, g$$

$$\therefore f^*(L)g(L) = f^*(0)g(0)$$

take $f=g=\psi$:

$$\psi^* \psi \Big|_{x=0} = \psi^* \psi \Big|_{x=L}$$

$$\therefore \frac{g(L)}{g(0)} = \left[\frac{f(L)}{f(0)} \right]^{-1*} = e^{i\alpha}$$

CONSTANT indep of f, g

$$\frac{\psi(L)}{\psi(0)} = \text{CONST} = \left[\frac{\psi(L)}{\psi(0)} \right]^{*-1} = e^{i\alpha}$$

FOR DEFINITENESS,

take $\alpha=0$ →
 CONDITION ON WAVEFN'S:

$$\psi(L) = \psi(0)$$

(will ensure later this choice doesn't matter!)

Return to plane wave basis: $\psi(x) = \frac{e^{ik_x x}}{\sqrt{L}}$

Impose $\psi(L) = \psi(0)$:

$$e^{ik_x L} = 1 \Rightarrow k_x = \frac{2\pi n}{L} \Rightarrow \hbar k_x = \boxed{p_x = n_x \cdot \frac{2\pi \hbar}{L}}$$

periodic boundary cond. *

⊕ "called periodic B.C."

∴ plane waves restricted to

$$= \frac{2\pi \hbar}{L}, \frac{4\pi \hbar}{L}, \frac{6\pi \hbar}{L}, \dots$$



"wall paper" boundary conditions

Want $n(E_x)$...

For PHOTONS, $(E_x = p_x c)$... drop x subscript for brevity:

$$n(E) \equiv \frac{dn}{dE} \stackrel{3D}{=} \frac{dn_x dn_y dn_z}{dE} = \frac{d^3 n}{dE} \dots \text{now relate } dn \text{ to } dp:$$

$$\star \rightarrow d^3 n = \left(dp \cdot \frac{L}{2\pi \hbar} \right)^3 = \left(\frac{L}{2\pi \hbar} \right)^3 \cdot \underbrace{d^3 p}_{\rightarrow d^3 p = p^2 dp d\Omega_p}$$

$$d^3 n = 2 \left(\frac{L}{2\pi \hbar} \right)^3 \cdot d\Omega_p \frac{E^2 dE}{c^3} \quad p = \frac{E}{c} \therefore dp = \frac{dE}{c}$$

$$\therefore d^3 p = \frac{E^2 dE}{c^3} \cdot d\Omega_p$$

⊕ x 2 independent POLARIZATION STATES of LIGHT

Electric field amplitude $\vec{E}_0 e^{i\vec{k}\cdot\vec{r}}$ from Maxwell's equ's in vacuum, $\vec{E}_0 \perp \vec{k} \therefore$ in a plane, spanned by two indep directions

e.g. $\vec{E}_0 e^{ikz}$ with $\vec{E}_0 = E_{0x} \hat{x} + E_{0y} \hat{y}$
 $\vec{k} \parallel z$

$$n(E) = \frac{d^3 n}{dE} = 2 \cdot \frac{L^3}{8\pi^3 \hbar^3} \cdot d\Omega_p \cdot \frac{E^2}{c}$$

integrate over angles of \vec{p} : $\int d\Omega_p = 4\pi$

$$n(E) = \frac{L^3}{\pi^2 \hbar^3} \frac{E^2}{c^3}$$

number density of photon states in vacuum of $L \times L \times L$

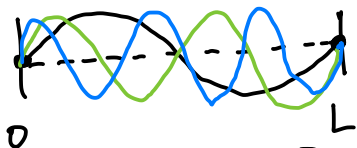
bound. cond. was:

~~$\psi(L) = \psi(0) e^{i\alpha}$~~
 $\psi(L) = \psi(0) e^{i\alpha}$
 CONST

... suppose we chose instead $\alpha = \pi/2$ (rather than $\alpha = 0$)

$$\alpha = \pi/2 : \psi(L) = \psi(0) \cdot e^{i\pi/2}$$

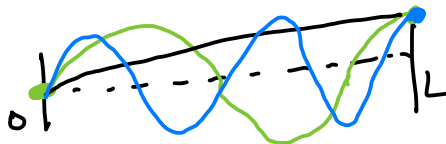
$\alpha = 0$



$$kL = \alpha + n2\pi$$

$$\therefore p = \hbar k = \hbar/L [\alpha + 2\pi n]$$

$$dp = dn \cdot (2\pi \hbar/L)$$



SPACING BETWEEN allowed momenta is THE SAME

Collect everything into F.G.R:

transition rate $W_{i \rightarrow f} = \frac{2\pi}{\hbar} |V_{fi}|^2 n(E_f | E_i \pm \hbar\omega)$ **

now we have this!

from (1) (2) for E1 radiation: $|V_{fi}|^2 = \frac{E_0^2}{3} |\langle f | q\vec{r} | i \rangle|^2$
 \vec{p}_{fi} : transition electric dipole moment

$\Rightarrow W_{i \rightarrow f} = \frac{2\pi}{\hbar} \frac{E_0^2}{3} |\vec{p}_{fi}|^2 \cdot \frac{L^3}{\pi^2 \hbar^3} \frac{E_\gamma^2}{c^3}$

** $E_\gamma = E_f - E_i$ on resonance

recall: $\sum_0 \underbrace{E^2}_{\text{field}} = \frac{\text{energy of EM wave}}{\text{volume}} = \frac{E_\gamma}{L^3} = \frac{\hbar\omega_{fi}}{L^3}$

$U_{EM, \text{wave}} = U_E + U_B = 2U_E$

$\sum_0 \underbrace{E^2}_{\text{field}} \cdot L^3 = E_\gamma$

combination can be replaced with something physical / non-artificial $\rightarrow E_\gamma$

Our $\psi'(\vec{r}, t) = V(\vec{r}) [e^{i\omega t} + e^{-i\omega t}]$
 ...arises from $\vec{E}(\vec{r}, t) = (E_0 e^{i\vec{k}\cdot\vec{r}}) [e^{i\omega t} + e^{-i\omega t}]$
 $2 \cos \omega t$

$\langle U_{EM} \rangle = E_0^2 \cdot \langle (2 \cos \omega t)^2 \rangle$
 cycle-average $= E_0^2 \cdot 4 \cdot \frac{1}{2} = 2E_0^2 \rightarrow$

$$\therefore W_{i \rightarrow f} = \frac{1}{3\pi\epsilon_0} \frac{|\vec{p}_{fi}|^2}{\hbar c^3} \omega_{fi}^3 \equiv A = A_{i \rightarrow f} \text{ Einstein's coefficient}$$

prob/time = RATE of spontaneous photon emission via E1 radiation

⊕ crucial: artificial L is gme as it must be

⊕ USUAL formula is $A = \text{above} \times 4\pi\epsilon_0$

difference: Griffiths/us/experimentalists = MKS units
 theorists / \approx all other QM books = Gaussian units

SURVIVAL GUIDE to MKS vs G units

① $F_E = \left(\frac{q_1 q_2}{r^2} \right)_G = \left(\frac{q_1 q_2}{4\pi\epsilon_0 r^2} \right)_{\text{MKS}}$: $q_G \leftrightarrow \frac{q_{\text{MKS}}}{\sqrt{4\pi\epsilon_0}}$

② $F_E = (qE)_{\text{both}}$: $E_G \leftrightarrow E_{\text{MKS}} \sqrt{4\pi\epsilon_0}$

③ $F_B = \left(q \frac{v}{c} \times B \right)_G = \left(q v \times B \right)_{\text{MKS}}$: $B_G \leftrightarrow B_{\text{MKS}} \cdot c \sqrt{4\pi\epsilon_0}$

④ $\vec{B} = (\vec{\nabla} \times \vec{A})_{\text{both}}$: $B_{\text{MKS}} \cdot \sqrt{\frac{4\pi}{\mu_0}}$

$\vec{E} = \left(-\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right)_G = \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \right)_{\text{MKS}}$