\( \vec{j}(\vec{r}, t) = \text{Re} \left[ \psi^* \frac{\partial}{\partial \vec{r}} \psi \right] \)  

Complementary velocity to prob. density: \( \vec{p}(\vec{r}, t) = \psi^* \psi \)  

\{ p, \vec{j} \} probability analogue of charge, current density in E&M; linked by \( \vec{j}(\vec{r}, t) = \rho(\vec{r}, t) \vec{v}(\vec{r}, t) \)  

Disc 10: \( \vec{j} \) for \( \phi \)-part of \( Y_{lm}(\theta, \phi) \):  

\[ \vec{j} = \hat{\phi} \text{Re} \left[ (e^{i\phi})^* \frac{\partial}{\partial \phi} e^{i\phi} \right] = \text{Re} \left[ -i\hbar (e^{-i\phi}) \frac{\partial}{\partial \phi} e^{i\phi} \right] \]

\[ = \hat{\phi} \text{Re} \left[ \frac{m \hbar}{\text{mass}} e^{-i\phi} - e^{i\phi} \right] = \frac{m \hbar}{\text{mass}} \hat{\phi} \sim (m \hbar) = L_z \text{ e-value} \]

\[ Y_{lm} = Y_{l, m+2} \]

\( L \): cone of probability, \( \mp \) precessing/circulating around \( \pm \frac{\hbar}{2} \text{ axis when } m < 0 \) or \( \pm \frac{\hbar}{2} \text{ axis when } m > 0 \)

If we could have \( L_z = L \) then we would know \( L_x = L_y = 0 \) which would violate the uncertainty principle associated with \( [L_x, L_z] = -i\hbar \neq 0 \) \[ [L_y, L_z] = i\hbar \neq 0 \]
\[
Y_0^0 = \left( \frac{1}{4\pi} \right)^{1/2} \\
Y_1^0 = \left( \frac{3}{4\pi} \right)^{1/2} \cos \theta \\
Y_1^{\pm 1} = \mp \left( \frac{3}{8\pi} \right)^{1/2} \sin \theta e^{\pm i\phi} \\
Y_2^0 = \left( \frac{5}{16\pi} \right)^{1/2} (3 \cos^2 \theta - 1) \\
Y_2^{\pm 1} = \mp \left( \frac{15}{8\pi} \right)^{1/2} \sin \theta \cos \theta e^{\pm i\phi} \\
Y_2^{\pm 2} = \left( \frac{15}{32\pi} \right)^{1/2} \sin^2 \theta e^{\pm 2i\phi} \\
Y_3^0 = \left( \frac{7}{16\pi} \right)^{1/2} (5 \cos^3 \theta - 3 \cos \theta) \\
Y_3^{\pm 1} = \mp \left( \frac{21}{64\pi} \right)^{1/2} \sin \theta (5 \cos^2 \theta - 1) e^{\pm i\phi} \\
Y_3^{\pm 2} = \left( \frac{105}{32\pi} \right)^{1/2} \sin^2 \theta \cos \theta e^{\pm 2i\phi} \\
Y_3^{\pm 3} = \mp \left( \frac{35}{64\pi} \right)^{1/2} \sin^3 \theta e^{\pm 3i\phi}
\]