The 1D Simple Harmonic Oscillator ™

Hamiltonian is $H(x)=\frac{p^2}{2m}+\frac{m\omega^2 x^2}{2}$ or $\frac{p^2}{2m}+\frac{k x^2}{2}$ with two given parameters: m and (ω or $k \equiv m\omega^2$).

The system has two intrinsic scales:

- an energy scale $\hbar\omega$
- a distance scale $\boxed{x_0 \equiv \sqrt{\frac{\hbar}{m\omega}}}$ → define dimensionless-position $\boxed{\xi \equiv \frac{x}{x_0} = x\sqrt{\frac{m\omega}{\hbar}}}$

Hamiltonian becomes : $H(\xi)=\frac{\hbar\omega}{2}\left(\xi^2-\frac{d^2}{d\xi^2}\right)$

Energy eigenvalues : $E_n=\left(n+\tfrac{1}{2}\right)\hbar\omega$

Energy eigenstates : $\psi_n(x)=\left(\frac{1}{\pi x_0^2}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\frac{x}{x_0}\right)e^{-\frac{x^2}{2x_0^2}}$ where $H_n(\xi)$ = Hermite Polynomials

Hermite Polynomials

$H_0(\xi)=1$

$H_1(\xi)=2\xi$

$H_2(\xi)=4\xi^2-2$

$H_3(\xi)=8\xi^3-12\xi$

Recursion Formula

$H_n(\xi)=a_i\xi^i+a_{i+2}\xi^{i+2}+a_{i+4}\xi^{i+4}+\cdots$

where $i=0$ or 1 and

$a_{j+2}=-a_j\,\frac{2(n-j)}{(j+1)(j+2)}$

Rodrigues Formula

$H_n(\xi)=(-1)^n e^{\xi^2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}$

Ladder operators for the energy eigenstates : $\hat{a}_\pm = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0}\mp i\frac{x_0\,\hat{p}}{\hbar}\right) = \frac{m\omega\hat{x}\mp i\hat{p}}{\sqrt{2\hbar m\omega}} = \frac{1}{\sqrt{2}}\left(\xi\mp\frac{d}{d\xi}\right)$ → $\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$

$[\hat{a}_-,\hat{a}_+]=1$ $\qquad \hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- +\frac{1}{2}\right)$ $\qquad |n\rangle = \frac{1}{\sqrt{n!}}(a_+)^n|0\rangle$

Useful substitutions : $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ +\hat{a}_-) = \frac{x_0}{\sqrt{2}}(\hat{a}_+ +\hat{a}_-)$

$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}_+ -\hat{a}_-) = \frac{i\hbar}{\sqrt{2}\,x_0}(\hat{a}_+ -\hat{a}_-)$