Useful Formulas

Quantum Mechanics Fundamentals

- Canonical Commutation Relations: $[\hat{x}, \hat{p}] = i\hbar$
- Position-basis representations: $\hat{x} f(x) = x f(x), \quad \hat{p} f(x) = -i\hbar \frac{df}{dx}$
- Schrödinger Equation: $i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$
- Euler's formula: $e^{ix} = \cos x + i \sin x$
- Fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$

Perturbation Theory

Time-independent perturbation theory

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \lambda \hat{H}' \\ |\psi_n\rangle &= \left|\psi_n^{(0)}\right\rangle + \lambda \left|\psi_n^{(1)}\right\rangle + \dots, \\ E_n &= E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots \end{aligned}$$

Non-degenerate case

$$E_n^{(1)} = \left\langle \psi_n^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle, \quad \left|\psi_n^{(1)}\right\rangle = -\sum_{m \neq n} \left|\psi_m^{(0)}\right\rangle \frac{\left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle}{E_m^{(0)} - E_n^{(0)}}, \quad E_n^{(2)} = -\sum_{m \neq n} \frac{\left|\left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle\right|^2}{E_m^{(0)} - E_n^{(0)}}$$

Degenerate case

Let $\left|\psi_{ni}^{(0)}\right\rangle, i = 1, 2, \dots d$ be the eigenstates of $\hat{H}_0$ that span the degenerate subspace. Find $\left|\psi_{nj}^{(0),\text{good}}\right\rangle, j = 1, 2, \dots d$, the eigenvectors of $\hat{H}'$ restricted to the degenerate subspace. Then $E_{nj}^{(1)} = \left\langle \psi_{nj}^{(0),\text{good}} \middle| \hat{H}' \middle| \psi_{ni}^{(0),\text{good}} \right\rangle$.

Heisenberg and Schrödinger Pictures

- For time-independent Hamiltonians, $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$
- Schrödinger picture: $|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$
- Heisenberg picture: $\hat{O}_H(t) = \hat{U}^\dagger(t) \hat{O}_H(0) \hat{U}(t)$
- Heisenberg equations of motion: $\frac{\partial \hat{O}_H}{\partial t} = \frac{i}{\hbar} \left[\hat{H}, \hat{O}_H(t)\right]$

Simple Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}),$$
$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} + i\hat{p}\right),$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} - i\hat{p}\right),$$
$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle,$$
$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle,$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Variational Principle

Let E_0 be the ground state energy of H. Then

$$R_H(|\psi\rangle) = \frac{\langle\psi| H |\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

for ANY $|\psi\rangle$.

Time-dependent Hamiltonians

$H = H_0 + H'(t)$ with $E_n, |\psi_n\rangle$ energies and eigenstates of H_0:

- $$|\psi(t)\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|\psi_n\rangle$$
$$i\hbar\dot{c}_n = \sum_m H'_{nm}(t)e^{i\omega_{nm}t}c_m$$
$$H'_{nm}(t) = \langle\psi_n| H'(t) |\psi_m\rangle,$$
$$\hbar\omega_{nm} = E_n - E_m$$

- If $|\psi(t=0)\rangle = |\psi_n\rangle$, then $P_{n\to m}(t) = |\langle\psi_m|\psi(t)\rangle|^2 = |c_m(t)|^2$
- Time-dependent perturbation theory: If $c_a(t=0) = 1, c_{m\neq a}(t=0) = 0$, then

$$c_a(t) \approx 1 - \frac{i}{\hbar}\int_0^t dt' H_{aa}(t')$$
$$c_{m\neq a} \approx -\frac{i}{\hbar}\int_0^t dt' H_{ma}(t')e^{i\omega_{ma}t}$$

- Fermi's Golden Rule: $R_{a\to b} = \frac{2\pi}{\hbar}|V_{ab}|^2\delta(\omega - \omega_{ba})$ where $H'_{ab}(t) = V_{ab}e^{i\omega t}$

- Adiabatic approximation: instantaneous eigenstates $H(t)\left|\psi_n(t)\right\rangle = E_n(t)\left|\psi_n(t)\right\rangle$. If $\left|\Psi(t=0)\right\rangle = \left|\psi_n(0)\right\rangle$ and if $H(t)$ varies slowly, then

$$\begin{aligned}\left|\Psi(t)\right\rangle &= e^{i\theta_n}e^{i\gamma_n}\left|\psi_n(t)\right\rangle \\ \theta_n &= -\frac{1}{\hbar}\int_0^t dt' E_n(t') \\ \gamma_n &= i\int_0^t dt' \left\langle \psi_n(t') \middle| \dot{\psi}_n(t') \right\rangle\end{aligned}$$

Angular Momentum

- Commutation relations: $[J_i, J_j] = i\hbar\sum_k \epsilon_{ijk}J_k$, where i, j, k are all one of x, y, z
- Eigenstates: $\left|jm\right\rangle$, $J^2\left|jm\right\rangle = \hbar^2 j(j+1)\left|jm\right\rangle, J_z\left|jm\right\rangle = \hbar m\left|jm\right\rangle$. j is integer or half-integer and $m \in \{-j, -j+1, \ldots, j-1, j\}$.
- Adding two angular momenta $\vec{J} = \vec{J}_1 + \vec{J}_2$:

$$\begin{aligned}J^2 &= J_1^2 + J_2^2 + 2\vec{J}_1\cdot\vec{J}_2 \\ \left[J_1^2, \vec{J}_1\right] &= \left[J_1^2, \vec{J}_2\right] = \left[J_2^2, \vec{J}_1\right] = \left[J_2^2, \vec{J}_2\right] = 0\end{aligned}$$

Complete set of commuting observables for total angular momentum: J_1^2, J_2^2, J^2, J_z. Clebsch-Gordan coefficients give

$$\begin{aligned}\left|j_1 j_2 j m_j\right\rangle &= \sum_{m_1, m_2} C^{jm_j}_{j_1 j_2 m_1 m_2}\left|j_1 j_2 m_1 m_2\right\rangle \\ C^{jm_j}_{j_1 j_2 m_1 m_2} &= \left\langle j_1 j_2 j m_j \middle| j_1 j_2 m_1 m_2\right\rangle\end{aligned}$$

Gaussian Integrals

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-bx^2} &= \sqrt{\frac{\pi}{b}} \\ \int_{-\infty}^{\infty} x^{2n}e^{-bx^2} &= (-1)^n\sqrt{\pi}\frac{d^n}{db^n}b^{-1/2} \\ \int_{-\infty}^{\infty} x^{2n+1}e^{-bx^2} &= 0\end{aligned}$$

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2	1
		+1
+1/2	+1/2	1

m_1	m_2	1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1
		−1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2
		+3/2
+1	+1/2	1

m_1	m_2	3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2
		−3/2
−1	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

m_1	m_2	5/2
		+5/2
+2	+1/2	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2
		−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2
		+2
+3/2	+1/2	1

m_1	m_2	2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2
		−2
−3/2	−1/2	1

2×1

m_1	m_2	3
		+3
+2	+1	1

m_1	m_2	3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3
		−3
−2	−1	1

3/2×1

m_1	m_2	5/2
		+5/2
+3/2	+1	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2
		−5/2
−3/2	−1	1

1×1

m_1	m_2	2
		+2
+1	+1	1

m_1	m_2	2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2
		−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

3/2×3/2

m_1	m_2	3
		+3
+3/2	+3/2	1

m_1	m_2	3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3
		−3
−3/2	−3/2	1

2×3/2

m_1	m_2	7/2
		+7/2
+2	+3/2	1

m_1	m_2	7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2
		−7/2
−2	−3/2	1

2×2

m_1	m_2	4
		+4
+2	+2	1

m_1	m_2	4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4
		−4
−2	−2	1