# Physics 487 Final Exam Fall 2020
## Thursday December 17, 1:30 pm – 4:30 pm

This is a closed book exam. No use of calculators or any other electronic devices is allowed. Work the problems only in your answer booklets only. The exam questions will *not* be collected at the end, so **anything you write on these question pages will NOT be graded**

You have **3 hours** to work the problems.

1) **Show your work** and/or reasoning. **Answers with no work or explanation get no points.** But ...
2) **Don't write long essays** explaining your reasoning. We only need to see enough work to confirm that you **understand** what you're doing and are **not just guessing**. (If you *are* guessing, explain that, then *verify* your guess explicitly.) A good **annotated sketch** is often the best explanation of all!
3) **All question parts on this exam are independent**: you can get full points on any part even if your answers to all the other parts are incorrect. You should **attempt all the question parts!** If you get stuck, move on to the next one and come back later. The worst thing you can do is stall on one question and not get to others whose solution may be very simple.
4) Partial credit will be given for incorrect answers if the work is understandable and some of it is correct. IMPORTANT: If you think you've made a mistake but can't find it, **explain what you think is wrong** ➔ you may well get partial credit for noticing your error!
5) It is fine to leave answers as **radicals or irreducible fractions** (e.g. $10\sqrt{3}$ or 5/7), but you will lose points for not simplifying answers to an **irreducible form** (e.g. $24(x^2 - y^2)/(\sqrt{9}x - \sqrt{9}y)$ is unacceptable.)

**Academic Integrity:**

*The giving of assistance to or receiving of assistance from another person, or the use of unauthorized materials during University Examinations can be grounds for disciplinary action, up to and including expulsion from the University.*

*Please be aware that prior to or during an examination, the instructional staff may wish to rearrange the student seating. Such action does not mean that anyone is suspected of inappropriate behavior.*

**Problem 1**

At time $t = 0$, a small sinusoidal perturbation $H'(\vec{r},t) = V(r)\, e^{i\omega t}$ is applied to a system with unperturbed Hamiltonian $H_0(\vec{r})$. It is found that the probability for the system to transition from initial state $|i\rangle$ to final state $|f\rangle$ is largest when $\omega_{fi} \equiv \omega_f - \omega_i \equiv (E_f - E_i)/\hbar$ matches the "driving" frequency $\omega$ of the applied perturbation. Estimate the FWHM (full-width at half maximum) width $\Delta\omega$ of the resonance at $\omega_{fi} = \omega$ as a function of time.

**Problem 2**

Two identical particles are placed in a 1D container. One of them is in state $\psi_a(x)$ and the other in state $\psi_b(x)$, where $\psi_a$ and $\psi_b$ are orthonormal to each other. Use $x_1$ as the coordinate of particle 1 and $x_2$ as the coordinate of particle 2.

(a) Calculate the expectation value of the squared-separation, $\left\langle (x_1 - x_2)^2 \right\rangle$, between the two particles in the following two cases :

- if the identical particles are fermions, and
- if the identical particles are bosons.

Express your answers in terms of the single-particle (single-wavefunction) expectation values $\langle x\rangle_a, \langle x\rangle_b, \langle x^2\rangle_a$, and/or $\langle x^2\rangle_b$ *and one other term.*

(By single-particle expectation values, we mean $\langle Q\rangle_a \equiv \langle \psi_a(x)|Q(x)|\psi_a(x)\rangle$, i.e. the expectation value if there was only one particle and it was in the state given by the subscript.)

(b) No matter what answer you got for (a), do you expect $\left\langle (x_1 - x_2)^2 \right\rangle$ to be larger for identical fermions or identical bosons, or should they be the same? Give some brief physical reasoning for your answer.

**Problem 3**

A spin-½ particle with spin $\vec{S}$ and magnetic dipole moment $\mu$ interacts with a constant magnetic field $\vec{B} = B_0\,\hat{z}$ through the interaction $H = -\vec{\mu}\cdot\vec{B} = -\mu\vec{\sigma}\cdot\vec{B}$ where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices. At time $t = 0$, the particle's spin state is represented by the spinor

$$\chi\big|_{t=0} = \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{pmatrix}$$

in the basis of $S_z$ eigenstates, where $\alpha$ is a known dimensionless quantity.
Calculate the expectation value of $S_x$ as a function of time.

**Problem 4**

The hydrogenic wavefunctions on your formula sheet were derived using a potential energy that treats the nucleus as a *point* of zero radius :

$$V_{\text{point}}(r) = -\frac{e^2}{4\pi\varepsilon_0 r} .$$

To estimate the effect of the proton's *actual* radius of $b \approx 10^{-15}$ m on the energy levels of hydrogen, let's instead treat the proton as a hollow spherical shell of radius $b$. This produces the following potential energy:

$$V_{\text{out}}(r) = -\frac{e^2}{4\pi\varepsilon_0 r} \quad \text{for } r > b \qquad \text{and} \qquad V_{\text{in}}(r) = \text{constant} = -\frac{e^2}{4\pi\varepsilon_0 b} \quad \text{for } r \le b .$$

**Using perturbation theory**, calculate the first-order correction to the ground state energy of an electron in hydrogen caused by this change in $V(r)$. The proton's radius $b$ is $10^5$ times smaller than the Bohr radius $a_0$ of the electron, so approximate your result to lowest non-vanishing order in the tiny quantity $b/a_0 << 1$.

▶ HINT 1: Your final answer should like like this:

$$\frac{\Delta E_{\text{gs}}}{E_{\text{gs}}} = A\left(\frac{b}{a_0}\right)^n \quad \text{where } A \text{ and } n \text{ are for you to determine.}$$

▶ HINT 2: Leave the $1/4\pi\varepsilon_0$ factor in the potential alone, i.e. don't try to *translate* it into something involving the Bohr radius; $a_0$ will show up naturally from the hydrogen wave functions.

▶ HINT 3: These integrals may help you :

$$\int x\, e^{-ax}\, dx = -\frac{e^{-ax}}{a^2}(ax+1) \qquad \int x^2\, e^{-ax}\, dx = -\frac{e^{-ax}}{a^3}\left(a^2x^2 + 2ax + 2\right) \qquad \int_0^\infty x^n\, e^{-x}\, dx = \Gamma(n+1) = n!$$

**Problem 5**

A particle of mass $m$ bounces elastically between two infinite walls, one located at $x = 0$ and the other at $x = L$. The particle is in its ground state, with energy $E$, and its potential energy is $V = 0$ inside the well. The familiar energy eigenstates for the 1D InfiniteWell™ are

$$\psi_n(x) = \sqrt{\frac{2}{L}}\, \sin\left(\frac{n\pi x}{L}\right).$$

(a) The separation between the walls is slowly increased to $2L$. Determine the change in the expectation value of the energy in the adiabatic approximation.

(b) New situation! Assume instead that the separation between the walls is increased rapidly, with the right-hand wall moving from $x = L$ to $x = 2L$ at a speed much greater than $\sqrt{E/m}$. Classically there is no change in the particle's energy since the wall is moving faster than the particle and cannot be struck by the particle while the wall is moving. Determine the quantum-mechanical change in the expectation value of the energy from before the well changed to right after its width doubled.

(c) Compute the probability that the particle remains in its lowest-possible energy state when the wall is moved rapidly as in part (b).

**Problem 6**

(a) Consider a hydrogen atom. If an electron is found in the 2p state of hydrogen, it will decay to the 1s groud state via spontaneous E1 radiation. Estimate the lifetime $\tau$ of the 2p $\to$ 1s transition in seconds. To do so, use the Bohr model of the atom (i.e. the no-fine-structure formulae from 486) to get the energies of the 2p and 1s states, and for the electric dipole transition matrix element, approximate $\left|\langle 2p|\vec{r}|1s\rangle\right| \approx a_0$ (good enough for an order-of-magnitude estimate!). Useful formulae for the Bohr radius $a_0$ and other things may be found in the "Constants" section of the 486 formula sheet. You will also need the electron mass : $m_e c^2 = 0.5$ MeV.

You only need to calculate the ORDER OF MAGNITUDE of the result!

(b) Consider a helium atom. Here are the electron configurations and term symbols for some of the He states found in the NIST database :

| | | |
|---|---|---|
| state #1: | $2p^2$ | ${}^3P_1$ |
| state #2a: | 1s 2p | ${}^3P_2$ |
| state #2b: | 1s 2p | ${}^3P_1$ |
| state #2c: | 1s 2p | ${}^1P_1$ |
| state #3: | 1s 2s | ${}^1S_0$ |
| state #4: | $1s^2$ | ${}^1S_0$ |

Which transitions between these states are "allowed" (i.e. can occur via E1 radiation) and which are "forbidden"? Treat states #2a,b,c as having the same energy so that you can ignore transitions between them; that leaves you with only 12 possible transitions to consider. Remember to give some reason for your answers (no work, no points).

(c) Consider helium states #2b and #2c given above. These states do not have quite the same energy. Which one has the lower energy? Explain your answer briefly but clearly.

$$|\vec{v}| \equiv \sqrt{\vec{v}\cdot\vec{v}} \qquad \vec{v} = \sum_{i=1}^{3} (\vec{v}\cdot\hat{r}_i)\,\hat{r}_i \qquad d\vec{l}_{path} = \frac{d\vec{l}}{du}\,du \qquad d\vec{A} = \left(\frac{\partial\vec{l}}{\partial u}\times\frac{\partial\vec{l}}{\partial v}\right) du\,dv$$

$$df(x_1,\ldots,x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,dx_i \qquad dV = \left(\frac{\partial\vec{l}}{\partial u}\times\frac{\partial\vec{l}}{\partial v}\right)\cdot\frac{\partial\vec{l}}{\partial w}\,du\,dv\,dw$$

Conceptual version:

$$d\vec{l}_u \equiv \frac{\partial\vec{l}}{\partial u}\,du$$

$$d\vec{l}_{path} = d\vec{l}_u \qquad d\vec{A} = d\vec{l}_u \times d\vec{l}_v \qquad dV = (d\vec{l}_u \times d\vec{l}_v)\cdot d\vec{l}_w$$

## Taylor

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

1st order approx for $x \ll 1$:

- $(1+x)^n \approx 1+nx$
- $\sin x \approx x$
- $\sin^{-1} x \approx x$
- $\cos x \approx 1 - \frac{x^2}{2}$
- $\cos^{-1} x \approx \frac{\pi}{2} - x$
- $\tan x \approx x$
- $\tan^{-1} x \approx x$
- $e^x \approx 1+x$
- $\ln(1+x) \approx x$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} \qquad \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}$$

| | **0°** | **30°** | **45°** | **60°** | **90°** |
|---|---|---|---|---|---|
| **sin** | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| **cos** | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| **tan** | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ |

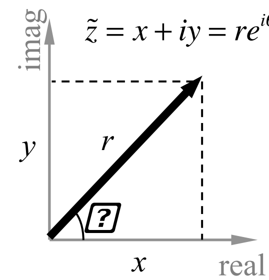
$$\sin a\,\sin b = \tfrac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$
$$\cos a\,\cos b = \tfrac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$
$$\sin a\,\cos b = \tfrac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

## Complex Numbers

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\tilde{z} = x + iy = re^{i\theta}$$




$$\tilde{z}^* \equiv x - iy = re^{-i\theta}$$

$$|\,\tilde{z}\,| \equiv \sqrt{\tilde{z}^*\tilde{z}} = r$$

## Integral Table

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right) \qquad \int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left(x+\sqrt{x^2\pm a^2}\right) \qquad \int \frac{x\,dx}{\sqrt{a^2\pm x^2}} = \pm\sqrt{a^2\pm x^2}$$

$$\int_0^{2\pi} \sin^2\phi\,d\phi = \int_0^{2\pi} \cos^2\phi\,d\phi = \pi \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \qquad \int \frac{x\,dx}{(a\pm x)^2} = \frac{a}{a\pm x} + \ln(a\pm x)$$

$$\int \sin^2\phi\,d\phi = \frac{\phi}{2} - \frac{\sin(2\phi)}{4} \qquad \int \frac{dx}{(a\pm x)^2} = \mp\frac{1}{a\pm x} \qquad \int \frac{x\,dx}{a^2\pm x^2} = \pm\frac{1}{2}\ln\left(a^2\pm x^2\right)$$

$$\int \cos^2\phi\,d\phi = \frac{\phi}{2} + \frac{\sin(2\phi)}{4} \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \int \frac{x\,dx}{\left(a^2\pm x^2\right)^{3/2}} = \mp\frac{1}{\sqrt{a^2\pm x^2}}$$

$$\int \sin^3\theta\,d\theta = \frac{\cos^3\theta}{3} - \cos\theta \qquad \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) \qquad \int \ln(ax)\,dx = x\ln(ax) - x$$

$$\int \cos^3\theta\,d\theta = \sin\theta - \frac{\sin^3\theta}{3} \qquad \int \frac{dx}{(a^2\pm x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2\pm x^2}} \qquad \int \frac{\ln(ax)}{x}\,dx = \frac{1}{2}\left[\ln(ax)\right]^2$$

$$\int \cos^n\theta\,\sin\theta\,d\theta = -\frac{\cos^{n+1}\theta}{n+1}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) \qquad \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$\int \sqrt{x^2\pm a^2}\,dx = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right| \qquad \int \frac{(x-a\cos\theta)\sin\theta\,d\theta}{(x^2+a^2-2ax\cos\theta)^{3/2}} = \frac{1}{x^2}\frac{a-x\cos\theta}{\sqrt{x^2+a^2-2ax\cos\theta}}$$