

Coupled Linear Oscillators $\mathbf{M}\ddot{\vec{q}} = -\mathbf{K}\vec{q}$ $T = \frac{1}{2}\mathbf{M}_{ij}\dot{q}_i\dot{q}_j$ $U = \frac{1}{2}\mathbf{K}_{ij}q_iq_j$

$\mathbf{M}_{ij} = \left. \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \right|_{\dot{\vec{q}}=0}$ $\mathbf{K}_{ij} = \left. \frac{\partial^2 U}{\partial q_i \partial q_j} \right|_{\vec{q}=0}$ **Transverse oscillations** of taut string: $k_T = \left. \frac{\text{tension}}{\text{length}} \right|_{\text{equilib}}$

Inner Product Space description and **Normal Coordinates** \vec{q} = column vector of generalized coord

- **Space** : $|\vec{q}(t)\rangle \equiv$ solutions of a linear oscillator system
- **Inner Product** : $\langle \vec{q}_1 | \vec{q}_2 \rangle \equiv \vec{q}_1^T \mathbf{M} \vec{q}_2$ and associated **magnitude** : $|\vec{q}|^2 \equiv \langle \vec{q} | \vec{q} \rangle$
- **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\vec{a}_m = \omega_m^2 \mathbf{M}\vec{a}_m$ and normalization $\hat{a}_m \equiv \vec{a}_m / |\vec{a}_m|$
- **Basis is Orthonormal** : $\langle \hat{a}_n | \hat{a}_m \rangle = \delta_{nm}$
- **Completeness** for $\vec{q}(t)$ and **Normal Coordinates** ξ_m :

ξ_m is the **component** of \vec{q} along mode m : $|\vec{q}(t)\rangle = \sum_{\text{modes } m} |\hat{a}_m\rangle \langle \hat{a}_m | \vec{q}(t)\rangle \equiv \sum_{\text{modes } m} \hat{a}_m \xi_m(t)$

ξ_m is **projected out** of \vec{q} by : $\xi_m(t) = \langle \hat{a}_m | \vec{q}(t)\rangle = A_m \cos(\omega_m t - \delta_m)$

- **Transformation** between q -space and ξ -space :

vectors : $\vec{\xi} = \mathbf{R}\vec{q}$ $\vec{q} = \mathbf{R}^{-1}\vec{\xi}$ $\mathbf{R}^{-1} = \begin{pmatrix} | & | & | \\ \hat{a}_1 & \hat{a}_2 & \dots \\ | & | & | \end{pmatrix}$ $\mathbf{R} = (\mathbf{R}^{-1})^T \mathbf{M}$

tensors : $\mathbf{M}^\xi = (\mathbf{R}^{-1})^T \mathbf{M} \mathbf{R}^{-1} \rightarrow \mathbf{M}_{mn}^\xi = \delta_{mn}$ & $\mathbf{K}_{mn}^\xi = \omega_m^2 \delta_{mn}$

inhomogeneous EOM : $\mathbf{M}\ddot{\vec{q}} + \mathbf{K}\vec{q} = \vec{F}$ in q -space \rightarrow $\mathbf{M}^\xi \ddot{\vec{\xi}} + \mathbf{K}^\xi \vec{\xi} = (\mathbf{R}^{-1})^T \vec{F}$ in ξ -space

Fourier Series as an Inner Product Space

- **Space** : $|f\rangle \equiv$ **τ -periodic functions** $f(t)$ that are periodic over $t = [-\tau/2 \rightarrow \tau/2]$, with $\omega = 2\pi/\tau$

• **Basis #1** : $|n\rangle \equiv \begin{cases} \sin(n\omega t) & n = 1, \dots, \infty \\ 1/\sqrt{2} & n = 0 \\ \cos(n\omega t) & n = -1, \dots, -\infty \end{cases}$

• **Inner Product #1** : $\langle g | f \rangle \equiv \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} g(t) f(t) dt$

• **Basis #2** : $|n\rangle = e^{in\omega t}$

• **Inner Product #2** : $\langle \tilde{g} | \tilde{f} \rangle \equiv \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} g^*(t) f(t) dt$

\rightarrow **Basis is Orthonormal** : $\langle n | m \rangle = \delta_{nm}$

\rightarrow **Completeness** : any $|f\rangle = \sum_{n=-\infty}^{+\infty} |n\rangle \langle n | f \rangle$

Miscellanea $d\vec{l} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}$ $d\vec{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi}$

$\vec{a} = \hat{s}[\ddot{s} - s\dot{\phi}^2] + \hat{\phi}[s\ddot{\phi} + 2\dot{s}\dot{\phi}] + \hat{z}[\ddot{z}]$

$\vec{a} = \hat{r}[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta] + \hat{\theta}[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta \cos\theta] + \hat{\phi}[\sin\theta(r\ddot{\phi} + 2\dot{r}\dot{\phi}) + \cos\theta(2r\dot{\theta}\dot{\phi})]$

$L = T - U$ $\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right)$ $H = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$ canonical / generalized $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ $Q_i \equiv \frac{\partial L}{\partial q_i}$

Accel Frames : $\vec{f} \equiv \vec{F}/m \rightarrow \vec{f}_{\text{lin}}^* = -\vec{A}_0$ $\vec{f}_{\text{cf}}^* = (\vec{\Omega} \times \vec{r}^*) \times \vec{\Omega} = \Omega^2 s^* \hat{s}^*$ $\vec{f}_{\text{Cor}}^* = 2\vec{v}^* \times \vec{\Omega}$ $\vec{f}_{\text{azim}}^* = \vec{r}^* \times \dot{\vec{\Omega}}$



2-Body Central Force Problems & Scattering



• **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r}$, $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $\mu = \frac{m_1 m_2}{M}$

• **Centrifugal force & PE** : $\vec{F}_{cf} = \frac{L^2}{\mu r^3} \hat{r}$, $U_{cf} = \frac{L^2}{2\mu r^2}$, effective radial $U^* = U + U_{cf}$

• **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu \ddot{r} = F(r) + F_{cf}(r)$, $E = T + U = \frac{1}{2} \mu \dot{r}^2 + U_{cf} + U$

• **Path Equation** : $u(\phi) \equiv 1/r(\phi) \rightarrow u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu \dot{r}}{L}$

• **Conics** : With (r, ϕ) centered on a focal point and E \equiv Ellipse, H \equiv Hyperbola

$$\frac{1}{r} = \frac{a}{b^2} (\pm 1 + e \cos \phi) \text{ with } \begin{cases} +: \text{E or H-near-branch} \\ -: \text{H-far-branch} \end{cases}, \quad e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} \text{ with } \begin{cases} -: \text{E} \\ +: \text{H} \end{cases}$$

• **Kepler Orbits** $F = -\frac{\gamma}{r^2}$: $r(\phi) = \frac{r_0}{\text{sgn}[\gamma] + e \cos \phi}$ with $r_0 = \frac{L^2}{\mu |\gamma|} = \frac{b^2}{a} = a |1 - e^2|$, $E = \mp \frac{|\gamma|}{2a} = \frac{|\gamma|(e^2 - 1)}{2r_0}$

Bounded orbits: $\tau^2 = \frac{4\pi^2 \mu}{\gamma} a^3$, $r_0 = \frac{2r_{\min} r_{\max}}{r_{\min} + r_{\max}}$, $e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$

Unbounded orbits: scattering angle $\theta = \pi - 2\alpha$ with $\tan \alpha = \frac{b}{a}$, impact parameter $b =$ semi-minor axis $b \odot$

• **XSec** : $d\Omega \equiv \frac{dA}{r^2} = \left\{ \frac{\sin \theta d\theta d\phi}{d\theta_x d\theta_y}, \frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| \right\}$ with $\theta =$ scattering angle • **Lumi**: $\mathcal{L} = n_A N_e$, $\frac{dN_{ev}}{dt} = \mathcal{L} \sigma$

• **Earth Specs**: $R_\oplus = 6.4 \times 10^6$ m, $GM_\oplus = gR_\oplus^2$, $g = 10$ m/s²

• **Earth Orbit**: $r_0 = a = 1$ A.U. = 1.5×10^{11} m, $v_\oplus = 3 \times 10^4$ m/s

• **Atomic Data**: 1 amu \approx mass of 1 nucleon (proton or neutron) = 1.66×10^{-27} kg = $(5/3) \times 10^{-24}$ g
Gas at STP has $N_{Avog} = 6.02 \times 10^{23}$ molecules in 22.4 liters; 1 barn = 10^{-28} m² = 10^{-24} cm²

Inertia Tensor

$$I_{ij} = \int dm (\delta_{ij} r^2 - r_i r_j) \quad \mathbf{I} = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$$

Principal Axes : $\mathbf{I} \hat{e} = \lambda \hat{e}$ $\mathbf{I} = \mathbf{I}_{CM} + \mathbf{I}'$

$\vec{L}^{(B)} = \mathbf{I}^{(B)} \vec{\omega}$ \forall body-fixed point B $T = \frac{1}{2} \vec{\omega}^T \mathbf{I} \vec{\omega}$

$\mathbf{I}^{(B)}$ simplifies for lamina, reflection symmetry, n-fold axisymmetry $\vec{\omega}_{1,3} = \vec{\omega}_{1,2} + \vec{\omega}_{2,3}$

Euler Equations

$$\vec{\tau} = \dot{\vec{L}} = \vec{\omega} \times \vec{L} + \dot{\vec{L}} \Big|_{\text{within body}}$$

$$\tau_1 = I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3$$

Euler Angles

$$\vec{\omega} = \dot{\phi} \hat{z} + \dot{\theta} \hat{e}'_2 + \dot{\psi} \hat{e}_3 \quad ; \quad \hat{z} = \cos \theta \hat{e}_3 - \sin \theta \hat{e}'_1$$

$$\vec{\omega} = (-\dot{\phi} \sin \theta \hat{e}'_1 + \dot{\theta} \hat{e}'_2) + (\dot{\psi} + \dot{\phi} \cos \theta) \hat{e}_3$$

Rotations: $\vec{v}^* = \mathbf{R} \vec{v}$, $\mathbf{I}^* = \mathbf{R} \mathbf{I} \mathbf{R}^{-1}$, passive rotⁿ :

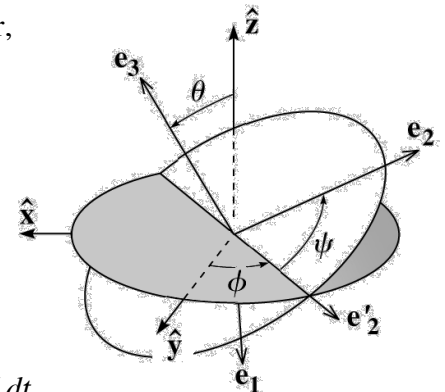
from S frame $\{\hat{x}, \hat{y}, \hat{z}\}$ to S* frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ $\mathbf{R}^{-1} = \begin{pmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{pmatrix}$

Free symmetric top : precession of $\vec{\omega}$ is

$$\vec{\Omega}^* = [(I_3/I_1) - 1] \omega_3 \hat{e}_3 \text{ body, } \vec{\Omega} = \vec{L} / I_1 \text{ lab}$$

$\vec{L}, \vec{\omega}, \hat{e}_3$ coplanar,

$$\vec{\omega} = -\vec{\Omega}^* + \vec{\Omega}$$



General Relativity $\left. \begin{matrix} d\tau^2 \\ -d\sigma^2 \end{matrix} \right\} = dt^2 (1 - 2M/r) - \frac{dr^2}{(1 - 2M/r)} - r^2 d\phi^2$

$$M \equiv GM_{kg} / c^2$$

$$t \equiv t_{\text{sec}} c$$

$$L_{SI} = -mc^2 d\tau / dt$$