Old Quantum Theory (1900–1925)

$E = hf = \hbar\omega$, $p = h/\lambda = \hbar k$

Quantization Rules : $E = nh$, $\oint_{\text{one period}} p_q \cdot dq = n_q h$

Correspondence Principle : CM is recovered in the limit of large quantum #s ($n \to \infty$)

Probability and some 3D Calculus

for a probability distribution $P(x)$: mean $\langle x \rangle = \int_{x_{\min}}^{x_{\max}} P(x)\, x\, dx$, variance $\sigma_x^2 \equiv \left\langle \left(x - \langle x \rangle\right)^2 \right\rangle = \langle x^2 \rangle - \langle x \rangle^2$, $\sigma_x \equiv$ standard deviation

3D operators in Cartesian coord's : $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ $\quad \vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ $\quad \nabla^2 \equiv \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

GGS Theorems : $\int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$ $\quad \int_{\text{Surf}} \left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{A} = \oint_{\partial\text{Surf}} \vec{E}\cdot d\vec{l}$ $\quad \int_{\text{Vol}} \left(\vec{\nabla}\cdot\vec{E}\right) dV = \oint_{\partial\text{Vol}} \vec{E}\cdot d\vec{A}$

Wave Mechanics

Physical observables Q correspond to linear **Hermitian operators** $\hat{Q}$, which are defined by these properties :

1. $\langle Q \rangle^* = \langle Q \rangle$ 2. $\left\langle \Psi \middle| \hat{Q}\Psi \right\rangle = \left\langle \hat{Q}\Psi \middle| \Psi \right\rangle$ 3. the eigenstates of $\hat{Q}$ are complete over Hilbert space

Schrödinger Equation : $\hat{H}\Psi = \hat{E}\Psi$

Operators in 3D $\vec{r}$-space : $\hat{E} = i\hbar\frac{\partial}{\partial t}$, $\hat{\vec{p}} = \frac{\hbar}{i}\vec{\nabla}$, $\hat{\vec{r}} = \vec{r}$, $\hat{H} = -\frac{\hbar^2\nabla^2}{2m} + V(\vec{r})$

Operators in 1D x-space : $\hat{E} = i\hbar\frac{\partial}{\partial t}$, $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$, $\hat{x} = x$, $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

Plane-wave eigenfunctions of $\hat{p},\hat{x}$ with Dirac normalization: $\psi_p(x) = e^{ipx/\hbar}/\sqrt{2\pi\hbar}$, $\psi_{x'}(x) = \delta(x - x')$

Boundary Conditions on wavefunctions:

a. Wavefunctions are always **continuous.**

b. Wavefunctions have **continuous derivatives**, **except** at points where $V = \pm\infty$

where $\lim_{\varepsilon\to 0} \psi'(x+\varepsilon) - \psi'(x-\varepsilon) = \left(2m/\hbar^2\right) \lim_{\varepsilon\to 0} \int_{x-\varepsilon}^{x+\varepsilon} V(x)\psi(x)\,dx$

c. Wavefunctions are **zero** in any region where $V = \infty$.

Probability density $\rho(\vec{r},t) = \left|\Psi(\vec{r},t)\right|^2 = \Psi^*\Psi$

Prob. current density $\vec{j}(\vec{r},t) = \text{Re}\left[\Psi^* \frac{\hat{\vec{p}}}{m}\Psi\right]$

Continuity Equation : $-\frac{\partial\rho}{\partial t} = \vec{\nabla}\cdot\vec{j}$ $\qquad R,T = \frac{\vec{j}_{\text{re,tr}}\cdot\vec{A}}{\vec{j}_{\text{in}}\cdot\vec{A}}$

Expectation Value $\langle Q \rangle$ of observable $Q(\vec{r},\vec{p})$: $\langle Q \rangle \equiv \left\langle \Psi \middle| \hat{Q}\Psi \right\rangle \equiv \int_{-\infty}^{+\infty} \Psi^*\, \hat{Q}\left(\vec{r}, -i\hbar\vec{\nabla}\right)\Psi\; d^3\vec{r}$

Ehrenfest's Theorem : Expectation values follow classical laws. $\frac{\langle p \rangle}{m} = \frac{d\langle x \rangle}{dt}$, $\frac{d\langle p \rangle}{dt} = \left\langle -\frac{dV}{dx} \right\rangle$

Virial Theorem : $2\langle T \rangle = \left\langle x\frac{dV}{dx} \right\rangle$

Wavefunction $\psi(q)$ in eigenbasis $\left|e_q\right\rangle = |q\rangle$ of any Hermitian operator $\hat{Q}$: $\psi(q) = \left\langle e_q \middle| \psi \right\rangle$

$\to$ Probability of obtaining eigenvalue q from measurement of $\hat{Q}$: $P(q) = |\psi(q)|^2 = \left|\left\langle e_q \middle| \psi \right\rangle\right|^2$

$\to$ wavefunction conversion between x- and p-space : $\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} e^{ipx/\hbar}\phi(p)\,dp \iff \phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} e^{-ipx/\hbar}\psi(x)\,dx$

$\to$ Operators in 1D p-space : $\hat{E} = i\hbar\frac{\partial}{\partial t}$, $\hat{p} = p$, $\hat{x} = i\hbar\frac{\partial}{\partial p}$, $\hat{H} = \frac{p^2}{2m} + V\left(i\hbar\frac{\partial}{\partial p}\right)$

Commutator : $\left[\hat{A},\hat{B}\right]\equiv\hat{A}\hat{B}-\hat{B}\hat{A}$ Theorem : Operators that commute share a common set of eigenfunctions.

Uncertainty Principle : $\sigma_A\sigma_B \ge \left|\frac{1}{2i}\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|$ e.g. $\sigma_x\sigma_p \ge \frac{\hbar}{2}$ Time-dep. of Expec. Value : $\frac{d\langle\hat{Q}\rangle}{dt}=\frac{i}{\hbar}\left\langle\left[\hat{H},\hat{Q}\right]\right\rangle+\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$

Axioms of Wave Mechanics

1. The state of a particle is represented by a vector $|\Psi(t)\rangle$ in a Hilbert space.

2. The independent variables x and p of classical mechanics are represented by linear Hermitian operators $\hat{x}$ and $\hat{p}$ with the following matrix elements in x-space (i.e., in the eigenbasis of $\hat{x}$): $\hat{x}(x)=x$ & $\hat{p}(x)=\frac{\hbar}{i}\frac{\partial}{\partial x}$. The operators corresponding to dependent variables $Q(x,p)$ are the linear Hermitian operators $\hat{Q}(\hat{x},\hat{p})$.

3. If the particle is in a state $|\psi\rangle$, measurement of the variable Q will yield one of its eigenvalues q with probability $P(q)=|\langle q|\psi\rangle|^2$ where $|q\rangle$ (often denoted $|e_q\rangle$) is the normalized eigenstate with eigenvalue q. The state of the system will change from $|\psi\rangle$ to $|q\rangle$ as a result of the measurement.

4. The state $|\Psi(t)\rangle$ obeys the Schrödinger Equ. $i\hbar\frac{d}{dt}|\Psi(t)\rangle=\hat{H}|\Psi(t)\rangle$ where $\hat{H}=H(\hat{x},\hat{p})$ is the Hamiltonian.

Miscellaneous Math

Gaussian probability distribution: $P(x;x_0,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-x_0)^2/2\sigma^2}$ Sums: $\sum_{j=0}^{\mu}1=\mu+1$, $\sum_{j=0}^{\mu}j=(\mu+1)\frac{\mu}{2}$

Dirac δ function : $\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{iqx}\,dq$ Defining Properties : 1. $\delta(x)=0$ when $x\neq0$; 2. $\delta(x)=\infty$ when $x=0$; 3. $\int_{-\infty}^{+\infty}\delta(x)\,dx=1$ OR $\int_{-\infty}^{+\infty}\delta(x)f(x)\,dx=f(0)$

Fourier Integrals: $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}A(k)e^{ikx}\,dk$ where $A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)e^{-ikx}\,dx$

Sinusoidal Integrals: $\int_0^{\pi}\begin{matrix}\sin^2(a\phi)\\ \cos^2(a\phi)\end{matrix}\,d\phi=\frac{\pi}{2}-\frac{\sin(2\pi a)}{4a}$ $\int_0^{\pi}\begin{matrix}\sin(n\phi)\sin(m\phi)\\ \cos(n\phi)\cos(m\phi)\end{matrix}\,d\phi=\delta_{nm}$ $\int_0^{\pi}\sin(n\phi)\cos(m\phi)\,d\phi=0$

Exponential Integrals: $\int_0^{\infty}x^n e^{-x}\,dx=\Gamma(n+1)=n!$ $\int x e^{-ax}\,dx=-\frac{e^{-ax}}{a^2}(ax+1)$ $\int x^2 e^{-ax}\,dx=-\frac{e^{-ax}}{a^3}(a^2x^2+2ax+2)$

Gaussian Integrals: $\int_{-\infty}^{+\infty}e^{-ax^2-bx}\,dx=\sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}}$ $\int_{-\infty}^{+\infty}x e^{-ax^2-bx}\,dx=-\frac{\sqrt{\pi}\,b}{2a^{3/2}}e^{\frac{b^2}{4a}}$ $\int_{-\infty}^{+\infty}x^2 e^{-ax^2-bx}\,dx=\frac{\sqrt{\pi}}{4a^{5/2}}(2a+b^2)e^{\frac{b^2}{4a}}$

Classical Mechanics security blanket ☺

$L(q_i,\dot{q}_i,t)=T-U$ Lagrange EOM: $\frac{\partial L}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)$

$H\equiv\dot{q}_i(\partial L/\partial\dot{q}_i)-L$ equals $T+U$ when $\vec{r}_a=\vec{r}_a(q_i)$

$dH/dt=-\partial L/\partial t$

Common Forces : $F_{\text{grav}}=\frac{Gm_1m_2}{r^2}$, $F_{\text{elec}}=\frac{q_1q_2}{4\pi\varepsilon_0 r^2}$, $F_{\text{cf}}=\frac{mv^2}{r}$

Generalized momentum $p_i\equiv\frac{\partial L}{\partial\dot{q}_i}$, force $Q_i\equiv\frac{\partial L}{\partial q_i}$

Hamilton's EOM: $-\frac{\partial H}{\partial q_i}=\frac{dp_i}{dt}$, $\frac{\partial H}{\partial p_i}=\frac{dq_i}{dt}$

Special Relativity: $E^2=(pc)^2+(mc^2)^2$

$\gamma=\frac{1}{\sqrt{1-(v/c)^2}}$, $E=\gamma mc^2$, $p=\gamma mv$, $v=\frac{pc^2}{E}$

Constants : $m_e c^2 = 0.511 \text{ MeV}$ $\quad \hbar c \begin{array}{l} = 197 \text{ MeV}\cdot\text{fm} \\ \approx 200 \text{ MeV}\cdot\text{fm}\end{array}$ $\quad \alpha \equiv \dfrac{e^2}{4\pi\varepsilon_0 \hbar c} = \dfrac{1}{137}$ $\quad a_0 \equiv \dfrac{4\pi\,\varepsilon_0\,\hbar^2}{m_e e^2} = \dfrac{(\hbar c)}{\alpha\,(m_e c^2)}$

Angular Momentum $\quad L^2 = -\hbar^2\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right)+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}\right]$

$$L_x = +i\hbar\left(\sin\phi\frac{\partial}{\partial\theta}+\frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right),\quad L_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta}-\frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right),\quad L_z = -i\hbar\frac{\partial}{\partial\phi}$$

Spin & Angular Momentum : L,l can be replaced by S,s $\quad [L^2, L_{x,y,z}] = 0$, $\quad \left[L_x, L_y\right] = i\hbar L_z$, etc

$$L^2|l\,m\rangle = \hbar^2 l(l+1)|l\,m\rangle,\quad L_z|l\,m\rangle = \hbar m|l\,m\rangle,\quad L_\pm = L_x \pm iL_y,\quad L_\pm|l\,m\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\,|l(m\pm1)\rangle$$

Pauli Spin Matrices $\left\{S_x, S_y, S_z\right\} = \dfrac{\hbar}{2}\left\{\sigma_x, \sigma_y, \sigma_z\right\}$ where $\sigma_x, \sigma_y, \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Spherical Harmonics $Y_l^m(\theta,\phi)$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \qquad Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta\, e^{\pm2i\phi}$$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \qquad Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2}(5\cos^3\theta - 3\cos\theta)$$

$$Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi} \qquad Y_3^{\pm1} = \mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \qquad Y_3^{\pm2} = \left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta\, e^{\pm2i\phi}$$

$$Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta\, e^{\pm i\phi} \qquad Y_3^{\pm3} = \mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta\, e^{\pm3i\phi}$$

H-like atom : radial wavefunctions $R_{nl}(r)$

$Ze \equiv$ nuclear charge (Z=1 is hydrogen)

$$R_{10} = 2\left(\frac{Z}{a_0}\right)^{3/2}\exp\left(-\frac{Zr}{a_0}\right)$$

$$R_{20} = \left(\frac{Z}{2a_0}\right)^{3/2}\left(2-\frac{Zr}{a_0}\right)\exp\left(-\frac{Zr}{2a_0}\right)$$

$$R_{21} = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)\exp\left(-\frac{Zr}{2a_0}\right)$$

$$E_n = -\frac{(Z\alpha)^2}{2n^2}\left(m_e c^2\right)$$

Spherical Coordinates

Line Element: $d\vec{l} = dr\ \hat{r} + r\,d\theta\ \hat{\theta} + r\sin\theta\,d\phi\ \hat{\phi}$

$$x = r\sin\theta\cos\phi \qquad \hat{x} = \sin\theta\cos\phi\ \hat{r} + \cos\theta\cos\phi\ \hat{\theta} - \sin\phi\ \hat{\phi}$$

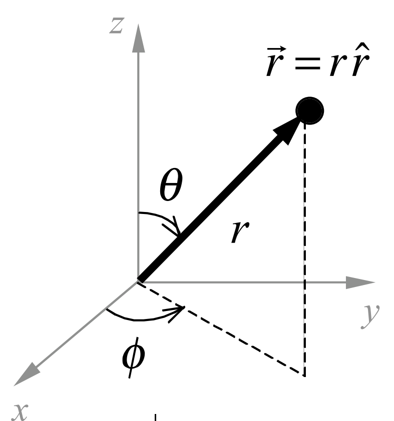
$$y = r\sin\theta\sin\phi \qquad \hat{y} = \sin\theta\sin\phi\ \hat{r} + \cos\theta\sin\phi\ \hat{\theta} + \cos\phi\ \hat{\phi}$$

$$z = r\cos\theta \qquad \hat{z} = \cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \hat{r} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}$$

$$\theta = \tan^{-1}(\sqrt{x^2+y^2}\,/\,z) \qquad \hat{\theta} = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}$$

$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$$



Gradient: $\quad \vec{\nabla}V = \dfrac{\partial V}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial V}{\partial\theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial V}{\partial\phi}\hat{\phi}$

Laplacian: $\quad \nabla^2 V = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial V}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 V}{\partial\phi^2}$

Divergence: $\quad \vec{\nabla}\cdot\vec{E} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 E_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\ E_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial E_\phi}{\partial\phi}$

	∂_r	∂_θ	∂_ϕ
$\hat{r}$	0	$\hat{\theta}$	$\sin\theta\hat{\phi}$
$\hat{\theta}$	0	$-\hat{r}$	$\cos\theta\hat{\phi}$
$\hat{\phi}$	0	0	$-\sin\theta\hat{r}$ $-\cos\theta\hat{\theta}$

Curl: $\quad \vec{\nabla}\times\vec{E} = \dfrac{\hat{r}}{r\sin\theta}\left[\dfrac{\partial}{\partial\theta}(\sin\theta\ E_\phi) - \dfrac{\partial E_\theta}{\partial\phi}\right] + \dfrac{\hat{\theta}}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial E_r}{\partial\phi} - \dfrac{\partial}{\partial r}(rE_\phi)\right] + \dfrac{\hat{\phi}}{r}\left[\dfrac{\partial}{\partial r}(rE_\theta) - \dfrac{\partial E_r}{\partial\theta}\right]$

Acceleration: $\quad \vec{a} = \hat{r}\left[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right] + \hat{\theta}\left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right] + \hat{\phi}\left[\sin\theta(r\ddot{\phi} + 2\dot{r}\dot{\phi}) + \cos\theta(2r\dot{\theta}\dot{\phi})\right]$

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

$$|\vec{v}| \equiv \sqrt{\vec{v}\cdot\vec{v}} \qquad \vec{v} = \sum_{i=1}^{3} (\vec{v}\cdot\hat{r}_i)\,\hat{r}_i \qquad d\vec{l}_{path} = \frac{d\vec{l}}{du}\,du \qquad d\vec{A} = \left(\frac{\partial\vec{l}}{\partial u}\times\frac{\partial\vec{l}}{\partial v}\right)du\,dv$$

$$df(x_1,\ldots,x_n) = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\,dx_i \qquad dV = \left(\frac{\partial\vec{l}}{\partial u}\times\frac{\partial\vec{l}}{\partial v}\right)\cdot\frac{\partial\vec{l}}{\partial w}\,du\,dv\,dw$$

Conceptual version:

$$d\vec{l}_u \equiv \frac{\partial\vec{l}}{\partial u}du \qquad d\vec{l}_{path} = d\vec{l}_u \qquad d\vec{A} = d\vec{l}_u \times d\vec{l}_v \qquad dV = (d\vec{l}_u\times d\vec{l}_v)\cdot d\vec{l}_w$$

Taylor

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

1st order approx for $x \ll 1$:

- $(1+x)^n \approx 1+nx$
- $\sin x \approx x$
- $\sin^{-1} x \approx x$
- $\cos x \approx 1 - \dfrac{x^2}{2}$
- $\cos^{-1} x \approx \dfrac{\pi}{2} - x$
- $\tan x \approx x$
- $\tan^{-1} x \approx x$
- $e^x \approx 1+x$
- $\ln(1+x) \approx x$

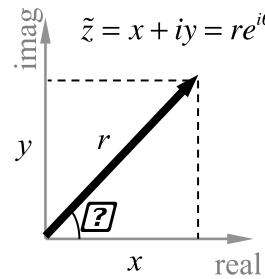
	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

$$\sin a\,\sin b = \tfrac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$
$$\cos a\,\cos b = \tfrac{1}{2}\left[\cos(a+b)+\cos(a-b)\right]$$
$$\sin a\,\cos b = \tfrac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$$

Complex Numbers

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\tilde{z} = x + iy = re^{i\theta}$$



$$\tilde{z}^* \equiv x - iy = re^{-i\theta}$$

$$|\tilde{z}| \equiv \sqrt{\tilde{z}^*\tilde{z}} = r$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} \qquad \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

Integral Table

$$\int\frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right)$$

$$\int_0^{2\pi}\sin^2\phi\,d\phi = \int_0^{2\pi}\cos^2\phi\,d\phi = \pi$$

$$\int\sin^2\phi\,d\phi = \frac{\phi}{2} - \frac{\sin(2\phi)}{4}$$

$$\int\cos^2\phi\,d\phi = \frac{\phi}{2} + \frac{\sin(2\phi)}{4}$$

$$\int\sin^3\theta\,d\theta = \frac{\cos^3\theta}{3} - \cos\theta$$

$$\int\cos^3\theta\,d\theta = \sin\theta - \frac{\sin^3\theta}{3}$$

$$\int\cos^n\theta\,\sin\theta\,d\theta = -\frac{\cos^{n+1}\theta}{n+1}$$

$$\int\frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left(x+\sqrt{x^2\pm a^2}\right)$$

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int\frac{dx}{(a\pm x)^2} = \mp\frac{1}{a\pm x}$$

$$\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int\frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$$

$$\int\frac{dx}{(a^2\pm x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2\pm x^2}}$$

$$\int\frac{x\,dx}{\sqrt{a^2\pm x^2}} = \pm\sqrt{a^2\pm x^2}$$

$$\int\frac{x\,dx}{(a\pm x)^2} = \frac{a}{a\pm x} + \ln\left(a\pm x\right)$$

$$\int\frac{x\,dx}{a^2\pm x^2} = \pm\frac{1}{2}\ln\left(a^2\pm x^2\right)$$

$$\int\frac{x\,dx}{\left(a^2\pm x^2\right)^{3/2}} = \mp\frac{1}{\sqrt{a^2\pm x^2}}$$

$$\int\ln(ax)\,dx = x\ln(ax) - x$$

$$\int\frac{\ln(ax)}{x}\,dx = \frac{1}{2}\left[\ln(ax)\right]^2$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$\int\sqrt{x^2\pm a^2}\,dx = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|$$

$$\int\frac{(x-a\cos\theta)\sin\theta\,d\theta}{(x^2+a^2-2ax\cos\theta)^{3/2}} = \frac{1}{x^2}\frac{a-x\cos\theta}{\sqrt{x^2+a^2-2ax\cos\theta}}$$