PHYSICS 335 SUMMARY

MATH

- Differential calculus $\longrightarrow$ div, grad, & curl
  $\longrightarrow$ Gauss-Green-Stokes theorem
- Integral calculus $\longrightarrow$ line, surface, & volume integrals
  $\longrightarrow$ irrotational & divergenceless fields
- Dirac δ-function $\longrightarrow$ defining prop: ① $\delta^3(\vec{r}) = \begin{cases} \emptyset & \vec{r} \neq \emptyset \\ \infty & \vec{r} = \emptyset \end{cases}$ ② $\int \delta^3(\vec{r})\, d\tau =$
  $\longrightarrow$ key prop: $\int \delta^3(\vec{r}-\vec{a})\, f(\vec{r}) = f(\vec{a})$
  $\longrightarrow$ $\vec{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\, \delta^{(3)}(\vec{\mathscr{r}})$

ALL ABOUT $\vec{E}$ $\vec{F} \equiv q\vec{E}$ ... 3 equiv formulations of electrostatics:

① DIFFER^L FORM

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E} = \emptyset$$

(Helmholtz Thm: this + BC $\rightarrow$ uniquely spec. $\vec{E}$)

② INTEG^L FORM

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\oint \vec{E} \cdot d\vec{l} = \emptyset$$

(direct consequ. of GGS Thm)

③ "BRUTE FORCE" SOLUTION

$$\vec{E}(\vec{r}) = \int \frac{dq}{4\pi\varepsilon_0} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\boxed{\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}_q}$

$\boxed{dq = \begin{matrix} \lambda\, dl \\ \sigma\, dA \\ \rho\, d\tau \end{matrix}}$

POTENTIAL $\vec{E} = -\vec{\nabla} V \longleftrightarrow V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$

Reformulate electrostatics in terms of potential:

① $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ ... ③ $V = \int \frac{dq}{4\pi\varepsilon_0 \mathscr{r}}$

- Work and energy:
  - Work to move q from $a \rightarrow b$: $W_{a\to b} = q\left(V(\vec{b}) - V(\vec{a})\right)$
  - Work to move q from reference point where $V = \emptyset$ to location $\vec{r}$: $W = q\, V(\vec{r})$

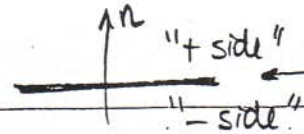
  here, V due to OTHER charges

  - Work to assemble charge dist^n: $W = \frac{1}{2} \int d\tau\, \rho \cdot V$
    $= \frac{\varepsilon_0}{2} \int_{ALL} E^2\, d\tau$

  here, V and E due to ENTIRE charge distribution

$* \; E_\perp \equiv \vec{E}\cdot\hat{n}$

BOUNDARY CONDITIONS



surface, carrying charge density
... for any point $\vec{r}_0$ on surface

$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} \implies$ ① $E_\perp^+ - E_\perp^- \Big|_{\vec{r}_0} \overset{*}{=} \frac{\sigma}{\varepsilon_0}\Big|_{\vec{r}_0}$ ... ③ $V^+ - V^- \Big|_{\vec{r}_0} = \emptyset$

$\oint \vec{E}\cdot d\vec{l} = \emptyset \implies$ ② $E_\parallel^+ - E_\parallel^- \Big|_{\vec{r}_0} = \emptyset$ ④ $\frac{\partial V^+}{\partial n} - \frac{\partial V^-}{\partial n}\Big|_{\vec{r}_0} = \frac{-\sigma}{\varepsilon_0}\Big|_{\vec{r}_0}$

CONDUCTORS

$\vec{E} = \emptyset$ in conductor → conductor is equipotential
→ $E^{OUTSIDE}$ is perpendicular to any metal surface
... and $E_\perp^{OUTSIDE} = \frac{\sigma}{\varepsilon_0}$ (since $E_\perp^{INSIDE} = \emptyset$)

UNIQUENESS THM's

Solutions V of Poisson's equ. unique in $\mathbb{R}$ given:

① Dirichlet BC: ρ in $\mathbb{R}$ and V on $\partial\mathbb{R}$

② Neumann BC: ρ in $\mathbb{R}$ and $\frac{\partial V}{\partial n}$ on $\partial\mathbb{R}$

③ If $\partial\mathbb{R}$ = conducting: ρ in $\mathbb{R}$ and Q_i on $\partial\mathbb{R}_i$

} V unique up to additive constant

METHOD of IMAGES

Satisfy BC on V in $\mathbb{R}$ by adding $V = V_\rho + V_{images}$

due to charges $\rho(\vec{r})$ in $\mathbb{R}$ ... due to image charges OUTSIDE $\mathbb{R}$ = Solutions of Laplace equ. IN $\mathbb{R}$

3 simple problems: ① charges near conducting plates
② point charge outside conducting sphere
③ ∞ line charge outside ∞ conducting cylinder

SEPARATION of VARIABLES

$V = \sum_i a_i V_i$ where V_i are separated solutions of Laplace's eqn:

Rectangular forms:

$$V_0 = \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\y\end{pmatrix}\begin{pmatrix}1\\z\end{pmatrix}$$

$$V_k = \begin{pmatrix}\sin kx\\\cos kx\end{pmatrix}\begin{pmatrix}e^{ky}\\e^{-ky}\end{pmatrix}\begin{pmatrix}1\\z\end{pmatrix}$$

$$V_{kl} = \begin{pmatrix}\sin kx\\\cos kx\end{pmatrix}\begin{pmatrix}\sin ly\\\cos ly\end{pmatrix}\begin{pmatrix}e^{\sqrt{k^2+l^2}\,z}\\e^{-\sqrt{k^2+l^2}\,z}\end{pmatrix}$$

$$V_{kl} = \begin{pmatrix}e^{kx}\\e^{-kx}\end{pmatrix}\begin{pmatrix}e^{ly}\\e^{-ly}\end{pmatrix}\begin{pmatrix}\sin\sqrt{k^2+l^2}\,z\\\cos\sqrt{k^2+l^2}\,z\end{pmatrix}$$

Spherical form:

$$V_l = \begin{pmatrix}r^l\\\frac{1}{r^{l+1}}\end{pmatrix}P_l(\cos\theta)$$

Cylindrical forms:

$$V_0 = \begin{pmatrix}1\\\ln s\end{pmatrix}(1)$$

$$V_n = \begin{pmatrix}s^n\\s^{-n}\end{pmatrix}\begin{pmatrix}\sin n\phi\\\cos n\phi\end{pmatrix}$$

STEPS ... ∅: Write down all BC

1: Choose form of V_i

2: Reduce form of V_i using "simple" BC's

3: Form linear combination $V = \sum_i a_i V_i$ and apply remaining BC's

☆ $\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a} = \frac{a}{2}\delta$ (also for cos)

$\int_0^a \cos\frac{n\pi x}{a}\sin\frac{m\pi x}{a} = \emptyset$

and $\int_0^\pi (\sin\theta\, d\theta) P_l(\cos\theta) P_m(\cos\theta) = \delta_{lm}\frac{2}{2l+1}$

SUMMARY of METHODS for finding $\vec{E}$, V...

Do we know full charge distribution in entire universe?

Ⓐ YES ... Does the system have enough symmetry? (spheres, ∞ cylinders, ∞ planes)

(A1) YES: Find $\vec{E}$ by Gauss' Law, then get $V = -\int_{\text{ref. pt.}}^{r} \vec{E}\cdot d\vec{l}$

(A2) No: Integrate $V = \int \frac{dq}{4\pi\varepsilon_0 \mathscr{r}}$, then get $\vec{E} = -\vec{\nabla}V$

Ⓑ NO ... Do we know ρ in $\mathcal{R}$ of interest + sufficient BC's on $\partial\mathcal{R}$?

If NOT, no chance to solve problem ... otherwise, use:

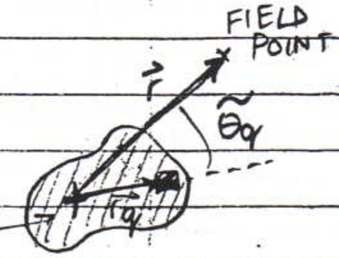
(B1) Method of Images or (B2) Separation of Variables

☆ Remember SUPERPOSITION ... works for E & V

☆ Need to calculate unknown charge distribution? → $\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = -\varepsilon_0 \nabla^2 V$

→ $\sigma = \varepsilon_0 (E_\perp^+ - E_\perp^-) = -\varepsilon_0\left(\frac{\partial V^+}{\partial n} - \frac{\partial V^-}{\partial n}\right)$

MULTIPOLES



- Far-field ($r \gg r_q$) expansion of potential with respect to a chosen ORIGIN

  $V(\vec{r}) = V_0 + V_1 + V_2 \cdots$

- Version #1: $V_n = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^{n+1}} \int r_q^n \, P_n(\underbrace{\cos\tilde{\theta}_q}_{\equiv \hat{r}\cdot\hat{r}_q}) \, dq$ $\left(dq = \begin{matrix} \rho(\vec{r}_q)\, d\tau_q \\ \sigma(\vec{r}_q)\, dA_q \\ \lambda(\vec{r}_q)\, dl_q \end{matrix} \right)$

- Version #2: coordinate-free form involving MOMENTS of charge distribution

$V_0 = \frac{Q_{total}}{4\pi\varepsilon_0 r}$ monopole moment $Q_{total} = \int dq$ (total charge)

$V_1 = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$ dipole moment $\vec{p} \equiv \int \vec{r}_q \, dq$ (charge imbalance $\oplus \uparrow \vec{p} \ominus$)

$V_2 = \frac{1}{4\pi\varepsilon_0 r^5} \sum_{ij} \frac{1}{2} Q_{ij} x_i x_j$ quadrupole moment $Q_{ij} = \int dq \, (3x_i^q x_j^q - \delta_{ij} r_q^2)$ (oblateness of distrib...)

DIPOLES in $\vec{E}$ FIELDS

- For ideal dipoles: $\vec{\tau} = \vec{p}\times\vec{E}$ ... $U = -\vec{p}\cdot\vec{E}$ ... $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} = -\vec{\nabla}U$

  preferred ORIENTATION is $\vec{p} \parallel \vec{E}$

  $F \neq 0$ only in NON-uniform E fields

POLARIZATION

- POLARIZABILITY α of object: $\vec{p} = \alpha\vec{E}_{ext}$ (equilibrium when $\vec{E}_{ext} + \vec{E}_{restoring} = \ldots$

  $\Rightarrow$ linear relation ... may require approx. for small displacements d

- POLARIZATION $\vec{P}$ = dipole moment / unit volume

- V of POLARIZED OBJECTS ① $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\hat{\boldsymbol{\imath}}\cdot\vec{P}(\vec{r}_q)\, d\tau_q}{\boldsymbol{\imath}^2}$

  ② Bound charges: $\sigma_B = \vec{P}\cdot\hat{n}$ ... $\rho_B = -\vec{\nabla}\cdot\vec{P}$

  ③ Shifted charge distributions: Two copies of $\pm\rho$ separated by $\vec{d}$, with $\vec{P} = \rho\cdot\vec{d}$

LINEAR DIELECTRICS

- Definitions: $\vec{P} = \varepsilon_0 \chi \vec{E}$ ... $\varepsilon = \varepsilon_0(1+\chi)$ ... $\varepsilon_r = \varepsilon/\varepsilon_0$

  UNITS! DIMENSIONLESS!

  ★ $\varepsilon_r > 1$ always

  ★ Useful check: $\varepsilon_r \to \infty$ = limit of perfect conductor

- New equations for $\vec{E}$ in terms of FREE CHARGE

$\vec{\nabla}\cdot\vec{E} = \dfrac{\rho_f}{\varepsilon}$	$\varepsilon^+ E_\perp^+ - \varepsilon^- E_\perp^- = \sigma_f$
$\vec{\nabla}\times\vec{E} = \emptyset$	$E_\parallel^+ - E_\parallel^- = \emptyset$
$\nabla^2 V = -\dfrac{\rho_f}{\varepsilon}$	$V^+ - V^- = \emptyset$ $\varepsilon^+ \dfrac{\partial V^+}{\partial n} - \varepsilon^- \dfrac{\partial V^-}{\partial n} = -\sigma_f$

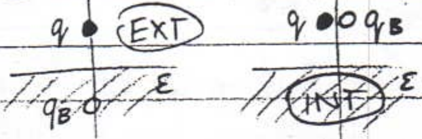
! Remember the EPSILON !
(Travels with $\vec{E}$ or V)

- CALCULATIONS

  ① Gauss' Law: $\oint \varepsilon \vec{E} \cdot d\vec{A} = Q_f^{enc}$

  ② Separation of Variables: same as before, but with new bound. cond'ns

  ③ Method of Images



  $q_B = q\left(\dfrac{1-\varepsilon_r}{1+\varepsilon_r}\right)$

  Always place image charge q_B OUTSIDE region where you're calculating V

  ⊗ Coulomb's Law (direct integration) does NOT work with free charge only !

- CAPACITORS .... capacitance LARGER when dielectric between terminals

  Partially-filled capacitors: → symmetry of $\vec{E}$ usually preserved because terminals are equipotential surfaces (→ Gauss' Law)

  → can often build system as series / parallel combination

- FORCE on dielectric: $\vec{F} = -\vec{\nabla} W$

  Be careful to include all sources of work! (e.g. battery, if Q changing)

LORENTZ FORCE

- $\vec{F} = q\vec{v}\times\vec{B}$ (point charge) ..... $d\vec{F} = \vec{I}\,dl\times\vec{B} = \vec{K}\,dA\times\vec{B} = \vec{J}\,d\tau\times\vec{B}$

  ★ total $\vec{F} = \vec{I} L\times\vec{B}$ for straight wire in uniform $\vec{B}$

- Circular motion: $R = mv_\perp / qB$

- Magnetic force does NO WORK! (always points $\perp$ to path of charge $\therefore \int \vec{F}\cdot d\vec{l} = \emptyset$

CURRENT DISTRIBUTIONS

total $I \equiv$ charge/unit time passing through given surface

- Distrib^{n}s: $\vec{K} \equiv \dfrac{d\vec{I}}{dl_\perp}$ (surface current) ... $\vec{J} \equiv \dfrac{d\vec{I}}{dA_\perp}$ (volume current)
- For moving charge dist^{ns}: $\vec{I} = \lambda\vec{v}$ ... $\vec{K} = \sigma\vec{v}$ ... $\vec{J} = \rho\vec{v}$
- TOTAL CURRENT I ...

  through curve C: $I_c = \int_c K\, dl_\perp = \int_c K_\perp\, dl = \int_c |\vec{K} \times d\vec{l}|$

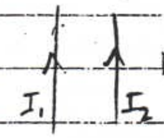
  through surface S: $I_s = \int_s J\, dA_\perp = \int_s J_\perp\, dA = \int_s \vec{J}\cdot d\vec{A}$
- CONTINUITY $\vec{\nabla}\cdot\vec{J} = -\dfrac{\partial\rho}{\partial t}$ ... $= \emptyset$ for STEADY CURRENTS (i.e. time-independent)

BIOT-SAVART

- $d\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{\vec{I}dl \times \hat{r}}{r^2}$
  - INTUITION: right-hand rule

  $\hookrightarrow$ $\vec{I}dl$ could be $\vec{K}dA$ or $\vec{J}d\tau$

  (wires) (surfaces) (volumes)
- Can easily calculate force between 2 wires:



IS ATTRACTIVE

DIFFERENTIAL EQU's for $\vec{B}$

- $\vec{\nabla}\cdot\vec{B} = \emptyset$ ... $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$
- AMPERE's LAW $\oint \vec{B}\cdot d\vec{l} = \mu_0 I^{enc}$ $\Longrightarrow$
  1. ∞ straight wires/cylinders
  2. ∞ planes/slabs
  3. ∞ solenoids
  4. toroids

  ★ First step: Determine DIRECTION and functional DEPENDENCE of $\vec{B}$ from intuition & symmetry
- BOUNDARY CONDITIONS $B_\perp^+ - B_\perp^- = \emptyset$

  $\vec{B}_{\parallel}^+ - \vec{B}_{\parallel}^- = \mu_0 \vec{K}\times\hat{n}$

MAGNETIC VECTOR POTENTIAL

- Definition: $\vec{B} = \vec{\nabla} \times \vec{A} \longrightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$
- Gauge Invariance: Adding a gradient $\vec{\nabla}\lambda$ to any $\vec{A}$ does not affect any physical observable
- Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = \phi$ ... can be achieved for any $\vec{A}_0$ by constructing $\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda$ with $\lambda = \frac{1}{4\pi}\int \frac{\vec{\nabla} \cdot \vec{A}_0}{\mathscr{r}} d\tau'$

$\Longrightarrow$ In this gauge: ① $\nabla^2 \vec{A} = -\mu_0 \vec{J} \longrightarrow \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}_q)}{\mathscr{r}} d\tau_q$ ⊛

② Simple boundary condition $\longrightarrow$ $\vec{A}$ continuous @ any boundary

- Calculating $\vec{A}$ from $\vec{J}$: almost NEVER possible in analytic form using ⊛ ... instead, calculate $\vec{B}$ first from current distrib^{n}, then...
- Calculating $\vec{A}$ from $\vec{B}$: ☆ KEY $\longrightarrow$ $\vec{A}$ follows $\vec{J}$ (from ⊛)

① Fix $\vec{A}$'s direction ($\vec{A} \parallel \vec{J}$) & functional dependence (from symmetry)

(2a) Solve diff. eq. $\vec{\nabla} \times \vec{A} = \vec{B}$ ... check Coulomb gauge ... apply boundary condition ($\vec{A}$ continuous)

OR (2b) Use $\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{A}$

MAGNETIC MULTIPOLES

$\vec{A} = \vec{A}_0 + \vec{A}_1 + \dots$ (far-field expansion in $\left(\frac{r_q}{r}\right)^n$)

☆ Our expansion is for steady currents only

- Monopole term $\vec{A}_0 = \phi$
- Dipole term:

$$\vec{A}_1(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \quad \text{with} \quad \vec{m} = \frac{1}{2}\int \vec{r}_q \times \vec{J}(\vec{r}_q) d\tau_q \quad \text{(magnetic dipole moment)}$$

$$\longrightarrow \vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}\right]$$

(same form as electric dipole field!)

MAGNETIC DIPOLES

General expression: $\vec{m} = \frac{1}{2}\int \vec{r}_q \times \begin{Bmatrix} \vec{J}(\vec{r}_q)\, d\tau_q \\ \vec{K}(\vec{r}_q)\, dA_q \\ \vec{I}(\vec{r}_q)\, dl_q \end{Bmatrix}$

- Dipole moment of current loop: $\vec{m} = I\int d\vec{A}_q \xrightarrow[\text{flat loop}]{} I \cdot \vec{A}$

- Calculating $\vec{m}$: either
  1. Use general expression above
  2. Decompose object into current loops: $\vec{m} = \int_{\text{loops}} d\vec{m}$ where $d\vec{m} = dI \cdot \vec{A}$ for each loop

- Dipoles in B fields experience torques and forces:

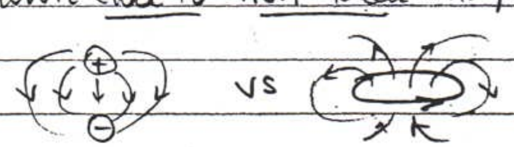
  ① $\vec{\tau} = \vec{m} \times \vec{B}$ ② $U = -\vec{m}\cdot\vec{B}$ ③ $\vec{F} = -\vec{\nabla} U$

☆ Gilbert model: Ideal (pure) dipoles $\vec{m}$ LOOK LIKE electric dipoles $\vec{p}$ — (N) "N" = positive "magnetic charge", (S) "S" = negative "magnetic charge"

These "magnetic charges" (= "poles") → experience electric-like force $\vec{F}_B = q_B \vec{B}$; → produce electric-like fields

⇒ Analogy useful, but breaks down close to non-ideal magnetic dip

MAGNETIZATION



- Magnetization $\vec{M}$ of material = mag. dipole moment / unit volume
- Magnetic materials acquire $\vec{M}$ in presence of external field $\vec{B}_{ext}$
  1. paramagnetic: $\vec{M} \parallel \vec{B}_{ext}$ ~~(similar to dielectrics)~~
  2. diamagnetic: $\vec{M} \parallel -\vec{B}_{ext}$
  3. Ferromagnetic: "spontaneous" $\vec{M}$ can exist even when $\vec{B}_{ext} = \emptyset$
- Calculating the $\vec{B}$ field of magnetized objects
  1. Bound Currents: $\vec{K}_b = \vec{M}\times\hat{n}$ ... $\vec{J}_b = \vec{\nabla}\times\vec{M}$ → calculate $\vec{B}$ due to these curren
  2. Mag. Pole Densities: $\vec{B} = \mu_0 \vec{M} + \vec{B}^*$

     Calculate $\vec{B}^*$ as you would an ELECTRIC field, using $\left(\mu_0 \text{ instead of } \frac{1}{\varepsilon_0}\right)$ and "charges" $\begin{pmatrix} \sigma_b^* = \vec{M}\cdot\hat{n} \\ \rho_b^* = -\vec{\nabla}\cdot\vec{M} \end{pmatrix}$

PHYSICS 336 SUMMARY

LINEAR MAGNETIC MATERIALS

- Definition $\vec{M} = \vec{B}\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)$ ... $\mu > \mu_0$: paramagnetic ($\vec{M} \parallel \vec{B}$)
  $\mu < \mu_0$: diamagnetic ($\vec{M} \parallel -\vec{B}$)
- The H field define $\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}}{\mu}$ analogous to $\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E}$
- Field equations in terms of FREE currents

$$\nabla \cdot B = \emptyset \quad \Longrightarrow \quad B_\perp^+ - B_\perp^- = \emptyset \qquad \text{OR} \qquad \nabla \cdot H = -\nabla \cdot M \quad \Longrightarrow \quad H_\perp^+ - H_\perp^- = -(M_\perp^+ - M_\perp^-)$$

$$\nabla \times \frac{B}{\mu} = J_f \qquad \frac{B_\parallel^+}{\mu^+} - \frac{B_\parallel^-}{\mu^-} = \vec{K}_f \times \hat{n} \qquad \qquad \nabla \times H = J_f \qquad H_\parallel^+ - H_\parallel^- = \vec{K}_f \times \hat{n}$$

- Technique #1: Ampère's Law — If currents have enough SYMMETRY, use $\oint \frac{\vec{B}}{\mu} \cdot dl = I_f^{enc}$
- Technique #2: Magnetic Scalar Potential — If $J_f = \emptyset$, and linear materials only,

$$\nabla \times H = 0, \quad \nabla \cdot H = \nabla \cdot \frac{B}{\mu} = 0 \longrightarrow \text{define } \vec{H} = -\vec{\nabla} V^* \longrightarrow$$

Solve Laplace's equ $\nabla^2 V^* = \emptyset$ as in electrostatics, but using magnetic boundary conditions

OHMIC CONDUCTORS

- Definition $\vec{J} = \sigma \vec{E} = \frac{\vec{E}}{\rho}$ (σ = conductivity, ρ = resistivity)
- Resistors $\equiv$ 2 metal terminals separated by weakly-conducting material

  ⇨ Calculate $R \equiv \frac{\Delta V}{I}$ where total $I = \int \vec{J} \cdot \vec{da} = \int \frac{E}{\rho} \cdot da$ integrated over xsec of resistor

  ... special cases:
  ① wire with UNIFORM xsec $\longrightarrow R = \rho \cdot L/A$
  ② all elec. flux contained in material $\longrightarrow RC = \rho\varepsilon$

- Additional Boundary Condition on $\vec{E}$ for STEADY CURRENTS

$$\vec{\nabla} \cdot \vec{J} = \emptyset \longrightarrow J_\perp^+ - J_\perp^- = \emptyset \longrightarrow \frac{E_\perp^+}{\rho^+} - \frac{E_\perp^-}{\rho^-} = \emptyset$$

(continuity when $\dot{\rho} = \emptyset$)

- Relaxation Time for dissipation of local charge excesses: $\tau = \rho\varepsilon$

EMF & INDUCTION

- Generalized Ohmic Relation $\vec{J} = \sigma \vec{f}$ with $f \equiv \frac{\text{force}}{\text{unit charge}}$ of ANY origin
- EMF $\mathcal{E} \equiv \oint \vec{f} \cdot d\vec{l}$ = work to move unit charge once around loop
- Faraday's Law $\mathcal{E} = -\frac{d\Phi}{dt}$ where $\Phi \equiv \int \vec{B} \cdot d\vec{a}$ (magnetic FLUX)

  - origin #1: (changing area) motional EMF $\rightarrow$ f = Lorentz force on charges within moving conductor
  - origin #2: (changing B) $f = E$ INDUCED by $\dot{B}$

- Induced $\vec{E}$-fields

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$\rightarrow \therefore \oint \vec{E} \cdot dl = -\oint \dot{\vec{B}} \cdot da$ (Faraday's Law)

$\rightarrow \therefore \vec{E} = -\frac{\mu_0}{4\pi} \int \frac{\dot{\vec{J}}(\vec{r}\,')}{\mathscr{r}} d\tau' \longrightarrow$ DIRECTION: Induced $\vec{E}$ "follows" $-\dot{\vec{J}}$

Technique:
- find $\vec{B}$ if necessary... static techniques ok if $\frac{\dot{J}}{J} \ll \frac{\mathscr{r}}{c}$
- intuit direction of $\vec{E}$
- solve for $\vec{E}$ via Faraday's Law

- Inductance

  self-inductance of one loop: $L \equiv \frac{\Phi_1}{I_1} \longrightarrow \mathcal{E} = -L\frac{dI}{dt}$, $W = \frac{1}{2} L I^2$

  mutual inductance of two loops: $M \equiv \frac{\Phi_2}{I_1} = \frac{\Phi_1}{I_2}$ (symmetric, by Neumann formula)

MAXWELL'S EQUATIONS

in terms of ALL sources	in terms of FREE sources only
$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ $\nabla \times E = -\frac{\partial B}{\partial t}$	$\nabla \cdot D = \rho_f$ $\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \cdot B = 0$ $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$	$\nabla \cdot B = 0$ $\nabla \times H = J_f + \frac{\partial D}{\partial t}$
$\rightarrow$ added displacement current $J_D \equiv \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ (needed for internal consistency)	$\rightarrow$ defining $\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}$, $\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}$
$\rightarrow$ These 4 + Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ = ALL of E&M!	$\rightarrow$ in linear materials, $\vec{D} = \varepsilon \vec{E}$, $\vec{H} = \frac{\vec{B}}{\mu}$

- Corollary #1 of Maxwell's equations: <u>Boundary Conditions</u>

$E_\perp^+ - E_\perp^- = \frac{\sigma}{\varepsilon_0}$ $\quad E_\parallel^+ - E_\parallel^- = \varnothing$	$D_\perp^+ - D_\perp^- = \sigma_f$ $\quad E_\parallel^+ - E_\parallel^- = \varnothing$
$B_\perp^+ - B_\perp^- = \varnothing$ $\quad B_\parallel^+ - B_\parallel^- = \mu_0 \vec{K} \times \hat{n}$	$B_\perp^+ - B_\perp^- = \varnothing$ $\quad H_\parallel^+ - H_\parallel^- = \vec{K}_f \times \hat{n}$

- Corollary #2: <u>Continuity Equ.</u> $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$ (built in via displacem. current)

- <u>Potential Formulation</u> of Max's Equ, with full time-dependence

$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$ ..... In Lorentz gauge, $\vec{\nabla}\cdot\vec{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$, $\Longrightarrow$ $\Box^2 V = -\frac{\rho}{\varepsilon_0}$, $\Box^2 \vec{A} = -\mu_0 \vec{J}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

with $\Box^2 \equiv \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}$

$\boxed{\text{SUPERCONDUCTORS}}$
- perfect conductivity $\longrightarrow E = \varnothing$
- perfect flux exclusion $\longrightarrow B = \varnothing$

★ flux exclusion can be modelled as due to ① free currents $(M = -H = 0)$ ② bound currents $(M = -H \neq 0)$ = perfect diamagnetism: $\mu = \varnothing$ $\Longrightarrow$ physically indistinguishable

$\boxed{\text{MAGNETIC MONOPOLES}}$ add "magnetic charge" to Maxwell's equ's:

- $\vec{\nabla}\cdot\vec{B} = \mu_0 \rho_m$ $\longrightarrow$ $\vec{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{r}$ for point monopole

- $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{J}_m$ (from continuity) $\longrightarrow$ Faraday's law changes: $\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} - \mu_0 I_m^{enc}$

$\boxed{\text{WAVES}}$

- <u>1D</u>: wave equ. $\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \varnothing$ solved by $f = g(z - vt) + h(z + vt)$

wave going right, wave going left, both with speed v

HARMONIC solution:

$f = f_0 \cos(kz - \omega t + \delta) = \mathrm{Re}\left[\tilde{f}_0 e^{i(kz - \omega t)}\right]$

with $v = \frac{\omega}{k}$ $\rightarrow$ "DISPERSION RELATION" between ω and k fixes SPEED of wave to that described by wave equ.

3D: wave equ. $\nabla^2 \vec{A} - \frac{1}{V^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \emptyset$

$\Rightarrow$ PLANE WAVE solution: $\vec{A} = \mathrm{Re}\left[\tilde{\vec{A}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$ with $v = \frac{\omega}{|\vec{k}|}$

$\Rightarrow$ POLARIZATION of plane wave:
- Linear $\rightarrow$ $\tilde{\vec{A}}_0 = A_0 \hat{p}$
- Circular $\rightarrow$ $\tilde{\vec{A}}_0 = A_0(\hat{p} + i\hat{s})$ with $\hat{p} \perp \hat{s}$
- Elliptical $\rightarrow$ $\tilde{\vec{A}}_0 = A_{0p}\hat{p} + A_{0s} e^{i\phi}\hat{s}$

$\boxed{\text{ENERGY OF EM FIELDS}}$ (general relations, NOT just for waves)

- Energy density: $u_E = \frac{\varepsilon}{2}\vec{E}\cdot\vec{E}$, $u_B = \frac{\vec{B}\cdot\vec{B}}{2\mu}$
- Transported power/time: $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu}$

★ These are the REAL ε, μ ... have nothing to do with the $\mathrm{Im}(\hat{\varepsilon})$ ~ term we introduce for the special case of waves in conductors

- Local energy conservation: $\frac{\partial u_{EM}}{\partial t} = -\vec{\nabla}\cdot\vec{S} - \frac{\partial u_{mech}}{\partial t}$ with $u_{EM} \equiv u_E + u_B$ & $\frac{\partial u_{mech}}{\partial t} = \vec{E}\cdot\vec{J}$

★ NOTES for WAVES: These energy equations involve PRODUCTS of vector fields...

① Addition $\longrightarrow$ careful! $\vec{S}_{1+2} = \frac{(\vec{E}_1+\vec{E}_2)\times(\vec{B}_1+\vec{B}_2)}{\mu_0} \neq \vec{S}_1 + \vec{S}_2$ in general!

② Real parts $\longrightarrow$ $\mathrm{Re}(\hat{z}_1)\cdot\mathrm{Re}(\hat{z}_2) \neq \mathrm{Re}(\hat{z}_1\cdot\hat{z}_2)$! Take REAL PARTS first:

eg. $\vec{S} = \frac{\mathrm{Re}(\vec{E})\times\mathrm{Re}(\vec{B})}{\mu_0}$ ... $u_E = \frac{\varepsilon_0}{2}\mathrm{Re}(\vec{E})\cdot\mathrm{Re}(\vec{E})$ ... etc.

③ Averaging over cycles $\longrightarrow$ for any waves A, B with the same $\vec{k}$ and ω,

$$\langle \tilde{\vec{A}}\cdot\tilde{\vec{B}}\rangle = \frac{1}{2}\mathrm{Re}(\tilde{\vec{A}}^*\cdot\tilde{\vec{B}}) \;\ldots\; \langle \tilde{\vec{A}}\times\tilde{\vec{B}}\rangle = \frac{1}{2}\mathrm{Re}(\tilde{\vec{A}}^*\times\tilde{\vec{B}})$$

RELATIVITY

Transformation rules from static frame S to frame S′ moving with speed βc

	INDEX NOTATION	MATRIX NOT$^{\underline{n}}$
scalars:	$\phi' = \phi$	$\phi' = \phi$
4-vectors:	$A^{\mu\prime} = \Lambda^\mu{}_\nu A^\nu$	$A' = \Lambda A$
4-tensors:	$F^{\mu\nu\prime} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$	$F' = \Lambda F \Lambda^T$

where $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

when $\vec{\beta}$ along $\hat{x}$ direction

... and $\gamma \equiv 1/\sqrt{1-\beta^2}$

Covariant vs Contravariant

UPPER index = contravariant $\longrightarrow$ transform with Λ

LOWER index = covariant $\longrightarrow$ transform with $\bar{\Lambda} = (\Lambda^{-1})^T = \Lambda$ with sign of β reversed

Changing covariant $\Leftrightarrow$ contrav.: $A_\mu = g_{\mu\nu} A^\nu$, $A^\mu = g^{\mu\nu} A_\nu$ where metric tensor $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ (i.e. change sign of spatial components 1,2,3)

★ SCALAR PRODUCT $A^\mu \cdot B_\mu$ is always Lorentz-invariant for any 4-vectors A, B

Physical Quantities in "covariant form"

- proper time $\tau = \frac{t}{\gamma_u}$ (scalar) for object w speed u
- space/time $x^\mu = [ct, \vec{x}]$
- derivatives $\partial_\mu = [\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}]$
- sources $J^\mu = [c\rho, \vec{J}]$
- potentials $A^\mu = [\frac{V}{c}, \vec{A}]$
- velocity $\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma_u [c, \vec{u}]$
- force $K^\mu = \frac{dp^\mu}{d\tau} = \gamma_u [\frac{\vec{F}\cdot\vec{u}}{c}, \vec{F}]$
- energy / momentum $p^\mu = [\frac{E}{c}, \vec{p}]$
- fields (anti-sym. tensors) $F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$ $G^{\mu\nu} = \begin{pmatrix} 0 & -B_z & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$

Electrodynamic Equations

- continuity $\partial_\mu J^\mu = \emptyset$
- Maxwell $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ (inhomog) & $\partial_\mu G^{\mu\nu} = \emptyset$ (homog.)
- Lorentz gauge $\partial_\mu A^\mu = \emptyset$
- Maxwell i.t.o. potentials $\Box^2 A^\mu = \partial_\nu \partial^\nu A^\mu = -\mu_0 J^\mu$
- Lorentz force $K^\mu = q F^{\mu\nu} \eta_\nu$
- Fields from potentials $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
- Momentum i.t.o. velocity $p^\mu = m\eta^\mu$ where $p^\mu = [\frac{E}{c}, \vec{p}]$

ELECTROMAG. PLANE WAVES	For a linear material with ε, μ, and Ohmic conductivity σ. (eg. vacuum with $\varepsilon_0, \mu_0, \sigma=0$), Max's Eq's WITHOUT SOURCES yield:

The Wave Equations

$$\nabla^2 \vec{E} = \tilde{\varepsilon}\mu \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \tilde{\varepsilon}\mu \frac{\partial^2 \vec{B}}{\partial t^2}$$

..... with the plane wave solution form

$$\vec{E}(\vec{r},t) = \tilde{\vec{E}}_0 e^{i(\tilde{\vec{k}}\cdot\vec{r}-\omega t)} \qquad \vec{B}(\vec{r},t) = \tilde{\vec{B}}_0 e^{i(\tilde{\vec{k}}\cdot\vec{r}-\omega t)}$$

3 new symbols:

- $\tilde{\varepsilon} \equiv \varepsilon + \frac{i\sigma}{\omega} \longrightarrow \varepsilon$ in insulator
- $\tilde{\vec{k}} \equiv \vec{k} + i\vec{K}$ ... skindepth $\delta \equiv \frac{1}{K}$
- $\tilde{n} \equiv \sqrt{\frac{\tilde{\varepsilon}\mu}{\varepsilon_0\mu_0}} \approx \sqrt{\frac{\tilde{\varepsilon}}{\varepsilon_0}} = n + i n_i$ (refractive index, complex in conduc.'s

Apply Maxwell's Equ's to plane wave solution form:

① Wave equ. $\longrightarrow \sqrt{\tilde{\vec{k}}\cdot\tilde{\vec{k}}} = \frac{\tilde{n}\omega}{c}$ (dispersion relation)

② Divergence equ's $\longrightarrow \tilde{\vec{k}}\cdot\vec{E} = \emptyset$ and $\tilde{\vec{k}}\cdot\vec{B} = \emptyset$

③ Curl equations $\longrightarrow \vec{B} = \frac{\tilde{\vec{k}}}{\omega}\times\vec{E}$ ... now some SPECIAL CASES ...

CASE #1: Good Insulator $\varepsilon \gg \sigma/\omega \longrightarrow \tilde{\varepsilon}, \tilde{\vec{k}}$, and $\tilde{n}$ are purely REAL

① $k = \frac{\omega n}{c}$ ② $\vec{k} \perp \vec{E}$ and $\vec{k} \perp \vec{B}$ ③ $\vec{B} = \frac{\vec{k}}{\omega}\times\vec{E} \longrightarrow \vec{B}\perp\vec{E}$

CASE #2: Good Conductor $\varepsilon \ll \sigma/\omega \longrightarrow \tilde{\varepsilon}$ pure IMAGINARY, and $n \approx n_i \gg 1$

① $k \approx K \rightarrow \delta \approx \frac{\lambda}{2\pi}$ ② ______ ③ ______

CASE #3: $\vec{k} \,/\!/\, \vec{K}$ eg. normal incidence on any conductor, or oblique inc. on excellent cond

① $k = \frac{\omega n}{c}$, $K = \frac{\omega n_i}{c}$ ② $\vec{k}\perp\vec{E}$ and $\vec{k}\perp\vec{B}$ ③ $\vec{B}\perp\vec{E}$ IF linearly polarized

REFLECTION When a wave encounters a boundary between materials, we have (1) incident, (1') reflected, & (2) transmitted waves

	$\vec{k}_2$ REAL	$\tilde{\vec{k}}_2$ COMPLEX (material #2 conducting, or T.I.R.)
Match phases:	• $\omega_1 = \omega_1' = \omega_2$	• (same)
	• $\vec{k}_1 \times \hat{n} = \vec{k}_1' \times \hat{n} = \vec{k}_2 \times \hat{n}$	• (same) ... (with $\tilde{\vec{k}}_2$, that is)
	∘ $\longrightarrow \theta_1' = \theta_1$	∘ (same)
	∘ $\longrightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$	∘ Now $\tilde{\theta}_2 \equiv \cos^{-1}\left(\frac{\tilde{\vec{k}}_2 \cdot \hat{n}}{\lvert\tilde{\vec{k}}_2\rvert}\right)$ COMPLEX, useless for geom. interpretation
	-- ∡ critical angle $\sin\theta_c = n_2/n_1$	-- but we find: $\vec{k}_2 // \vec{K}_2 // \hat{n}$ for GOOD CONDUCTOR $\vec{k}_2 // \vec{K}_2 // \hat{n}$ for NORMAL INCIDENCE $\vec{K}_2 // \hat{n}$, $\vec{k}_2$ along boundary for T.I.R.
Boundary Conditions:	• $E_{//}$ continuous (always)	• (same)
	• $B_{//}$ continuous ($\because K_F = \emptyset$)	• (same) ... (because in Ohmic material, $J = \sigma E$)
Match amplitudes:	• $r_s = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$ etc...	• (same) ... but complex: $\tilde{r}_s = \frac{n_1\cos\theta_1 - \tilde{n}_2\cos\tilde{\theta}_2}{n_1\cos\theta_1 + \tilde{n}_2\cos\tilde{\theta}_2}$
		...but: $\cos\tilde{\theta}_2 = 1$ for GOOD CONDUCTOR $\cos\tilde{\theta}_2 = 1$ for NORMAL INCIDENCE $\tilde{r}_s, \tilde{r}_p$ just a phase for T.I.R.
$R \equiv \frac{\langle -\vec{S}_1' \cdot \hat{n}\rangle}{\langle \vec{S}_1 \cdot \hat{n}\rangle}$:	• $R = r^2$	• $R = \tilde{r}^* \tilde{r}$
$T \equiv \frac{\langle \vec{S}_2 \cdot \hat{n}\rangle}{\langle \vec{S}_1 \cdot \hat{n}\rangle}$:	• $T = \frac{n_2\cos\theta_2}{n_1\cos\theta_1} t^2$ and $1 - R = T$	• For decaying wave, define $A \equiv 1 - R$ --- note: $A_n \approx \frac{2}{n_2}$ for GOOD CONDUCTOR at NORMAL INCIDENCE from AIR

DISPERSION

- Dispersion relation of system is the function $\omega(k)$
- Phase velocity $v_p = \frac{\omega}{k} \longrightarrow$ speed of "wavefronts" of constant phase

  Group velocity $v_g = \frac{d\omega}{dk}$ $\longrightarrow$ speed of energy propagation
  $\longrightarrow$ speed of wave packet "envelope" (i.e. of "pulse")

GUIDED WAVES

New wave solution forms: $\vec{E}(\vec{r},t) = \vec{E}_0(x,y)\, e^{i(kz-\omega t)}$
$\vec{B}(\vec{r},t) = \vec{B}_0(x,y)\, e^{i(kz-\omega t)}$

TE waves $\longrightarrow E_z = \emptyset$
TM waves $\longrightarrow B_z = \emptyset$

(amplitudes are now FUNCTIONS of coord's $\perp$ to propagation direc$^{\underline{n}}$)

Solution Tactic (for TE waves)

1. Apply wave equ. to B_z and solve via separation of variables
2. Get other E, B components from B_z & Max's CURL equ's ($\bigstar\ \vec{B} \neq \frac{\hat{k}}{\omega} \times \vec{E}$ here!)
3. Apply BOUNDARY CONDITIONS at waveguide walls (metal) $E_{\parallel} = \emptyset$ and $B_{\perp} = \emptyset$

(+) DISPERSION REL$^{\underline{n}}$ comes from ① ... or apply wave equ. to ANY component of E, B

- **Rectangular Waveguides**
  - amplitude $B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$
  - dispersion: $k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$ ... cutoff frequ. $\omega_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$

- **Miscellaneous**
  - no TEM modes possible in any hollow waveguide
  - $v_p \cdot v_g = c^2$ for all guided waves

RETARDED POTENTIALS & FIELDS

Find general solution to electrodynamic potential equ's $\Box^2 V = -\frac{\rho}{\varepsilon_0}$, $\Box^2 \vec{A} = -\mu_0 \vec{J}$

- **Retarded time** $t_r \equiv t - \frac{\mathscr{r}}{c}$ with $\mathscr{r} = |\vec{r} - \vec{r}\,'|$ as always

$\Rightarrow$ always evaluate sources @ $(\vec{r}\,', t_r)$, to account for finite speed c with which field information travels from source pt. $\longrightarrow$ field pt. $(\vec{r}, t)$

- **Retarded potentials** $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\, d\tau'$, $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\, d\tau'$

- **Retarded fields** (Jefimenko's Equ's)

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \left[\frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}\,',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c^2\mathscr{r}}\right] d\tau'$$

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \left[\frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{\mathscr{r}c}\right] \times \hat{\mathscr{r}}\, d\tau'$$

(recover Biot-Savart law in quasistatic regime $\frac{\dot{J}}{J} \ll \frac{\mathscr{r}}{c}$)

MOVING POINT CHARGE — Charge q moving along trajectory $\vec{r}' = \vec{w}(t_r)$

- Liénard-Wiechert potentials

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0 \mathscr{r}^*}, \quad \vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2} V(\vec{r},t) \quad \text{where}$$

- $\mathscr{r}^* \equiv \mathscr{r}(1 - \hat{\mathscr{r}}\cdot\vec{v}/c)$
- $\mathscr{r}, \vec{v}$ evaluated at t_r

Tactic:
① $\mathscr{r} = c(t - t_r)$
② $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$
③ $\vec{v} = \dot{\vec{w}}(t_r)$

Solve for $\mathscr{r}, \vec{v}$ of UNIQUE source point in communication with field point $(\vec{r}, t)$

- Fields $\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$, $\vec{B}(\vec{r},t) = \frac{\hat{\mathscr{r}}}{c}\times\vec{E}$

where $\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}$ ... acceleration $\vec{a} \equiv \dot{\vec{v}}$ ... and everything evaluated @ t_r

* NOTE: accelerating charge produces $E, B \sim \frac{1}{\mathscr{r}}$ $\longrightarrow$ $S \sim \frac{1}{\mathscr{r}^2}$ $\longrightarrow$ Power $= \int S\cdot da$ survives out to $\mathscr{r} = \infty$!
(this is "RADIATION")

- Special case: constant velocity $\vec{v}_0$

  - Potentials: $\mathscr{r}^* = \frac{1}{c}\sqrt{(c^2 t - \vec{r}\cdot\vec{v}_0)^2 - (c^2 - v_0^2)(c^2 t^2 - r^2)}$ $\longrightarrow$ insert into LW Potentials

  - Fields: $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{R}}{R^2}\frac{(1 - v_0^2/c^2)}{\left(1 - \frac{v_0^2 \sin^2\theta}{c^2}\right)^{3/2}}$, $\vec{B} = \frac{\vec{v}_0}{c^2}\times\vec{E}$ with $\vec{R} = \vec{r} - \vec{v}_0 t$, $\cos\theta = \hat{v}_0\cdot\hat{R}$

$\Rightarrow$ $\vec{E} \parallel \hat{R}$ points away from CONCURRENT (not retarded) location of charge q

RADIATION — for LOCALIZED source distributions

- Radiated Power $P_{rad} \equiv \lim_{r\to\infty} \oint_{\text{sphere of radius } r} \vec{S}\cdot d\vec{a}$ $\longrightarrow$ rad$^{\underline{n}}$ ONLY occurs when $E, B \sim \frac{1}{r}$ $\longrightarrow$ eg. static sources CANNOT radiate

- Radiation Zone $r' \ll \lambda \sim \frac{c}{\omega} \ll r$ ... take 0th order in r'/r, expand in $r'\omega/c$ ...
  - at all orders: $\vec{E} = -\hat{r}c\times\vec{B}$ ... $\vec{E} \perp \hat{r} \perp \vec{B}$ ... $\vec{S} = \frac{c}{\mu_0}B^2\hat{r}$ (radial power xport)
  - electric dipole (E1): $\vec{B} = -\frac{\mu_0}{4\pi rc}\left[\hat{r}\times\ddot{\vec{p}}(t_0)\right]$ ... $P_{rad} = \frac{\mu_0 |\ddot{\vec{p}}(t_0)|^2}{6\pi c}$ where $t_0 \equiv t - \frac{r}{c}$ (retarded time to ORIGIN)
  - magnetic dipole (M1): $\vec{B} = -\frac{\mu_0}{4\pi rc^2}\hat{r}\times\left[\hat{r}\times\ddot{\vec{m}}(t_0)\right]$ ... $P_{rad} = \frac{\mu_0 |\ddot{\vec{m}}(t_0)|^2}{6\pi c^3}$

Complex Refractive Index

$$\tilde{n} \equiv \sqrt{\frac{\tilde{\epsilon}}{\epsilon_0}}, \quad \tilde{\epsilon} \equiv \epsilon + i\frac{\sigma}{\omega} \quad \Rightarrow \quad n^2 = \frac{\epsilon}{2\epsilon_0}\left[\ 1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}\right], \quad n_i^2 = \frac{\epsilon}{2\epsilon_0}\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}\right]$$

Fresnel coefficients: (any of the values may be complex)

$$r_s = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} \qquad r_p = \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}$$

$$t_s = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} \qquad t_p = \frac{2n_1\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}$$

Field Boosts:

$$E'_\parallel = E_\parallel \qquad B'_\parallel = B_\parallel$$

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}_\perp)$$

$$\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \frac{\mathbf{v}\times\mathbf{E}_\perp}{c^2})$$

Lorentz Boost Matrix:

EM Field Tensors:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \quad G^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & \frac{E_z}{c} & -\frac{E_y}{c} \\ B_y & -\frac{E_z}{c} & 0 & \frac{E_x}{c} \\ B_z & \frac{E_y}{c} & -\frac{E_x}{c} & 0 \end{pmatrix}$$

get $\overline{\Lambda}_\mu{}^\nu$ by reversing β with $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \partial_\mu F^{\mu\nu} = \mu_0 J^\nu, \quad \partial_\mu G^{\mu\nu} = 0$

TE$_{mn}$ Modes: Equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \frac{\omega^2}{c^2}\right)B_z = 0$$

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

... and Solutions in Rectangular Wave Guide:

$$B_z = B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$E_x = \frac{-i\omega}{(\omega/c)^2 - k^2}\left(\frac{n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_y = \frac{+i\omega}{(\omega/c)^2 - k^2}\left(\frac{m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$B_x = \frac{-ik}{(\omega/c)^2 - k^2}\left(\frac{m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$B_y = \frac{-ik}{(\omega/c)^2 - k^2}\left(\frac{n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Retarded Fields

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int d\tau'\left(\frac{\rho}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\mathbf{J}}}{c^2\mathscr{r}}\right)$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int d\tau'\left(\frac{\mathbf{J}}{\mathscr{r}^2} + \frac{\dot{\mathbf{J}}}{c\mathscr{r}}\right)\times\hat{\mathscr{r}}$$

... for a moving point charge

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]$$

$$\mathbf{B} = \frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\times\mathbf{E} \qquad \text{with } \mathbf{u} \equiv c\,\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

... for constant $\mathbf{v}_0 = \beta_0 c$

$$\mathscr{r}^* = \frac{1}{c}\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v}_0)^2 - (c^2 - v_0^2)(c^2t^2 - r^2)}$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3}\frac{1 - \beta_0^2}{(1 - \beta_0^2\sin^2\theta)^{3/2}} \qquad \mathbf{B} = \frac{\mathbf{v}_0}{c^2}\times\mathbf{E} \qquad \mathbf{R} \equiv \mathbf{r} - \mathbf{v}_0 t$$

Gauss-Green-Stokes Theorems

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} V \cdot d\vec{l} = V(\vec{b}) - V(\vec{a})$$ "Gradient = Green's Theorem"

$$\int_{Surface} (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = \oint_{\partial Surface} \vec{E} \cdot d\vec{l}$$ "Curl = Stokes' Theorem"

$$\int_{Volume} (\vec{\nabla} \cdot \vec{E})\, dV = \oint_{\partial Volume} \vec{E} \cdot d\vec{A}$$ "Divergence = Gauss' Theorem"

Vector-Calculus Identities

Triple Product Rules

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \qquad \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Product Rules

$$\vec{\nabla}(fg) = f(\vec{\nabla} g) + g(\vec{\nabla} f) \qquad \vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}$$

$$\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} f) \qquad \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times \vec{\nabla} f \qquad \vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

Second Derivatives

$$\vec{\nabla} \times (\vec{\nabla} f) = 0 \qquad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \qquad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

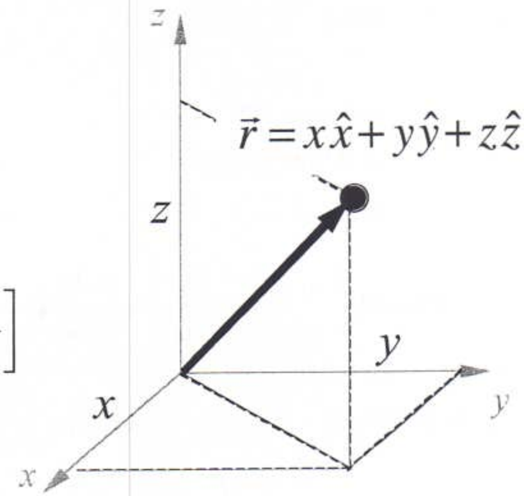
Cartesian Coordinates

Gradient: $$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$

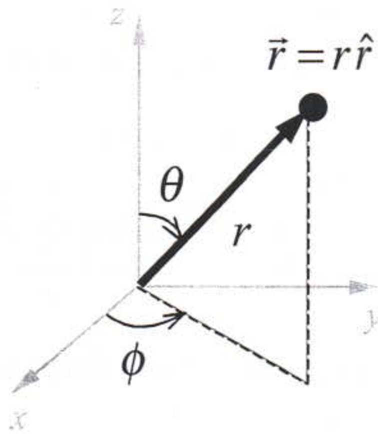
Divergence: $$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Curl: $$\vec{\nabla} \times \vec{E} = \hat{x}\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] + \hat{y}\left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right] + \hat{z}\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]$$

Laplacian: $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$



Spherical Coordinates



$$x = r\sin\theta\cos\phi \qquad \hat{x} = \sin\theta\cos\phi\;\hat{r} + \cos\theta\cos\phi\;\hat{\theta} - \sin\phi\;\hat{\phi}$$
$$y = r\sin\theta\sin\phi \qquad \hat{y} = \sin\theta\sin\phi\;\hat{r} + \cos\theta\sin\phi\;\hat{\theta} + \cos\phi\;\hat{\phi}$$
$$z = r\cos\theta \qquad \hat{z} = \cos\theta\;\hat{r} - \sin\theta\;\hat{\theta}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \hat{r} = \sin\theta\cos\phi\;\hat{x} + \sin\theta\sin\phi\;\hat{y} + \cos\theta\;\hat{z}$$
$$\theta = \tan^{-1}(\sqrt{x^2+y^2}\,/\,z) \qquad \hat{\theta} = \cos\theta\cos\phi\;\hat{x} + \cos\theta\sin\phi\;\hat{y} - \sin\theta\;\hat{z}$$
$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\;\hat{x} + \cos\phi\;\hat{y}$$

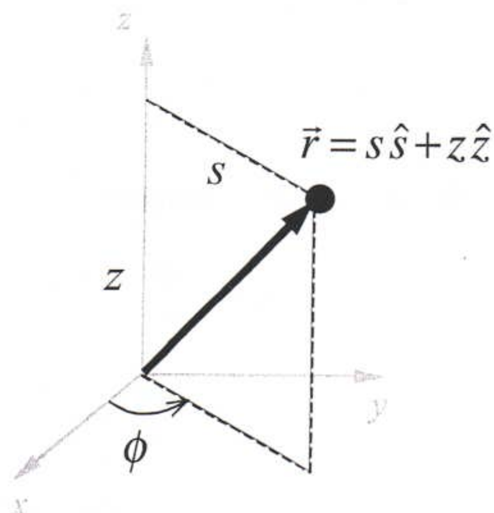
Gradient: $$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

Divergence: $$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\;E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

Curl: $$\vec{\nabla}\times\vec{E} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\;E_\phi) - \frac{\partial E_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}(rE_\phi)\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]$$

Laplacian: $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

Cylindrical Coordinates



$$x = s\cos\phi \qquad \hat{x} = \cos\phi\;\hat{s} - \sin\phi\;\hat{\phi}$$
$$y = s\sin\phi \qquad \hat{y} = \sin\phi\;\hat{s} + \cos\phi\;\hat{\phi}$$
$$z = z \qquad \hat{z} = \hat{z}$$

$$s = \sqrt{x^2+y^2} \qquad \hat{s} = +\cos\phi\;\hat{x} + \sin\phi\;\hat{y}$$
$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\;\hat{x} + \cos\phi\;\hat{y}$$
$$z = z \qquad \hat{z} = \hat{z}$$

Gradient: $$\vec{\nabla} V = \frac{\partial V}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial V}{\partial\phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

Divergence: $$\vec{\nabla}\cdot\vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) + \frac{1}{s}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}$$

Curl: $$\vec{\nabla}\times\vec{E} = \left[\frac{1}{s}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sE_\phi) - \frac{\partial E_s}{\partial\phi}\right]\hat{z}$$

Laplacian: $$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$