Lecture 20 supplement: April 1, 2021

PHYSICS 419 - Spring 2021

1 Greenberger-Horne-Zeilinger State

Bell's analysis requires a series of measurements and a statistical analysis of the results. Greenberger-Horne-Zeilinger (GHZ) devised a single 3-spin state on which a **single** measurement is sufficient to test the local hidden variables hypothesis. The GHZ state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\uparrow\rangle_2|\uparrow\rangle_3 + |\downarrow\rangle_1|\downarrow\rangle_2|\downarrow\rangle_3\right), \tag{1}$$

in which $|\uparrow(\downarrow)\rangle_i$ indicates the spin of particle i. We are using the convention that $|\uparrow\rangle$ represents a projection of the spin along the positive z axis and hence is the state $|+z\rangle$. The down-spin state is $|\downarrow\rangle = |-z\rangle$. We do not have to stick with this choice of axis, however. We can project the spin onto any axis we choose, for example x or y. Since these axes are perpendicular to one another, we need to represent the projection of the spin along these axes by a set of mutually orthogonal vectors. The basis that works for x is

$$|+x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \tag{2}$$

and for $-x$

$$|-x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right). \tag{3}$$

Here $+$ or $-$ stand for along the positive direction and along the negative direction of the x axis. We can now solve these equations by taking sums and differences to obtain explicit expressions for $|\uparrow\rangle$ or $|\downarrow\rangle$:

$$\begin{aligned} |\uparrow\rangle &= \frac{1}{\sqrt{2}}\left(|+x\rangle + |-x\rangle\right) \\ |\downarrow\rangle &= \frac{1}{\sqrt{2}}\left(|+x\rangle - |-x\rangle\right). \end{aligned} \tag{4}$$

Likewise, we can do the same for the y direction. The basis for the y axis which is orthogonal to the $x-$direction is

$$\begin{aligned} |+y\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right) \\ |-y\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - i|\downarrow\rangle\right). \end{aligned} \quad (5)$$

Similarly, we can express the spins in terms of the y-axis basis states:

$$\begin{aligned} |\uparrow\rangle &= \frac{1}{\sqrt{2}}\left(|+y\rangle + |-y\rangle\right) \\ |\downarrow\rangle &= \frac{1}{\sqrt{2}i}\left(|+y\rangle - |-y\rangle\right). \end{aligned} \quad (6)$$

In these states, $i^2 = -1$.

Now for the puchline. We can express the spins in the GHZ state in terms of the basis states above. For $|\uparrow\rangle_1$ we will substitute the first of Eqs. (4) (use the second one for the down-spin state) and for $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ the first and second of Eqs. (6) respectively with $i = 2, 3$. We then substitute them into the GHZ state

$$\begin{aligned} |\psi\rangle &= \frac{1}{2\sqrt{2}}\left(\left(|+x\rangle_1 + |-x\rangle_1\right)\left(|+y\rangle_2 + |-y\rangle_2\right)\left(|+|y\rangle_3 + |-y\rangle_3\right)\right. \\ &- \left.\left(|+x\rangle_1 - |-x\rangle_1\right)\left(|+y\rangle_2 - |-y\rangle_2\right)\left(|+y\rangle_3 - |-y\rangle_3\right)\right) \end{aligned} \quad (7)$$

and do all the multiplications. Note all the terms that contain $|-\alpha\rangle$ ($\alpha = x, y$) states in the first term enter with the opposite sign in the second term. Because there is an overall $-$ sign in front of the second term (coming from the two factors of i), there can be no terms with an even number of $|-\alpha\rangle$ states. The only non-zero terms have an odd number of such states. Each non-zero term enters twice. The result is

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}\left(|+x\rangle_1|+y\rangle_2|-y\rangle_3 + |+x\rangle_1|-y\rangle_2|+y\rangle_3 + |-x\rangle_1|+y\rangle_2|+y\rangle_3 + |-x\rangle_1|-y\rangle_2|-y\rangle_3\right) \\ &\equiv \frac{1}{\sqrt{2}}\left((++-) + (+-+) + (-++) + (---)\right), \end{aligned} \quad (8)$$

where we have used short-hand notation for a state that looks like $(++-) = |+x\rangle_1|+y\rangle_2|-y\rangle_3$. All of these states have the property that the product of the values of the spin projections onto the x and y axes is $X_1Y_2Y_3 = -1$. But we could have permuted the spins. If we do this, we reach the conclusion that

$$\begin{aligned} Y_1X_2Y_3 &= -1 \\ Y_1Y_2X_3 &= -1. \end{aligned} \tag{9}$$

Now let's take the product of these results

$$(X_1Y_2Y_3)(Y_1X_2Y_3)(Y_1Y_2X_3) = X_1X_2X_3 = -1, \tag{10}$$

if we treat the Y_i's as just numbers. In actuality they are not numbers but operators. So lets treat them as classically real. So classical realism gives us a prediction for the product $X_1X_2X_3 = -1$ which we can test against the rules of quantum mechanics. To proceed, we redo the argument assuming all of the spins are along the $x-$ axis. The details are just the same as before but with y replaced with x:

$$\begin{aligned} |\psi\rangle &= \frac{1}{2\sqrt{2}}\left(\left(|+x\rangle_1 + |-x\rangle_1\right)\left(|+x\rangle_2 + |-x\rangle_2\right)\left(|+x\rangle_3 + |-x\rangle_3\right)\right. \\ &+ \left.\left(|+x\rangle_1 - |-x\rangle_1\right)\left(|+x\rangle_2 - |-x\rangle_2\right)\left(|+x\rangle_3 - |-x\rangle_3\right)\right). \end{aligned} \tag{11}$$

Note the second term enters with a plus sign. Hence, only an even number of $x-$ projections involving the $|-x\rangle$ state survive:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left((+++) + (+--) + (--+) + (-+-)\right). \tag{12}$$

In this state $X_1X_2X_3 = +1$ not -1 as local hidden variables would have us believe. Hence, from a single state, we can debunk local hidden variables. We live in a world in which $X_1X_2X_3 = +1$. Hence, there is no reality in quantum mechanics independent of the probabilities.