Mathematics Formulary

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Dear reader,

This document contains 66 pages with mathematical equations intended for physicists and engineers. It is intended to be a short reference for anyone who often needs to look up mathematical equations.

This document can also be obtained from the author, Johan Wevers (`johanw@vulcan.xs4all.nl`).

It can also be found on the WWW on `http://www.xs4all.nl/~johanw/index.html`.

The C code for the rootfinding via Newtons method and the FFT in chapter 8 are from "*Numerical Recipes in C*", 2nd Edition, ISBN 0-521-43108-5.

The Mathematics Formulary is made with teTEX and LATEX version 2.09.

If you prefer the notation in which vectors are typefaced in boldface, uncomment the redefinition of the `\vec` command and recompile the file.

If you find any errors or have any comments, please let me know. I am always open for suggestions and possible corrections to the mathematics formulary.

Johan Wevers

Chapter 1

Basics

1.1 Goniometric functions

For the goniometric ratios for a point p on the unit circle holds:

$$\cos(\phi) = x_p \quad , \quad \sin(\phi) = y_p \quad , \quad \tan(\phi) = \frac{y_p}{x_p}$$

$\sin^2(x) + \cos^2(x) = 1$ and $\cos^{-2}(x) = 1 + \tan^2(x)$.

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \quad , \quad \sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}$$

The **sum formulas** are:

$$\begin{aligned}
\sin(p) + \sin(q) &= 2\sin(\tfrac{1}{2}(p+q))\cos(\tfrac{1}{2}(p-q)) \\
\sin(p) - \sin(q) &= 2\cos(\tfrac{1}{2}(p+q))\sin(\tfrac{1}{2}(p-q)) \\
\cos(p) + \cos(q) &= 2\cos(\tfrac{1}{2}(p+q))\cos(\tfrac{1}{2}(p-q)) \\
\cos(p) - \cos(q) &= -2\sin(\tfrac{1}{2}(p+q))\sin(\tfrac{1}{2}(p-q))
\end{aligned}$$

From these equations can be derived that

$$\begin{aligned}
2\cos^2(x) = 1 + \cos(2x) \quad &, \quad 2\sin^2(x) = 1 - \cos(2x) \\
\sin(\pi - x) = \sin(x) \quad &, \quad \cos(\pi - x) = -\cos(x) \\
\sin(\tfrac{1}{2}\pi - x) = \cos(x) \quad &, \quad \cos(\tfrac{1}{2}\pi - x) = \sin(x)
\end{aligned}$$

Conclusions from equalities:

$$\begin{aligned}
\underline{\sin(x) = \sin(a)} \quad &\Rightarrow \quad x = a \pm 2k\pi \text{ or } x = (\pi - a) \pm 2k\pi, \quad k \in I\!N \\
\underline{\cos(x) = \cos(a)} \quad &\Rightarrow \quad x = a \pm 2k\pi \text{ or } x = -a \pm 2k\pi \\
\underline{\tan(x) = \tan(a)} \quad &\Rightarrow \quad x = a \pm k\pi \text{ and } x \neq \frac{\pi}{2} \pm k\pi
\end{aligned}$$

The following relations exist between the inverse goniometric functions:

$$\arctan(x) = \arcsin\left(\frac{x}{\sqrt{x^2+1}}\right) = \arccos\left(\frac{1}{\sqrt{x^2+1}}\right) \quad , \quad \sin(\arccos(x)) = \sqrt{1-x^2}$$

1.2 Hyperbolic functions

The hyperbolic functions are defined by:

$$\sinh(x)=\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2}~~,~~\cosh(x)=\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}~~,~~\tanh(x)=\frac{\sinh(x)}{\cosh(x)}$$

From this follows that $\cosh^2(x)-\sinh^2(x)=1$. Further holds:

$$\mathrm{arsinh}(x)=\ln|x+\sqrt{x^2+1}|~~,~~\mathrm{arcosh}(x)=\mathrm{arsinh}(\sqrt{x^2-1})$$

1.3 Calculus

The derivative of a function is defined as:

$$\frac{df}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Derivatives obey the following algebraic rules:

$$d(x\pm y)=dx\pm dy~~,~~d(xy)=xdy+ydx~~,~~d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$$

For the derivative of the inverse function $f^{\rm inv}(y)$, defined by $f^{\rm inv}(f(x))=x$, holds at point $P=(x,f(x))$:

$$\left(\frac{df^{\rm inv}(y)}{dy}\right)_P\cdot\left(\frac{df(x)}{dx}\right)_P=1$$

Chain rule: if $f=f(g(x))$, then holds

$$\frac{df}{dx}=\frac{df}{dg}\frac{dg}{dx}$$

Further, for the derivatives of products of functions holds:

$$(f\cdot g)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}\cdot g^{(k)}$$

For the *primitive function* $F(x)$ holds: $F'(x)=f(x)$. An overview of derivatives and primitives is:

$y = f(x)$	$dy/dx = f'(x)$	$\int f(x)dx$
ax^n	anx^{n-1}	$a(n+1)^{-1}x^{n+1}$
$1/x$	$-x^{-2}$	$\ln\|x\|$
a	0	ax
a^x	$a^x\ln(a)$	$a^x/\ln(a)$
e^x	e^x	e^x
${}^a\log(x)$	$(x\ln(a))^{-1}$	$(x\ln(x)-x)/\ln(a)$
$\ln(x)$	$1/x$	$x\ln(x)-x$
$\sin(x)$	$\cos(x)$	$-\cos(x)$
$\cos(x)$	$-\sin(x)$	$\sin(x)$
$\tan(x)$	$\cos^{-2}(x)$	$-\ln\|\cos(x)\|$
$\sin^{-1}(x)$	$-\sin^{-2}(x)\cos(x)$	$\ln\|\tan(\frac{1}{2}x)\|$
$\sinh(x)$	$\cosh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$	$\sinh(x)$
$\arcsin(x)$	$1/\sqrt{1-x^2}$	$x\arcsin(x)+\sqrt{1-x^2}$
$\arccos(x)$	$-1/\sqrt{1-x^2}$	$x\arccos(x)-\sqrt{1-x^2}$
$\arctan(x)$	$(1+x^2)^{-1}$	$x\arctan(x)-\frac{1}{2}\ln(1+x^2)$
$(a+x^2)^{-1/2}$	$-x(a+x^2)^{-3/2}$	$\ln\|x+\sqrt{a+x^2}\|$
$(a^2-x^2)^{-1}$	$2x(a^2+x^2)^{-2}$	$\frac{1}{2a}\ln\|(a+x)/(a-x)\|$

The *curvature* ρ of a curve is given by: $\rho = \dfrac{(1+(y')^2)^{3/2}}{|y''|}$

The theorem of De 'l Hôpital: if $f(a)=0$ and $g(a)=0$, then is $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

1.4 Limits

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1\ ,\quad \lim_{x\to 0}\frac{e^x-1}{x}=1\ ,\quad \lim_{x\to 0}\frac{\tan(x)}{x}=1\ ,\quad \lim_{k\to 0}(1+k)^{1/k}=e\ ,\quad \lim_{x\to\infty}\left(1+\frac{n}{x}\right)^x=e^n$$

$$\lim_{x\downarrow 0}x^a\ln(x)=0\ ,\quad \lim_{x\to\infty}\frac{\ln^p(x)}{x^a}=0\ ,\quad \lim_{x\to 0}\frac{\ln(x+a)}{x}=a\ ,\quad \lim_{x\to\infty}\frac{x^p}{a^x}=0 \text{ als } |a|>1.$$

$$\lim_{x\to 0}\left(a^{1/x}-1\right)=\ln(a)\ ,\quad \lim_{x\to 0}\frac{\arcsin(x)}{x}=1\ ,\quad \lim_{x\to\infty}\sqrt[x]{x}=1$$

1.5 Complex numbers and quaternions

1.5.1 Complex numbers

The complex number $z = a + bi$ with a and $b \in I\!R$. a is the *real part*, b the *imaginary part* of z. $|z| = \sqrt{a^2+b^2}$. By definition holds: $i^2 = -1$. Every complex number can be written as $z = |z|\exp(i\varphi)$, with $\tan(\varphi)=b/a$. The *complex conjugate* of z is defined as $\overline{z}=z^*:=a-bi$. Further holds:

$$\begin{array}{rcl}(a+bi)(c+di) & = & (ac-bd)+i(ad+bc)\\ (a+bi)+(c+di) & = & a+c+i(b+d)\\ \dfrac{a+bi}{c+di} & = & \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}\end{array}$$

Goniometric functions can be written as complex exponents:

$$\begin{array}{rcl}\sin(x) & = & \dfrac{1}{2i}(\mathrm{e}^{ix}-\mathrm{e}^{-ix})\\ \cos(x) & = & \dfrac{1}{2}(\mathrm{e}^{ix}+\mathrm{e}^{-ix})\end{array}$$

From this follows that $\cos(ix)=\cosh(x)$ and $\sin(ix)=i\sinh(x)$. Further follows from this that $\mathrm{e}^{\pm ix}=\cos(x)\pm i\sin(x)$, so $\mathrm{e}^{iz}\neq 0\forall z$. Also the theorem of De Moivre follows from this: $(\cos(\varphi)+i\sin(\varphi))^n=\cos(n\varphi)+i\sin(n\varphi)$.

Products and quotients of complex numbers can be written as:

$$\begin{array}{rcl}z_1\cdot z_2 & = & |z_1|\cdot|z_2|(\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2))\\ \dfrac{z_1}{z_2} & = & \dfrac{|z_1|}{|z_2|}(\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2))\end{array}$$

The following can be derived:

$$|z_1+z_2|\leq|z_1|+|z_2|\quad,\quad|z_1-z_2|\geq|\;|z_1|-|z_2|\;|$$

And from $z=r\exp(i\theta)$ follows: $\ln(z)=\ln(r)+i\theta$, $\ln(z)=\ln(z)\pm 2n\pi i$.

1.5.2 Quaternions

Quaternions are defined as: $z=a+bi+cj+dk$, with $a,b,c,d\in I\!R$ and $i^2=j^2=k^2=-1$. The products of i,j,k with each other are given by $ij=-ji=k$, $jk=-kj=i$ and $ki=-ik=j$.

1.6 Geometry

1.6.1 Triangles

The sine rule is:

$$\frac{a}{\sin(\alpha)}=\frac{b}{\sin(\beta)}=\frac{c}{\sin(\gamma)}$$

Here, α is the angle opposite to a, β is opposite to b and γ opposite to c. The cosine rule is: $a^2=b^2+c^2-2bc\cos(\alpha)$. For each triangle holds: $\alpha+\beta+\gamma=180^\circ$.

Further holds:

$$\frac{\tan(\frac{1}{2}(\alpha+\beta))}{\tan(\frac{1}{2}(\alpha-\beta))}=\frac{a+b}{a-b}$$

The surface of a triangle is given by $\frac{1}{2}ab\sin(\gamma)=\frac{1}{2}ah_a=\sqrt{s(s-a)(s-b)(s-c)}$ with h_a the perpendicular on a and $s=\frac{1}{2}(a+b+c)$.

1.6.2 Curves

Cycloid: if a circle with radius a rolls along a straight line, the trajectory of a point on this circle has the following parameter equation:

$$x = a(t + \sin(t)) \quad , \quad y = a(1 + \cos(t))$$

Epicycloid: if a small circle with radius a rolls along a big circle with radius R, the trajectory of a point on the small circle has the following parameter equation:

$$x = a \sin\left(\frac{R+a}{a}t\right) + (R+a)\sin(t) \quad , \quad y = a\cos\left(\frac{R+a}{a}t\right) + (R+a)\cos(t)$$

Hypocycloid: if a small circle with radius a rolls inside a big circle with radius R, the trajectory of a point on the small circle has the following parameter equation:

$$x = a \sin\left(\frac{R-a}{a}t\right) + (R-a)\sin(t) \quad , \quad y = -a\cos\left(\frac{R-a}{a}t\right) + (R-a)\cos(t)$$

A hypocycloid with $a = R$ is called a **cardioid**. It has the following parameterequation in polar coordinates: $r = 2a[1 - \cos(\varphi)]$.

1.7 Vectors

The *inner product* is defined by: $\vec{a} \cdot \vec{b} = \sum\limits_i a_i b_i = |\vec{a}\,| \cdot |\vec{b}\,| \cos(\varphi)$

where φ is the angle between $\vec{a}$ and $\vec{b}$. The *external product* is in $I\!R^3$ defined by:

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Further holds: $|\vec{a} \times \vec{b}\,| = |\vec{a}\,| \cdot |\vec{b}\,| \sin(\varphi)$, and $\vec{a} \times (\vec{b} \times \vec{c}\,) = (\vec{a} \cdot \vec{c}\,)\vec{b} - (\vec{a} \cdot \vec{b}\,)\vec{c}$.

1.8 Series

1.8.1 Expansion

The Binomium of Newton is:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where $\binom{n}{k} := \dfrac{n!}{k!(n-k)!}$.

By subtracting the series $\sum\limits_{k=0}^{n} r^k$ and $r\sum\limits_{k=0}^{n} r^k$ one finds:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

and for $|r| < 1$ this gives the *geometric series*: $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$.

The *arithmetic series* is given by: $\displaystyle\sum_{n=0}^{N}(a+nV) = a(N+1) + \tfrac{1}{2}N(N+1)V$.

The expansion of a function around the point a is given by the *Taylor series*:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R$$

where the remainder is given by:

$$R_n(h) = (1-\theta)^n \frac{h^n}{n!} f^{(n+1)}(\theta h)$$

and is subject to:

$$\frac{mh^{n+1}}{(n+1)!} \leq R_n(h) \leq \frac{Mh^{n+1}}{(n+1)!}$$

From this one can deduce that

$$(1-x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

One can derive that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\ , \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}\ , \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}n(n+1)(2n+1)\ , \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}\ , \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \tfrac{1}{2}\ , \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}\ , \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}\ , \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$$

1.8.2 Convergence and divergence of series

If $\sum\limits_n |u_n|$ converges, $\sum\limits_n u_n$ also converges.

If $\lim\limits_{n\to\infty} u_n \neq 0$ then $\sum\limits_n u_n$ is divergent.

An alternating series of which the absolute values of the terms drop monotonously to 0 is convergent (Leibniz).

If $\int_p^\infty f(x)dx < \infty$, then $\sum\limits_n f_n$ is convergent.

If $u_n > 0\ \forall n$ then is $\sum\limits_n u_n$ convergent *if* $\sum\limits_n \ln(u_n+1)$ is convergent.

If $u_n = c_n x^n$ the radius of convergence ρ of $\sum\limits_n u_n$ is given by: $\displaystyle\frac{1}{\rho} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

If: $\lim_{n\to\infty} \frac{u_n}{v_n} = p$, than the following is true: if $p > 0$ than $\sum_n u_n$ and $\sum_n v_n$ are both divergent or both convergent, if $p = 0$ holds: if $\sum_n v_n$ is convergent, than $\sum_n u_n$ is also convergent.

If L is defined by: $L = \lim_{n\to\infty} \sqrt[n]{|n_n|}$, or by: $L = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$, then is $\sum_n u_n$ divergent if $L > 1$ and convergent if $L < 1$.

1.8.3 Convergence and divergence of functions

$f(x)$ is continuous in $x = a$ only if the upper - and lower limit are equal: $\lim_{x\uparrow a} f(x) = \lim_{x\downarrow a} f(x)$. This is written as: $f(a^-) = f(a^+)$.

If $f(x)$ is continuous in a *and*: $\lim_{x\uparrow a} f'(x) = \lim_{x\downarrow a} f'(x)$, *than* $f(x)$ is differentiable in $x = a$.

We define: $\|f\|_W := \sup(|f(x)| \,|x \in W)$, and $\lim_{x\to\infty} f_n(x) = f(x)$. Than holds: $\{f_n\}$ is uniform convergent if $\lim_{n\to\infty} \|f_n - f\| = 0$, or: $\forall(\varepsilon > 0)\exists(N)\forall(n \geq N)\|f_n - f\| < \varepsilon$.

Weierstrass' test: if $\sum \|u_n\|_W$ is convergent, than $\sum u_n$ is uniform convergent.

We define $S(x) = \sum_{n=N}^{\infty} u_n(x)$ and $F(y) = \int_a^b f(x, y)dx := F$. Than it can be proved that:

Theorem	**For**	**Demands on W**	**Than holds on W**
C	rows	f_n continuous, $\{f_n\}$ uniform convergent	f is continuous
	series	$S(x)$ uniform convergent, u_n continuous	S is continuous
	integral	f is continuous	F is continuous
I	rows	f_n can be integrated, $\{f_n\}$ uniform convergent	f_n can be integrated, $\int f(x)dx = \lim_{n\to\infty} \int f_n dx$
	series	$S(x)$ is uniform convergent, u_n can be integrated	S can be integrated, $\int Sdx = \sum \int u_n dx$
	integral	f is continuous	$\int Fdy = \iint f(x, y)dxdy$
D	rows	$\{f_n\} \in$C^{-1}; $\{f'_n\}$ unif.conv $\to \phi$	$f' = \phi(x)$
	series	$u_n \in$C^{-1}; $\sum u_n$ conv; $\sum u'_n$ u.c.	$S'(x) = \sum u'_n(x)$
	integral	$\partial f/\partial y$ continuous	$F_y = \int f_y(x, y)dx$

1.9 Products and quotients

For $a, b, c, d \in I\!\!R$ holds:
The **distributive property**: $(a+b)(c+d) = ac + ad + bc + bd$
The **associative property**: $a(bc) = b(ac) = c(ab)$ and $a(b+c) = ab + ac$
The **commutative property**: $a + b = b + a$, $ab = ba$.

Further holds:

$$\frac{a^{2n} - b^{2n}}{a \pm b} = a^{2n-1} \pm a^{2n-2}b + a^{2n-3}b^2 \pm \cdots \pm b^{2n-1} \quad , \quad \frac{a^{2n+1} - b^{2n+1}}{a+b} = \sum_{k=0}^{n} a^{2n-k}b^{2k}$$

$$(a \pm b)(a^2 \pm ab + b^2) = a^3 \pm b^3 \ , \ \ (a+b)(a-b) = a^2 + b^2 \ , \ \ \frac{a^3 \pm b^3}{a+b} = a^2 \mp ba + b^2$$

1.10 Logarithms

Definition: ${}^a\log(x) = b \Leftrightarrow a^b = x$. For logarithms with base e one writes $\ln(x)$.

Rules: $\log(x^n) = n\log(x)$, $\log(a) + \log(b) = \log(ab)$, $\log(a) - \log(b) = \log(a/b)$.

1.11 Polynomials

Equations of the type

$$\sum_{k=0}^{n} a_k x^k = 0$$

have n roots which may be equal to each other. Each polynomial $p(z)$ of order $n \geq 1$ has at least one root in $\mathbb{C}$. If all $a_k \in I\!\!R$ holds: when $x = p$ with $p \in \mathbb{C}$ a root, than p^* is also a root. Polynomials up to and including order 4 have a general analytical solution, for polynomials with order ≥ 5 there does not exist a general analytical solution.

For $a, b, c \in I\!\!R$ and $a \neq 0$ holds: the 2nd order equation $ax^2 + bx + c = 0$ has the general solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $a, b, c, d \in I\!\!R$ and $a \neq 0$ holds: the 3rd order equation $ax^3 + bx^2 + cx + d = 0$ has the general analytical solution:

$$\begin{aligned} x_1 &= K - \frac{3ac - b^2}{9a^2K} - \frac{b}{3a} \\ x_2 = x_3^* &= -\frac{K}{2} + \frac{3ac - b^2}{18a^2K} - \frac{b}{3a} + i\frac{\sqrt{3}}{2}\left(K + \frac{3ac - b^2}{9a^2K}\right) \end{aligned}$$

with $K = \left(\dfrac{9abc - 27da^2 - 2b^3}{54a^3} + \dfrac{\sqrt{3}\sqrt{4ac^3 - c^2b^2 - 18abcd + 27a^2d^2 + 4db^3}}{18a^2}\right)^{1/3}$

1.12 Primes

A *prime* is a number $\in I\!N$ that can only be divided by itself and 1. There are an infinite number of primes. Proof: suppose that the collection of primes P would be finite, than construct the number $q = 1 + \prod_{p\in P} p$, than holds $q = 1(p)$ and so Q cannot be written as a product of primes from P. This is a contradiction.

If $\pi(x)$ is the number of primes $\leq x$, than holds:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln(x)} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{\pi(x)}{\int\limits_2^x \frac{dt}{\ln(t)}} = 1$$

For each $N \geq 2$ there is a prime between N and $2N$.

The numbers $F_k := 2^k + 1$ with $k \in I\!N$ are called *Fermat numbers*. Many Fermat numbers are prime.

The numbers $M_k := 2^k - 1$ are called *Mersenne numbers*. They occur when one searches for *perfect numbers*, which are numbers $n \in I\!N$ which are the sum of their different dividers, for example $6 = 1+2+3$. There are 23 Mersenne numbers for $k < 12000$ which are prime: for $k \in \{2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213\}$.

To check if a given number n is prime one can use a sieve method. The first known sieve method was developed by Eratosthenes. A faster method for large numbers are the 4 Fermat tests, who don't prove that a number is prime but give a large probability.

1. Take the first 4 primes: $b = \{2, 3, 5, 7\}$,
2. Take $w(b) = b^{n-1} \bmod n$, for each b,
3. If $w = 1$ for each b, then n is probably prime. For each other value of w, n is certainly not prime.

Chapter 3

Calculus

3.1 Integrals

3.1.1 Arithmetic rules

The primitive function $F(x)$ of $f(x)$ obeys the rule $F'(x) = f(x)$. With $F(x)$ the primitive of $f(x)$ holds for the definite integral

$$\int_a^b f(x)dx = F(b) - F(a)$$

If $u = f(x)$ holds:

$$\int_a^b g(f(x))df(x) = \int_{f(a)}^{f(b)} g(u)du$$

Partial integration: with F and G the primitives of f and g holds:

$$\int f(x) \cdot g(x)dx = f(x)G(x) - \int G(x)\frac{df(x)}{dx}dx$$

A derivative can be brought under the intergral sign (see section 1.8.3 for the required conditions):

$$\frac{d}{dy}\left[\int_{x=g(y)}^{x=h(y)} f(x,y)dx\right] = \int_{x=g(y)}^{x=h(y)} \frac{\partial f(x,y)}{\partial y}dx - f(g(y),y)\frac{dg(y)}{dy} + f(h(y),y)\frac{dh(y)}{dy}$$

3.1.2 Arc lengts, surfaces and volumes

The arc length ℓ of a curve $y(x)$ is given by:

$$\ell = \int \sqrt{1 + \left(\frac{dy(x)}{dx}\right)^2}dx$$

The arc length ℓ of a parameter curve $F(\vec{x}(t))$ is:

$$\ell = \int Fds = \int F(\vec{x}(t))|\dot{\vec{x}}(t)|dt$$

with

$$\vec{t} = \frac{d\vec{x}}{ds} = \frac{\dot{\vec{x}}(t)}{|\dot{\vec{x}}(t)|} \quad , \quad |\vec{t}\,| = 1$$

$$\int (\vec{v}, \vec{t})ds = \int (\vec{v}, \dot{\vec{t}}(t))dt = \int (v_1 dx + v_2 dy + v_3 dz)$$

The surface A of a solid of revolution is:

$$A = 2\pi \int y \sqrt{1 + \left(\frac{dy(x)}{dx}\right)^2} dx$$

The volume V of a solid of revolution is:

$$V = \pi \int f^2(x)dx$$

3.1.3 Separation of quotients

Every rational function $P(x)/Q(x)$ where P and Q are polynomials can be written as a linear combination of functions of the type $(x-a)^k$ with $k \in \mathbb{Z}$, and of functions of the type

$$\frac{px+q}{((x-a)^2+b^2)^n}$$

with $b > 0$ and $n \in \mathbb{N}$. So:

$$\frac{p(x)}{(x-a)^n} = \sum_{k=1}^{n} \frac{A_k}{(x-a)^k} \quad , \quad \frac{p(x)}{((x-b)^2+c^2)^n} = \sum_{k=1}^{n} \frac{A_k x + B}{((x-b)^2+c^2)^k}$$

Recurrent relation: for $n \neq 0$ holds:

$$\int \frac{dx}{(x^2+1)^{n+1}} = \frac{1}{2n}\frac{x}{(x^2+1)^n} + \frac{2n-1}{2n}\int \frac{dx}{(x^2+1)^n}$$

3.1.4 Special functions

Elliptic functions

Elliptic functions can be written as a power series as follows:

$$\sqrt{1-k^2\sin^2(x)} = 1 - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!(2n-1)} k^{2n}\sin^{2n}(x)$$

$$\frac{1}{\sqrt{1-k^2\sin^2(x)}} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} k^{2n}\sin^{2n}(x)$$

with $n!! = n(n-2)!!$.

The Gamma function

The gamma function $\Gamma(y)$ is defined by:

$$\Gamma(y)=\int\limits_0^\infty e^{-x}x^{y-1}dx$$

One can derive that $\Gamma(y+1)=y\Gamma(y)=y!$. This is a way to define faculties for non-integers. Further one can derive that

$$\Gamma(n+\tfrac{1}{2})=\frac{\sqrt{\pi}}{2^n}(2n-1)!! \quad\text{and}\quad \Gamma^{(n)}(y)=\int\limits_0^\infty e^{-x}x^{y-1}\ln^n(x)dx$$

The Beta function

The betafunction $\beta(p,q)$ is defined by:

$$\beta(p,q)=\int\limits_0^1 x^{p-1}(1-x)^{q-1}dx$$

with p and $q>0$. The beta and gamma functions are related by the following equation:

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

The Delta function

The delta function $\delta(x)$ is an infinitely thin peak function with surface 1. It can be defined by:

$$\delta(x)=\lim_{\varepsilon\to 0}P(\varepsilon,x) \quad\text{with}\quad P(\varepsilon,x)=\begin{cases}0 & \text{for } |x|>\varepsilon\\ \dfrac{1}{2\varepsilon} & \text{when } |x|<\varepsilon\end{cases}$$

Some properties are:

$$\int\limits_{-\infty}^{\infty}\delta(x)dx=1 \quad,\quad \int\limits_{-\infty}^{\infty}F(x)\delta(x)dx=F(0)$$

3.1.5 Goniometric integrals

When solving goniometric integrals it can be useful to change variables. The following holds if one defines $\tan(\frac{1}{2}x):=t$:

$$dx=\frac{2dt}{1+t^2}\ ,\quad \cos(x)=\frac{1-t^2}{1+t^2}\ ,\quad \sin(x)=\frac{2t}{1+t^2}$$

Each integral of the type $\int R(x,\sqrt{ax^2+bx+c})dx$ can be converted into one of the types that were treated in **section 3.1.3**. After this conversion one can substitute in the integrals of the type:

$$\int R(x,\sqrt{x^2+1})dx \quad : \quad x=\tan(\varphi)\ , dx=\frac{d\varphi}{\cos(\varphi)} \ \text{ of } \ \sqrt{x^2+1}=t+x$$

$$\int R(x,\sqrt{1-x^2})dx \quad : \quad x=\sin(\varphi)\ , dx=\cos(\varphi)d\varphi \ \text{ of } \ \sqrt{1-x^2}=1-tx$$

$$\int R(x,\sqrt{x^2-1})dx \quad : \quad x=\frac{1}{\cos(\varphi)}\ , dx=\frac{\sin(\varphi)}{\cos^2(\varphi)}d\varphi \ \text{ of } \ \sqrt{x^2-1}=x-t$$

These definite integrals are easily solved:

$$\int\limits_0^{\pi/2}\cos^n(x)\sin^m(x)dx=\frac{(n-1)!!(m-1)!!}{(m+n)!!}\cdot\left\{\begin{array}{ll}\pi/2 & \text{when } m \text{ and } n \text{ are both even}\\ 1 & \text{in all other cases}\end{array}\right.$$

Some important integrals are:

$$\int\limits_0^\infty\frac{xdx}{e^{ax}+1}=\frac{\pi^2}{12a^2}\ ,\quad \int\limits_{-\infty}^\infty\frac{x^2dx}{(e^x+1)^2}=\frac{\pi^2}{3}\ ,\quad \int\limits_0^\infty\frac{x^3dx}{e^x+1}=\frac{\pi^4}{15}$$

3.2 Functions with more variables

3.2.1 Derivatives

The *partial derivative* with respect to x of a function $f(x,y)$ is defined by:

$$\left(\frac{\partial f}{\partial x}\right)_{x_0}=\lim_{h\to0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

The *directional derivative* in the direction of α is defined by:

$$\frac{\partial f}{\partial\alpha}=\lim_{r\downarrow0}\frac{f(x_0+r\cos(\alpha),y_0+r\sin(\alpha))-f(x_0,y_0)}{r}=(\vec{\nabla}f,(\sin\alpha,\cos\alpha))=\frac{\nabla f\cdot\vec{v}}{|\vec{v}|}$$

When one changes to coordinates $f(x(u,v),y(u,v))$ holds:

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

If $x(t)$ and $y(t)$ depend only on one parameter t holds:

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

The *total differential* df of a function of 3 variables is given by:

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$$

So

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$$

The *tangent* in point $\vec{x}_0$ at the surface $f(x,y) = 0$ is given by the equation $f_x(\vec{x}_0)(x - x_0) + f_y(\vec{x}_0)(y - y_0) = 0$.

The *tangent plane* in $\vec{x}_0$ is given by: $f_x(\vec{x}_0)(x - x_0) + f_y(\vec{x}_0)(y - y_0) = z - f(\vec{x}_0)$.

3.2.2 Taylor series

A function of two variables can be expanded as follows in a Taylor series:

$$f(x_0 + h, y_0 + k) = \sum_{p=0}^{n} \frac{1}{p!}\left(h\frac{\partial^p}{\partial x^p} + k\frac{\partial^p}{\partial y^p}\right) f(x_0, y_0) + R(n)$$

with $R(n)$ the residual error and

$$\left(h\frac{\partial^p}{\partial x^p} + k\frac{\partial^p}{\partial y^p}\right) f(a, b) = \sum_{m=0}^{p} \binom{p}{m} h^m k^{p-m} \frac{\partial^p f(a,b)}{\partial x^m \partial y^{p-m}}$$

3.2.3 Extrema

When f is continuous on a compact boundary V there exists a global maximum and a global minumum for f on this boundary. A boundary is called compact if it is limited and closed.

Possible extrema of $f(x,y)$ on a boundary $V \in \mathbb{R}^2$ are:

1. Points on V where $f(x,y)$ is not differentiable,
2. Points where $\vec{\nabla} f = \vec{0}$,
3. If the boundary V is given by $\varphi(x,y) = 0$, than all points where $\vec{\nabla} f(x,y) + \lambda\vec{\nabla}\varphi(x,y) = 0$ are possible for extrema. This is the multiplicator method of Lagrange, λ is called a multiplicator.

The same as in $\mathbb{R}^2$ holds in $\mathbb{R}^3$ when the area to be searched is constrained by a compact V, and V is defined by $\varphi_1(x,y,z) = 0$ and $\varphi_2(x,y,z) = 0$ for extrema of $f(x,y,z)$ for points (1) and (2). Point (3) is rewritten as follows: possible extrema are points where $\vec{\nabla} f(x,y,z) + \lambda_1\vec{\nabla}\varphi_1(x,y,z) + \lambda_2\vec{\nabla}\varphi_2(x,y,z) = 0$.

3.2.4 The ∇-operator

In cartesian coordinates (x,y,z) holds:

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \\
\text{grad} f &= \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z \\
\text{div }\vec{a} &= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}
\end{aligned}$$

$$\begin{aligned}
\text{curl } \vec{a} &= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z \\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

In cylindrical coordinates (r, φ, z) holds:

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\vec{e}_\varphi + \frac{\partial}{\partial z}\vec{e}_z \\
\text{grad} f &= \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi + \frac{\partial f}{\partial z}\vec{e}_z \\
\text{div } \vec{a} &= \frac{\partial a_r}{\partial r} + \frac{a_r}{r} + \frac{1}{r}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z} \\
\text{curl } \vec{a} &= \left(\frac{1}{r}\frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\varphi + \left(\frac{\partial a_\varphi}{\partial r} + \frac{a_\varphi}{r} - \frac{1}{r}\frac{\partial a_r}{\partial \varphi}\right)\vec{e}_z \\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

In spherical coordinates (r, θ, φ) holds:

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\vec{e}_\varphi \\
\text{grad} f &= \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi \\
\text{div } \vec{a} &= \frac{\partial a_r}{\partial r} + \frac{2a_r}{r} + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{a_\theta}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial \varphi} \\
\text{curl } \vec{a} &= \left(\frac{1}{r}\frac{\partial a_\varphi}{\partial \theta} + \frac{a_\theta}{r\tan\theta} - \frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial \varphi}\right)\vec{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial \varphi} - \frac{\partial a_\varphi}{\partial r} - \frac{a_\varphi}{r}\right)\vec{e}_\theta + \\
&\quad \left(\frac{\partial a_\theta}{\partial r} + \frac{a_\theta}{r} - \frac{1}{r}\frac{\partial a_r}{\partial \theta}\right)\vec{e}_\varphi \\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\tan\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}
\end{aligned}$$

General orthonormal curvilinear coordinates (u, v, w) can be derived from cartesian coordinates by the transformation $\vec{x} = \vec{x}(u, v, w)$. The unit vectors are given by:

$$\vec{e}_u = \frac{1}{h_1}\frac{\partial \vec{x}}{\partial u}\ ,\quad \vec{e}_v = \frac{1}{h_2}\frac{\partial \vec{x}}{\partial v}\ ,\quad \vec{e}_w = \frac{1}{h_3}\frac{\partial \vec{x}}{\partial w}$$

where the terms h_i give normalization to length 1. The differential operators are than given by:

$$\begin{aligned}
\text{grad} f &= \frac{1}{h_1}\frac{\partial f}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial f}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial f}{\partial w}\vec{e}_w \\
\text{div } \vec{a} &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u}(h_2 h_3 a_u) + \frac{\partial}{\partial v}(h_3 h_1 a_v) + \frac{\partial}{\partial w}(h_1 h_2 a_w)\right)
\end{aligned}$$

$$\begin{aligned}\text{curl }\vec{a} &= \frac{1}{h_2h_3}\left(\frac{\partial(h_3a_w)}{\partial v}-\frac{\partial(h_2a_v)}{\partial w}\right)\vec{e}_u+\frac{1}{h_3h_1}\left(\frac{\partial(h_1a_u)}{\partial w}-\frac{\partial(h_3a_w)}{\partial u}\right)\vec{e}_v+\\ &\quad \frac{1}{h_1h_2}\left(\frac{\partial(h_2a_v)}{\partial u}-\frac{\partial(h_1a_u)}{\partial v}\right)\vec{e}_w\\ \nabla^2 f &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)\right]\end{aligned}$$

Some properties of the ∇-operator are:

$\text{div}(\phi\vec{v})=\phi\text{div}\vec{v}+\text{grad}\phi\cdot\vec{v}$	$\text{curl}(\phi\vec{v})=\phi\text{curl}\vec{v}+(\text{grad}\phi)\times\vec{v}$	$\text{curl grad}\phi=\vec{0}$
$\text{div}(\vec{u}\times\vec{v})=\vec{v}\cdot(\text{curl}\vec{u})-\vec{u}\cdot(\text{curl}\vec{v})$	$\text{curl curl}\vec{v}=\text{grad div}\vec{v}-\nabla^2\vec{v}$	$\text{div curl}\vec{v}=0$
$\text{div grad}\phi=\nabla^2\phi$	$\nabla^2\vec{v}\equiv(\nabla^2v_1,\nabla^2v_2,\nabla^2v_3)$	

Here, $\vec{v}$ is an arbitrary vectorfield and ϕ an arbitrary scalar field.

3.2.5 Integral theorems

Some important integral theorems are:

Gauss: $$\oint\!\!\!\oint(\vec{v}\cdot\vec{n})d^2A=\iiint(\text{div}\vec{v})d^3V$$

Stokes for a scalar field: $$\oint(\phi\cdot\vec{e}_{\rm t})ds=\iint(\vec{n}\times\text{grad}\phi)d^2A$$

Stokes for a vector field: $$\oint(\vec{v}\cdot\vec{e}_{\rm t})ds=\iint(\text{curl}\vec{v}\cdot\vec{n})d^2A$$

this gives: $$\oint\!\!\!\oint(\text{curl}\vec{v}\cdot\vec{n})d^2A=0$$

Ostrogradsky: $$\oint\!\!\!\oint(\vec{n}\times\vec{v})d^2A=\iiint(\text{curl}\vec{v})d^3A$$

$$\oint\!\!\!\oint(\phi\vec{n})d^2A=\iiint(\text{grad}\phi)d^3V$$

Here the orientable surface $\iint d^2A$ is bounded by the Jordan curve $s(t)$.

3.2.6 Multiple integrals

Let A be a closed curve given by $f(x,y)=0$, than the surface A inside the curve in $\mathbb{R}^2$ is given by

$$A=\iint d^2A=\iint dxdy$$

Let the surface A be defined by the function $z=f(x,y)$. The volume V bounded by A and the xy plane is than given by:

$$V=\iint f(x,y)dxdy$$

The volume inside a closed surface defined by $z = f(x, y)$ is given by:

$$V = \iiint d^3V = \iint f(x,y)dxdy = \iiint dxdydz$$

3.2.7 Coordinate transformations

The expressions d^2A and d^3V transform as follows when one changes coordinates to $\vec{u} = (u, v, w)$ through the transformation $x(u, v, w)$:

$$V = \iiint f(x,y,z)dxdydz = \iiint f(\vec{x}(\vec{u})) \left|\frac{\partial\vec{x}}{\partial\vec{u}}\right| dudvdw$$

In $I\!R^2$ holds:

$$\frac{\partial\vec{x}}{\partial\vec{u}} = \left|\begin{array}{cc} x_u & x_v \\ y_u & y_v \end{array}\right|$$

Let the surface A be defined by $z = F(x, y) = X(u, v)$. Than the volume bounded by the xy plane and F is given by:

$$\iint\limits_S f(\vec{x})d^2A = \iint\limits_G f(\vec{x}(\vec{u})) \left|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}\right| dudv = \iint\limits_G f(x, y, F(x,y))\sqrt{1 + \partial_x F^2 + \partial_y F^2}dxdy$$

3.3 Orthogonality of functions

The inner product of two functions $f(x)$ and $g(x)$ on the interval $[a, b]$ is given by:

$$(f, g) = \int\limits_a^b f(x)g(x)dx$$

or, when using a weight function $p(x)$, by:

$$(f, g) = \int\limits_a^b p(x)f(x)g(x)dx$$

The *norm* $\|f\|$ follows from: $\|f\|^2 = (f, f)$. A set functions f_i is *orthonormal* if $(f_i, f_j) = \delta_{ij}$.

Each function $f(x)$ can be written as a sum of orthogonal functions:

$$f(x) = \sum_{i=0}^{\infty} c_i g_i(x)$$

and $\sum c_i^2 \leq \|f\|^2$. Let the set g_i be orthogonal, than it follows:

$$c_i = \frac{f, g_i}{(g_i, g_i)}$$

3.4 Fourier series

Each function can be written as a sum of independent base functions. When one chooses the orthogonal basis $(\cos(nx), \sin(nx))$ we have a Fourier series.

A periodical function $f(x)$ with period $2L$ can be written as:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Due to the orthogonality follows for the coefficients:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt \quad , \quad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \quad , \quad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

A Fourier series can also be written as a sum of complex exponents:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{inx}$$

with

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}dx$$

The *Fourier transform* of a function $f(x)$ gives the transformed function $\hat{f}(\omega)$:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\omega x}dx$$

The inverse transformation is given by:

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{i\omega x}d\omega$$

where $f(x^+)$ and $f(x^-)$ are defined by the lower - and upper limit:

$$f(a^-) = \lim_{x\uparrow a} f(x) \quad , \quad f(a^+) = \lim_{x\downarrow a} f(x)$$

For continuous functions is $\frac{1}{2}\left[f(x^+) + f(x^-)\right] = f(x)$.

Chapter 4

Differential equations

4.1 Linear differential equations

4.1.1 First order linear DE

The general solution of a linear differential equation is given by $y_A = y_H + y_P$, where y_H is the solution of the *homogeneous equation* and y_P is a *particular solution*.

A first order differential equation is given by: $y'(x) + a(x)y(x) = b(x)$. Its homogeneous equation is $y'(x) + a(x)y(x) = 0$.

The solution of the homogeneous equation is given by

$$y_H = k \exp\left(\int a(x)dx\right)$$

Suppose that $a(x) = a =$constant.

Substitution of $\exp(\lambda x)$ in the homogeneous equation leads to the *characteristic equation* $\lambda + a = 0$
$\Rightarrow \lambda = -a$.

Suppose $b(x) = \alpha \exp(\mu x)$. Than one can distinguish two cases:

1. $\lambda \neq \mu$: a particular solution is: $y_P = \exp(\mu x)$
2. $\lambda = \mu$: a particular solution is: $y_P = x \exp(\mu x)$

When a DE is solved by *variation of parameters* one writes: $y_P(x) = y_H(x) f(x)$, and than one solves $f(x)$ from this.

4.1.2 Second order linear DE

A differential equation of the second order with constant coefficients is given by: $y''(x) + ay'(x) + by(x) = c(x)$. If $c(x) = c =$constant there exists a particular solution $y_P = c/b$.

Substitution of $y = \exp(\lambda x)$ leads to the characteristic equation $\lambda^2 + a\lambda + b = 0$.

There are now 2 possibilities:

1. $\lambda_1 \neq \lambda_2$: than $y_H = \alpha \exp(\lambda_1 x) + \beta \exp(\lambda_2 x)$.
2. $\lambda_1 = \lambda_2 = \lambda$: than $y_H = (\alpha + \beta x) \exp(\lambda x)$.

If $c(x) = p(x)\exp(\mu x)$ where $p(x)$ is a polynomial there are 3 possibilities:

1. $\lambda_1, \lambda_2 \neq \mu$: $y_{\rm P} = q(x)\exp(\mu x)$.
2. $\lambda_1 = \mu, \lambda_2 \neq \mu$: $y_{\rm P} = xq(x)\exp(\mu x)$.
3. $\lambda_1 = \lambda_2 = \mu$: $y_{\rm P} = x^2 q(x)\exp(\mu x)$.

where $q(x)$ is a polynomial of the same order as $p(x)$.

When: $y''(x) + \omega^2 y(x) = \omega f(x)$ and $y(0) = y'(0) = 0$ follows: $y(x) = \int\limits_0^x f(x)\sin(\omega(x-t))dt$.

4.1.3 The Wronskian

We start with the LDE $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ and the two initial conditions $y(x_0) = K_0$ and $y'(x_0) = K_1$. When $p(x)$ and $q(x)$ are continuous on the open interval I there exists a unique solution $y(x)$ on this interval.

The general solution can than be written as $y(x) = c_1 y_1(x) + c_2 y_2(x)$ and y_1 and y_2 are linear independent. These are also *all* solutions of the LDE.

The *Wronskian* is defined by:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

y_1 and y_2 are linear independent if and only if on the interval I when $\exists x_0 \in I$ so that holds: $W(y_1(x_0), y_2(x_0)) = 0$.

4.1.4 Power series substitution

When a series $y = \sum a_n x^n$ is substituted in the LDE with constant coefficients $y''(x) + py'(x) + qy(x) = 0$ this leads to:

$$\sum_n \left[n(n-1)a_n x^{n-2} + pna_n x^{n-1} + qa_n x^n \right] = 0$$

Setting coefficients for equal powers of x equal gives:

$$(n+2)(n+1)a_{n+2} + p(n+1)a_{n+1} + qa_n = 0$$

This gives a general relation between the coefficients. Special cases are $n = 0, 1, 2$.

4.2 Some special cases

4.2.1 Frobenius' method

Given the LDE

$$\frac{d^2 y(x)}{dx^2} + \frac{b(x)}{x}\frac{dy(x)}{dx} + \frac{c(x)}{x^2}y(x) = 0$$

with $b(x)$ and $c(x)$ analytical at $x = 0$. This LDE has at least one solution of the form

$$y_i(x) = x^{r_i} \sum_{n=0}^{\infty} a_n x^n \quad \text{with} \quad i = 1, 2$$

with r real or complex and chosen so that $a_0 \neq 0$. When one expands $b(x)$ and $c(x)$ as $b(x) = b_0 + b_1 x + b_2 x^2 + ...$ and $c(x) = c_0 + c_1 x + c_2 x^2 + ...$, it follows for r:

$$r^2 + (b_0 - 1)r + c_0 = 0$$

There are now 3 possibilities:

1. $r_1 = r_2$: than $y(x) = y_1(x) \ln|x| + y_2(x)$.
2. $r_1 - r_2 \in \mathbb{N}$: than $y(x) = k y_1(x) \ln|x| + y_2(x)$.
3. $r_1 - r_2 \neq \mathbb{Z}$: than $y(x) = y_1(x) + y_2(x)$.

4.2.2 Euler

Given the LDE

$$x^2 \frac{d^2 y(x)}{dx^2} + ax \frac{dy(x)}{dx} + by(x) = 0$$

Substitution of $y(x) = x^r$ gives an equation for r: $r^2 + (a-1)r + b = 0$. From this one gets two solutions r_1 and r_2. There are now 2 possibilities:

1. $r_1 \neq r_2$: than $y(x) = C_1 x^{r1} + C_2 x^{r_2}$.
2. $r_1 = r_2 = r$: than $y(x) = (C_1 \ln(x) + C_2) x^r$.

4.2.3 Legendre's DE

Given the LDE

$$(1 - x^2) \frac{d^2 y(x)}{dx^2} - 2x \frac{dy(x)}{dx} + n(n-1)y(x) = 0$$

The solutions of this equation are given by $y(x) = aP_n(x) + by_2(x)$ where the *Legendre polynomials* $P(x)$ are defined by:

$$P_n(x) = \frac{d^n}{dx^n} \left(\frac{(1 - x^2)^n}{2^n n!} \right)$$

For these holds: $\|P_n\|^2 = 2/(2n+1)$.

4.2.4 The associated Legendre equation

This equation follows from the θ-dependent part of the wave equation $\nabla^2 \Psi = 0$ by substitution of $\xi = \cos(\theta)$. Than follows:

$$(1 - \xi^2) \frac{d}{d\xi} \left((1 - \xi^2) \frac{dP(\xi)}{d\xi} \right) + [C(1 - \xi^2) - m^2] P(\xi) = 0$$

Regular solutions exists only if $C = l(l+1)$. They are of the form:

$$P_l^{|m|}(\xi) = (1-\xi^2)^{m/2}\frac{d^{|m|}P^0(\xi)}{d\xi^{|m|}} = \frac{(1-\xi^2)^{|m|/2}}{2^l l!}\frac{d^{|m|+l}}{d\xi^{|m|+l}}(\xi^2-1)^l$$

For $|m| > l$ is $P_l^{|m|}(\xi) = 0$. Some properties of $P_l^0(\xi)$ zijn:

$$\int\limits_{-1}^{1} P_l^0(\xi)P_{l'}^0(\xi)d\xi = \frac{2}{2l+1}\delta_{ll'} \quad , \quad \sum_{l=0}^{\infty} P_l^0(\xi)t^l = \frac{1}{\sqrt{1-2\xi t+t^2}}$$

This polynomial can be written as:

$$P_l^0(\xi) = \frac{1}{\pi}\int\limits_0^{\pi}(\xi+\sqrt{\xi^2-1}\cos(\theta))^l d\theta$$

4.2.5 Solutions for Bessel's equation

Given the LDE

$$x^2\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx} + (x^2-\nu^2)y(x) = 0$$

also called *Bessel's equation*, and the Bessel functions of the first kind

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+\nu}m!\Gamma(\nu+m+1)}$$

for $\nu := n \in \mathbb{N}$ this becomes:

$$J_n(x) = x^n \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+n}m!(n+m)!}$$

When $\nu \neq \mathbb{Z}$ the solution is given by $y(x) = aJ_\nu(x) + bJ_{-\nu}(x)$. But because for $n \in \mathbb{Z}$ holds: $J_{-n}(x) = (-1)^n J_n(x)$, this does not apply to integers. The general solution of Bessel's equation is given by $y(x) = aJ_\nu(x) + bY_\nu(x)$, where Y_ν are the *Bessel functions of the second kind*:

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \quad \text{and} \quad Y_n(x) = \lim_{\nu\to n} Y_\nu(x)$$

The equation $x^2y''(x) + xy'(x) - (x^2+\nu^2)y(x) = 0$ has the modified Bessel functions of the first kind $I_\nu(x) = i^{-\nu}J_\nu(ix)$ as solution, and also solutions $K_\nu = \pi[I_{-\nu}(x) - I_\nu(x)]/[2\sin(\nu\pi)]$.

Sometimes it can be convenient to write the solutions of Bessel's equation in terms of the Hankel functions

$$H_n^{(1)}(x) = J_n(x) + iY_n(x) \quad , \quad H_n^{(2)}(x) = J_n(x) - iY_n(x)$$

4.2.6 Properties of Bessel functions

Bessel functions are orthogonal with respect to the weight function $p(x) = x$.

$J_{-n}(x) = (-1)^n J_n(x)$. The Neumann functions $N_m(x)$ are definied as:

$$N_m(x) = \frac{1}{2\pi} J_m(x) \ln(x) + \frac{1}{x^m} \sum_{n=0}^{\infty} \alpha_n x^{2n}$$

The following holds: $\lim\limits_{x\to 0} J_m(x) = x^m$, $\lim\limits_{x\to 0} N_m(x) = x^{-m}$ for $m \neq 0$, $\lim\limits_{x\to 0} N_0(x) = \ln(x)$.

$$\lim_{r\to\infty} H(r) = \frac{\mathrm{e}^{\pm ikr}\mathrm{e}^{i\omega t}}{\sqrt{r}} \quad , \quad \lim_{x\to\infty} J_n(x) = \sqrt{\frac{2}{\pi x}} \cos(x - x_n) \quad , \quad \lim_{x\to\infty} J_{-n}(x) = \sqrt{\frac{2}{\pi x}} \sin(x - x_n)$$

with $x_n = \frac{1}{2}\pi(n + \frac{1}{2})$.

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x) \quad , \quad J_{n+1}(x) - J_{n-1}(x) = -2\frac{dJ_n(x)}{dx}$$

The following integral relations hold:

$$J_m(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp[i(x\sin(\theta) - m\theta)]d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin(\theta) - m\theta)d\theta$$

4.2.7 Laguerre's equation

Given the LDE

$$x\frac{d^2y(x)}{dx^2} + (1-x)\frac{dy(x)}{dx} + ny(x) = 0$$

Solutions of this equation are the Laguerre polynomials $L_n(x)$:

$$L_n(x) = \frac{\mathrm{e}^x}{n!} \frac{d^n}{dx^n} \left(x^n \mathrm{e}^{-x}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \binom{n}{m} x^m$$

4.2.8 The associated Laguerre equation

Given the LDE

$$\frac{d^2y(x)}{dx^2} + \left(\frac{m+1}{x} - 1\right) \frac{dy(x)}{dx} + \left(\frac{n + \frac{1}{2}(m+1)}{x}\right) y(x) = 0$$

Solutions of this equation are the associated Laguerre polynomials $L_n^m(x)$:

$$L_n^m(x) = \frac{(-1)^m n!}{(n-m)!} \mathrm{e}^{-x} x^{-m} \frac{d^{n-m}}{dx^{n-m}} \left(\mathrm{e}^{-x} x^n\right)$$

4.2.9 Hermite

The differential equations of Hermite are:

$$\frac{d^2\mathrm{H}_n(x)}{dx^2}-2x\frac{d\mathrm{H}_n(x)}{dx}+2n\mathrm{H}_n(x)=0 \quad\text{and}\quad \frac{d^2\mathrm{He}_n(x)}{dx^2}-x\frac{d\mathrm{He}_n(x)}{dx}+n\mathrm{He}_n(x)=0$$

Solutions of these equations are the Hermite polynomials, given by:

$$\mathrm{H}_n(x)=(-1)^n\exp\left(\frac{1}{2}x^2\right)\frac{d^n(\exp(-\frac{1}{2}x^2))}{dx^n}=2^{n/2}\mathrm{He}_n(x\sqrt{2})$$

$$\mathrm{He}_n(x)=(-1)^n(\exp\left(x^2\right)\frac{d^n(\exp(-x^2))}{dx^n}=2^{-n/2}\mathrm{H}_n(x/\sqrt{2})$$

4.2.10 Chebyshev

The LDE

$$(1-x^2)\frac{d^2U_n(x)}{dx^2}-3x\frac{dU_n(x)}{dx}+n(n+2)U_n(x)=0$$

has solutions of the form

$$U_n(x)=\frac{\sin[(n+1)\arccos(x)]}{\sqrt{1-x^2}}$$

The LDE

$$(1-x^2)\frac{d^2T_n(x)}{dx^2}-x\frac{dT_n(x)}{dx}+n^2T_n(x)=0$$

has solutions $T_n(x)=\cos(n\arccos(x))$.

4.2.11 Weber

The LDE $W_n''(x)+(n+\frac{1}{2}-\frac{1}{4}x^2)W_n(x)=0$ has solutions: $W_n(x)=\mathrm{He}_n(x)\exp(-\frac{1}{4}x^2)$.

4.3 Non-linear differential equations

Some non-linear differential equations and a solution are:

$$\begin{array}{ll}
y'=a\sqrt{y^2+b^2} & y=b\sinh(a(x-x_0))\\
y'=a\sqrt{y^2-b^2} & y=b\cosh(a(x-x_0))\\
y'=a\sqrt{b^2-y^2} & y=b\cos(a(x-x_0))\\
y'=a(y^2+b^2) & y=b\tan(a(x-x_0))\\
y'=a(y^2-b^2) & y=b\coth(a(x-x_0))\\
y'=a(b^2-y^2) & y=b\tanh(a(x-x_0))\\
y'=ay\left(\dfrac{b-y}{b}\right) & y=\dfrac{b}{1+Cb\exp(-ax)}
\end{array}$$

4.4 Sturm-Liouville equations

Sturm-Liouville equations are second order LDE's of the form:

$$-\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)y(x)=\lambda m(x)y(x)$$

The boundary conditions are chosen so that the operator

$$L=-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)$$

is Hermitean. The normalization function $m(x)$ must satisfy

$$\int\limits_a^b m(x)y_i(x)y_j(x)dx=\delta_{ij}$$

When $y_1(x)$ and $y_2(x)$ are two linear independent solutions one can write the Wronskian in this form:

$$W(y_1,y_2)=\left|\begin{array}{cc} y_1 & y_2 \\ y_1' & y_2' \end{array}\right|=\frac{C}{p(x)}$$

where C is constant. By changing to another dependent variable $u(x)$, given by: $u(x)=y(x)\sqrt{p(x)}$, the LDE transforms into the *normal form*:

$$\frac{d^2u(x)}{dx^2}+I(x)u(x)=0 \quad\text{with}\quad I(x)=\frac{1}{4}\left(\frac{p'(x)}{p(x)}\right)^2-\frac{1}{2}\frac{p''(x)}{p(x)}-\frac{q(x)-\lambda m(x)}{p(x)}$$

If $I(x)>0$, than $y''/y<0$ and the solution has an oscillatory behaviour, if $I(x)<0$, than $y''/y>0$ and the solution has an exponential behaviour.

4.5 Linear partial differential equations

4.5.1 General

The *normal derivative* is defined by:

$$\frac{\partial u}{\partial n}=(\vec{\nabla}u,\vec{n})$$

A frequently used solution method for PDE's is *separation of variables*: one assumes that the solution can be written as $u(x,t)=X(x)T(t)$. When this is substituted two ordinary DE's for $X(x)$ and $T(t)$ are obtained.

4.5.2 Special cases

The wave equation

The *wave equation* in 1 dimension is given by

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

When the initial conditions $u(x,0)=\varphi(x)$ and $\partial u(x,0)/\partial t=\Psi(x)$ apply, the general solution is given by:

$$u(x,t)=\frac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right]+\frac{1}{2c}\int\limits_{x-ct}^{x+ct}\Psi(\xi)d\xi$$

The diffusion equation

The *diffusion equation* is:

$$\frac{\partial u}{\partial t}=D\nabla^2u$$

Its solutions can be written in terms of the propagators $P(x,x',t)$. These have the property that $P(x,x',0)=\delta(x-x')$. In 1 dimension it reads:

$$P(x,x',t)=\frac{1}{2\sqrt{\pi Dt}}\exp\left(\frac{-(x-x')^2}{4Dt}\right)$$

In 3 dimensions it reads:

$$P(x,x',t)=\frac{1}{8(\pi Dt)^{3/2}}\exp\left(\frac{-(\vec{x}-\vec{x}\,')^2}{4Dt}\right)$$

With initial condition $u(x,0)=f(x)$ the solution is:

$$u(x,t)=\int\limits_{\mathcal{G}}f(x')P(x,x',t)dx'$$

The solution of the equation

$$\frac{\partial u}{\partial t}-D\frac{\partial^2u}{\partial x^2}=g(x,t)$$

is given by

$$u(x,t)=\int dt'\int dx'g(x',t')P(x,x',t-t')$$

The equation of Helmholtz

The equation of Helmholtz is obtained by substitution of $u(\vec{x},t)=v(\vec{x})\exp(i\omega t)$ in the wave equation. This gives for v:

$$\nabla^2v(\vec{x},\omega)+k^2v(\vec{x},\omega)=0$$

This gives as solutions for v:

1. In cartesian coordinates: substitution of $v=A\exp(i\vec{k}\cdot\vec{x}\,)$ gives:

$$v(\vec{x}\,)=\int\cdots\int A(k)\mathrm{e}^{i\vec{k}\cdot\vec{x}}dk$$

with the integrals over $\vec{k}^{\,2}=k^2$.

2. In polar coordinates:

$$v(r,\varphi)=\sum_{m=0}^{\infty}(A_mJ_m(kr)+B_mN_m(kr))\mathrm{e}^{im\varphi}$$

3. In spherical coordinates:

$$v(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}J_{l+\frac{1}{2}}(kr)+B_{lm}J_{-l-\frac{1}{2}}(kr)]\frac{Y(\theta,\varphi)}{\sqrt{r}}$$

4.5.3 Potential theory and Green's theorem

Subject of the potential theory are the *Poisson equation* $\nabla^2u=-f(\vec{x}\,)$ where f is a given function, and the *Laplace equation* $\nabla^2u=0$. The solutions of these can often be interpreted as a potential. The solutions of Laplace's equation are called *harmonic functions*.

When a vector field $\vec{v}$ is given by $\vec{v}=\mathrm{grad}\varphi$ holds:

$$\int\limits_a^b(\vec{v},\vec{t}\,)ds=\varphi(\vec{b}\,)-\varphi(\vec{a}\,)$$

In this case there exist functions φ and $\vec{w}$ so that $\vec{v}=\mathrm{grad}\varphi+\mathrm{curl}\vec{w}$.

The *field lines* of the field $\vec{v}(\vec{x}\,)$ follow from:

$$\dot{\vec{x}}\,(t)=\lambda\vec{v}(\vec{x}\,)$$

The *first theorem of Green* is:

$$\iiint\limits_{\mathcal{G}}[u\nabla^2v+(\nabla u,\nabla v)]d^3V=\oiint\limits_{\mathcal{S}}u\frac{\partial v}{\partial n}d^2A$$

The *second theorem of Green* is:

$$\iiint\limits_{\mathcal{G}}[u\nabla^2v-v\nabla^2u]d^3V=\oiint\limits_{\mathcal{S}}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)d^2A$$

A harmonic function which is 0 on the boundary of an area is also 0 within that area. A harmonic function with a normal derivative of 0 on the boundary of an area is constant within that area.

The *Dirichlet problem* is:

$$\nabla^2u(\vec{x}\,)=-f(\vec{x}\,)\ \ ,\ \ \vec{x}\in R\ \ ,\ \ u(\vec{x}\,)=g(\vec{x}\,)\ \text{ for all }\ \vec{x}\in S.$$

It has a unique solution.

The *Neumann problem* is:

$$\nabla^2u(\vec{x}\,)=-f(\vec{x}\,)\ \ ,\ \ \vec{x}\in R\ \ ,\ \ \frac{\partial u(\vec{x}\,)}{\partial n}=h(\vec{x}\,)\ \text{ for all }\ \vec{x}\in S.$$

The solution is unique except for a constant. The solution exists if:

$$-\iiint\limits_R f(\vec{x})d^3V = \oiint\limits_S h(\vec{x})d^2A$$

A *fundamental solution* of the Laplace equation satisfies:

$$\nabla^2 u(\vec{x}) = -\delta(\vec{x})$$

This has in 2 dimensions in polar coordinates the following solution:

$$u(r) = \frac{\ln(r)}{2\pi}$$

This has in 3 dimensions in spherical coordinates the following solution:

$$u(r) = \frac{1}{4\pi r}$$

The equation $\nabla^2 v = -\delta(\vec{x} - \vec{\xi})$ has the solution

$$v(\vec{x}) = \frac{1}{4\pi|\vec{x} - \vec{\xi}|}$$

After substituting this in Green's 2nd theorem and applying the sieve property of the δ function one can derive Green's 3rd theorem:

$$u(\vec{\xi}) = -\frac{1}{4\pi}\iiint\limits_R \frac{\nabla^2 u}{r} d^3V + \frac{1}{4\pi}\oiint\limits_S \left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] d^2A$$

The *Green function* $G(\vec{x}, \vec{\xi})$ is defined by: $\nabla^2 G = -\delta(\vec{x} - \vec{\xi})$, and on boundary S holds $G(\vec{x}, \vec{\xi}) = 0$. Than G can be written as:

$$G(\vec{x}, \vec{\xi}) = \frac{1}{4\pi|\vec{x} - \vec{\xi}|} + g(\vec{x}, \vec{\xi})$$

Than $g(\vec{x}, \vec{\xi})$ is a solution of Dirichlet's problem. The solution of Poisson's equation $\nabla^2 u = -f(\vec{x})$ when on the boundary S holds: $u(\vec{x}) = g(\vec{x})$, is:

$$u(\vec{\xi}) = \iiint\limits_R G(\vec{x}, \vec{\xi}) f(\vec{x}) d^3V - \oiint\limits_S g(\vec{x})\frac{\partial G(\vec{x}, \vec{\xi})}{\partial n} d^2A$$