Homework #6

Instructions: There are 4 questions, each worth 5 points. Please upload solutions to Gradescope. These solutions can either be handwritten or typeset using something like LaTeX. In particular, LaTeX is a commonly used tool in physics, so you may find it to be a useful thing to learn. To encourage this, any assignment that is typeset instead of handwritten will get +1 as a bonus point (out of 20 for the full problem set).

You are strongly encouraged to work collaboratively and use all tools available to understand how to solve the problems. However you arrive at your understanding of how to do the problem, it is essential that your solution is independent work, not copied or generally plagiarizing the work of others. Plagiarism is a serious offence.

You can use computer algebra things like Mathematica, or Wolfram Alpha, or online integral sites, or anything else that helps.

Q 1
A lightly damped mass-spring system (with damping force $F_D = -cv$) is driven near resonance. Consider parameters $c = 6 \text{ kg/s}$, $k = 250 \text{ N/m}$, and $m = 10 \text{ kg}$ acted on by the force $F = 10 \sin(5t) \text{ N}$;

a) Write down the steady-state response to this forcing.

b) Write down the total solution (including the homogeneous solution) that satisfies the initial condition $x_0 = 2m, v_0 = 0.2m/s$ at $t = 0$).

Q 2
The canonical lightly damped mass-spring system that we have been discussing in lectures has the equation

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = \frac{F_o}{m} \cos(\omega_F t) \quad (1)$$

Take the amplitude of the oscillation at very low driving frequencies to be $A_o$. In units of $A_o$, plot the amplitude of the steady-state response (ignoring the cos modulation in $x(t)$, just use the amplitude) as a function of $\omega_F/\omega_o$ (over a range of 0.1 to 10) for varying quality factors ($Q \equiv \omega_o/\gamma$) of the oscillator, $Q = [2, 8, 16]$.

Q 3
The canonical lightly damped mass-spring system that we have been discussing in lectures has the equation

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = \frac{F_o}{m} \cos(\omega_F t) \quad (2)$$

The steady state response $x(t)$ is proportional to $\cos(\omega_F t - \phi)$, where $\phi$ is a phase shift. Plot $\phi$ as a function of $\omega_F/\omega_o$ (over a range of 0.1 to 10) for varying quality factors ($Q \equiv \omega_o/\gamma$) of the oscillator, $Q = [2, 8, 16]$.

Q 4
In this question we will get some practice on Fourier expansions to see how the convergence works. For each of the following periodic patterns, calculate the Fourier series expansion and plot the cumulative expansion for $n = 1, 2, 3, 4, 5$ (i.e., show the sum of terms up to $n = 1$, then sum of terms up to $n = 2$, then sum of terms up to $n = 3$, etc.). The idea is to make plots that look like the top of [http://mathworld.wolfram.com/FourierSeries.html](http://mathworld.wolfram.com/FourierSeries.html). If you want to add more terms, go for it.

(i) Square wave, where $f(t) = 1$ for $0 < t < T/2$ and $f(t) = -1$ for $T/2 < t < T$

(ii) Parabolic wave, where a periodic function has $f(t) = (t/T)^2$ for $-T/2 < t < T/2$

Hints: For the first one, the mathworld page will be helpful, for the second, try the Fourier series information on [math24.net](http://math24.net)