Phys 325: Midterm #2

Nov 14, 2019

There are 5 questions, each of which is worth an equal number of points. Feel free to take the question sheet with you, it will not be used for grading.

*Please use one separate booklet for each question.* Only the first question in any 4-page booklet will be graded (no exceptions!)

Several pages of formula sheets are attached. In addition, everyone is allowed a single piece of 8.5”x11” paper of handwritten notes. You can write on as many sides as you’d like.
1. A simple pendulum of mass \( m \) and length \( l \) is hanging from the roof of a car. Initially, the pendulum is hanging straight down, in a uniform gravitational field \( g \), and the car is at rest. The car then accelerates uniformly in the horizontal direction with acceleration \( \ddot{a} = a \hat{x} \).

   a) While the car accelerates in the \( +\hat{x} \) direction, what is the new equilibrium angle around which it oscillates?

   b) While the car accelerates in the \( +\hat{x} \) direction, what is the oscillation frequency of the pendulum? Assume that the oscillations around the equilibrium are small, but the equilibrium angle itself is not necessarily small.

2. A mass \( m \) is on a spring with spring constant \( k \) in the presence of a velocity-dependent force \( \vec{F} = -c\vec{v} \). The motion is constrained to be only in the \( \pm \hat{x} \) direction. The damping force is weak, such that \( c^2 << mk \) (i.e., the system is lightly damped).

   a) Show that the motion is described by \( \ddot{x} + \gamma \dot{x} + \omega_0^2 x \) for appropriate definitions of \( \gamma \) and \( \omega_0 \).

   b) At time \( t = 0 \), the system has \( x(0) = x_o \) and \( v(0) = 0 \). What is \( x(t) \)?

   c) At time \( t = 0 \), the system has \( x(0) = 0 \) and \( v(0) = v_o \). What is \( x(t) \)?

3. A lightly damped oscillator has natural frequency \( \omega_0 \) and a damping constant \( \gamma \). When not driven by an external force, it can be described by an equation of motion \( \ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \). The system has a force applied such that

   \[
   \vec{F}(t) = \begin{cases} 
   F_0\hat{x} & -5t_o < t < -2t_o \text{ time period A} \\
   0 & -2t_o < t < 2t_o \text{ time period B} \\
   -F_0\hat{x} & 2t_o < t < 5t_o \text{ time period C} \\
   0 & t > 5t_o \text{ time period D}
   \end{cases}
   \]

   The oscillator is at rest at \( t = -\infty \).

   a) What is \( x(t) \) during time period A (i.e., the first interval over which the force is being applied)?

   b) What is \( x(t) \) during time period C (i.e., the second interval over which the force is non-zero)?

   It is recommended that you use Green’s functions for these questions, and you do not need to explicitly evaluate any integrals. You should leave integral solutions in a form that would be easy to hand to a computer algebra system.

4. A lightly damped oscillator, with natural frequency \( \omega_0 \) and a damping constant
\( \gamma \), has a force applied such that \( F(t) = F_0 e^{-|t|/\tau_0} \), formally extending to \( \pm \infty \) in time, although it is exponentially small at large values of \( |t| \).

a) By directly doing the integral (i.e., starting from the definition of the Fourier transform), show that the Fourier transform of \( F(t) \) is

\[
\tilde{F}(\omega) = F_0 \frac{2\tau_0}{1 + \tau_0^2 \omega^2}
\]

as can be verified by using the appropriate Fourier transform pair given near the end of the formula sheet.

b) Find the Fourier transform \( \tilde{x}(\omega) \) of the steady-state response \( x(t) \) to the force and write down an integral expression for \( x(t) \). There is no need to try to explicitly evaluate the integral.

5. A baseball is launched to the north at angle \( \phi \) relative to the horizontal with initial speed \( v_0 \). We will calculate the effect of the Coriolis force using an iterative approach. For this calculation, take \( \hat{z} \) to be the vertical direction, and assume the horizontal motion of the ball (to the north) is in the \( \hat{y} \) direction, with \( \hat{x} \) pointing east. Assume that we are playing baseball at a latitude (i.e., measured from the equator) \( \lambda \) degrees. You should assume a uniform gravitational field \( \tilde{g} = -g \hat{z} \) and ignore the centrifugal acceleration.

a) Ignore the rotation of the Earth and for that case show that the resulting motion is

\[
[x(t), y(t), z(t)] = [0, v_0 t \cos(\phi), v_0 t \sin(\phi) - gt^2/2]
\]

We will call this solution the zeroth order solution.

b) Use the zeroth order solution to calculate the approximate equations of motion for the baseball in the presence of Earth rotation, where the Earth rotates around the north pole at angular speed \( \Omega \). You can ignore the centrifugal effect.

c) Integrate the equations of motion found in b) to find the displacement in the \( x \) direction (recall that positive \( x \) is to the east), assuming that the time of flight for the baseball is the same as for the zeroth order solution.

d) The baseball is launched at roughly 45° relative to the horizontal at mid-latitudes (\( \lambda \sim 45^\circ \), roughly correct for the now world-champion Montreal Expos), with an initial speed of 100 mph (45 m/s). What would be the right order of magnitude for the amount of deflection due to the Coriolis effect? (i.e., would this be a few microns or less, a bit less than a mm, a few cm, or a few m?). Answers just written down with no quantitative justification will not be given credit.
Useful formulae

Momentum of a particle:
\[ \vec{p} = m \frac{d\vec{r}}{dt} \]

Force on a particle:
\[ \vec{F} = \frac{d\vec{p}}{dt} \]

Center of mass:
\[ \vec{r}_{cm} = \frac{1}{M} \int \vec{r} dm = \frac{1}{M} \sum_i m_i \vec{r}_i \]

Change in momentum of center of mass in presence of external forces \( F_i \):
\[ \frac{d\vec{p}_{cm}}{dt} = \sum_i \vec{F}_i \]

Cylindrical coordinates:
\[ x = R \cos \phi \]
\[ y = R \sin \phi \]
\[ z = z \]
\[ R = \sqrt{x^2 + y^2} \]
\[ \vec{r} = \vec{\rho} + y \hat{j} + z \hat{k} \]
\[ \vec{R} = \cos \phi \hat{\rho} + \sin \phi \hat{j} \]
\[ \hat{\phi} = -\sin \phi \hat{\rho} + \cos \phi \hat{j} \]
\[ \frac{d\vec{R}}{dt} = \dot{\phi} \hat{\rho} \]
\[ \frac{d\hat{\phi}}{dt} = -\dot{R} \hat{\phi} \]
\[ \vec{v} = \vec{\dot{R}} \cdot \vec{R} + R \dot{\phi} \hat{\rho} + \ddot{z} \hat{k} \]
\[ \vec{a} = (\ddot{R} - R \ddot{\phi}^2) \hat{\rho} + (R \ddot{\phi} + 2 \dot{R} \dot{\phi}) \hat{j} + \ddot{z} \hat{k} \]
\[ \nabla^2 f = \frac{1}{R} \frac{\partial}{\partial R} \left( R^2 \frac{\partial f}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \]

Spherical coordinates (3D):
\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]
\[ \vec{r} = \hat{x} \hat{i} + y \hat{j} + z \hat{k} \]
\[ \hat{\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \]
\[ \hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j} \]
\[ \frac{d\vec{r}}{dt} = \dot{\theta} \hat{i} \sin \phi + \dot{\phi} \hat{j} + \dot{R} \hat{R} + R \dot{\phi} \hat{\phi} \]
\[ \frac{d\hat{\theta}}{dt} = \dot{R} \hat{R} + R \dot{\phi} \hat{\phi} \]
\[ \frac{d\hat{\phi}}{dt} = -\dot{R} \hat{\phi} \]

Curl of a vector field:
\[ \nabla \times \vec{F} = \hat{i} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{j} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{k} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]

Stokes’ Theorem, which turns a line integral around a closed path into an integral over the enclosed area, with \( d\vec{A} \) a vector pointing in the direction orthogonal to the area (for example, a
path that lies in the $x$-$y$ plane gets turned into an area integral with $d\vec{A}$ pointing in the $z$ direction, which picks out the $z$ component of the curl):

$$\int \vec{F} \cdot d\vec{r} = \int (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} \quad (32)$$

Simple harmonic motion:

$$U(x) = \frac{1}{2}k(x - x_o)^2 + constant \quad (33)$$
$$F(x) = -k(x - x_o) \quad (34)$$
$$\omega = \sqrt{\frac{k}{m}} \quad (35)$$

If $\ddot{x} = -\omega^2 x$ then $x = A \sin(\omega t + \phi) \quad (36)$

Taylor expansion:

$$f(x) \sim f(x_o) + \frac{df}{dx}(x_o)(x - x_o) + \frac{1}{2!} \frac{d^2 f}{dx^2}(x_o)(x - x_o)^2 + \frac{1}{3!} \frac{d^3 f}{dx^3}(x_o)(x - x_o)^3 + \ldots \quad (37)$$

Rocket-related useful equation (for exhaust relative velocity $\vec{u}$; careful with signs!):

$$\frac{d\vec{F}}{dt} = M \frac{d\vec{v}}{dt} + \vec{u} \frac{dm}{dt} \quad (38)$$

Center of mass frame scattering (relative velocity $\vec{V}$):

$$\vec{V} = \vec{v}_1 - \vec{v}_2 \quad (39)$$
$$\vec{p}_1 = \mu \vec{V} = -\vec{p}_2 \quad (40)$$
$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (41)$$

Potential energy:

Constant force : $U(x) = -Fx + C \quad (42)$

Spring : $U(x) = \frac{1}{2}kx^2 + C \quad (43)$

$1/r^2$ Gravity : $U(x) = -GMm/r + C \quad (44)$

Some useful approximations (for small arguments to first order):

$$\sin \theta \sim \theta - \ldots \quad (45)$$
$$\cos \theta \sim 1 - \ldots \quad (46)$$
$$\ln(1 + x) \sim x^2 + \ldots \quad (47)$$
$$e^x \sim 1 + x + \ldots \quad (48)$$
$$e^{\alpha x} \sim 1 + \alpha x + \ldots \quad (49)$$

Central force motion effective potential for reduced mass $\mu$ and angular momentum $\ell$ and potential $U(r)$:

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2} \quad (50)$$

Orbit in a potential $U = -C/r$:

$$r(\theta) = \frac{(\ell^2/\mu C)}{1 - \sqrt{1 + (2E\ell^2/\mu C^2) \sin(\theta - \theta_o)}} \quad (51)$$

Taking $\theta_o = -\pi/2$, and using the following definitions:

$$r = r_o \quad (52)$$
$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu C^2}} \quad (53)$$
$$r_o = \frac{\ell^2}{\mu C} \quad (54)$$

For a bound orbit, this is an ellipse, with the length of the longest axis $A$ (all the way across i.e., the thing that would be the diameter, not the radius, for a circle) and the short axis $B$:

$$A = \frac{2r_o}{1 - \epsilon^2} = \frac{C}{-E} \quad (55)$$
$$B = \frac{2\ell}{\sqrt{-2E}} \quad (56)$$

The longest and shortest distances from the central object:

$$r_{max} = \frac{r_o}{1 - \epsilon} \quad (57)$$
$$r_{min} = \frac{r_o}{1 + \epsilon} \quad (58)$$
Kepler’s third law relating period $T$ to length of long axis $A$:

$$T^2 = \frac{\pi^2}{2(M + m)G} A^3 \quad (59)$$

Vis-viva equation for Keplerian orbits:

$$v^2 = GM_{\text{rot}} \left( \frac{2}{r} - \frac{1}{a} \right) \quad (60)$$

Poisson’s equation:

$$\nabla^2 \Phi = 4\pi G \rho \quad (61)$$

Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\omega t) + b_n \sin(n\omega t) \right] \quad (62)$$

where $\omega \equiv \frac{2\pi}{T}$

$$a_n = \frac{2}{T} \int_{0}^{T} f(t) \cos(n\omega t) dt \quad (64)$$

$$b_n = \frac{2}{T} \int_{0}^{T} f(t) \sin(n\omega t) dt \quad (65)$$

Fourier transforms:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad (66)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega \quad (67)$$

Lightly damped harmonic motion (for $\phi$ and $A$ set by initial conditions):

$$\ddot{x} = -\gamma \dot{x} - \omega_0^2 x \quad (68)$$

$$x(t) = Ae^{-\gamma t/2} \cos(\omega' t + \phi) \quad (69)$$

$$\omega' \equiv \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \quad (70)$$

Forced damped harmonic motion: steady state solution (initial conditions are irrelevant after a sufficiently long time):

$$\ddot{x} = -\gamma \dot{x} - \omega_0^2 x + \frac{F_0}{m} \cos(\omega_F t) \quad (71)$$

$$x_{ss}(t) = \frac{F_0 \cos(\omega_F t - \phi)}{m \sqrt{(\omega_0^2 - \omega_F^2)^2 + \omega_F^2 \gamma^2}} \quad (72)$$

$$\tan(\phi) = \frac{\omega_F \gamma}{\omega_0^2 - \omega_F^2} \quad (73)$$

Green’s function for damped oscillator, where $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$:

$$G(t - \tau) = \frac{\sin[\omega'(t - \tau)]}{m\omega'} e^{-\gamma(t-\tau)/2} H(t - \tau) \quad (74)$$

$$|\tilde{G}(\omega)| = \frac{1}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} \quad (75)$$

$$x(t) = \int_{-\infty}^{t} F(\tau) G(t - \tau) d\tau \quad (76)$$

Fictitious force: uniform acceleration $\ddot{a}$ for mass $m$:

$$\ddot{F}_{\text{fict}} = -m \ddot{a} \quad (77)$$

Coriolis force, for system rotating with angular velocity $\Omega$ and for a velocity measured in the rotating frame of $\vec{v}_{\text{rot}}$:

$$\ddot{F}_{\text{cor}} = -2m \vec{\Omega} \times \vec{v}_{\text{rot}} \quad (78)$$

Centrifugal force, for system rotating with angular velocity $\Omega$:

$$\ddot{F}_{\text{cent}} = -m \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \quad (79)$$

Relation between velocities measured in rotating and inertial frames:

$$\left( \frac{d\vec{B}}{dt} \right)_{\text{inertial}} = \left( \frac{d\vec{B}}{dt} \right)_{\text{rotating}} + \vec{\Omega} \times \vec{B} \quad (80)$$

A few trig identities:

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \quad (81)$$

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \quad (82)$$

Some more calculus:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\vec{v}} \cdot \nabla f \quad (83)$$

$$\ddot{x} = \frac{1}{2} \frac{d}{dx} \chi^2 \quad (84)$$
Some integrals:

\[ \int \frac{dx}{x + a} = \ln(x + a) + C \]  
\[ (85) \]

\[ \int e^x \, dx = e^x + C \]  
\[ (86) \]

\[ \int \ln x \, dx = x \ln x - x + C \]  
\[ (87) \]

\[ \int \sin x \, dx = -\cos x + C \]  
\[ (88) \]

\[ \int \cos x \, dx = \sin x + C \]  
\[ (89) \]

\[ \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \]  
\[ (90) \]

\[ \int \frac{xdx}{\sqrt{x^2 + a^2}} = \sqrt{a^2 + x^2} \]  
\[ (91) \]

\[ \int \frac{dx}{(a + x)^n} = \frac{(a + x)^{1-n} + C}{1 - n} \]  
\[ (92) \]

\[ \int \frac{xdx}{(x + a)^3} = -\frac{a + 2x}{2(a + x)^2} \]  
\[ (93) \]

\[ \int_{-\infty}^{\infty} f(t) \delta(t - a) \, dt = f(a) \]  
\[ (94) \]

\[ \int_{-\infty}^{\infty} f(t) \delta(a - t) \, dt = f(a) \]  
\[ (95) \]

Some Fourier transform pairs:

\[ \int_{-\infty}^{\infty} e^{-i\omega t} e^{iat} \, dt = 2\pi \delta(\omega - a) \]  
\[ (96) \]

\[ \int_{-\infty}^{\infty} e^{-i\omega t} e^{-at^2} \, dt = \sqrt{\frac{\pi}{a}} e^{-\omega^2/(4a)} \]  
\[ (97) \]

\[ \int_{-\infty}^{\infty} e^{-i\omega t} e^{-a|t|} \, dt = \frac{2a}{a^2 + \omega^2} \]  
\[ (98) \]

\[ \int_{-\infty}^{\infty} e^{-i\omega t} \cos(at^2) \, dt = \sqrt{\frac{\pi}{a}} \cos\left(\frac{\omega^2}{4a} - \frac{\pi}{4}\right) \]  
\[ (99) \]

\[ \int_{-\infty}^{\infty} e^{-i\omega t} \sin(at^2) \, dt = -\sqrt{\frac{\pi}{a}} \sin\left(\frac{\omega^2}{4a} - \frac{\pi}{4}\right) \]  
\[ (100) \]