effects are most pronounced by performing multivariable extremizations. And why not include centrifugal acceleration?

Example 23.19: Pendulum Oscillation

a) Find the equation of motion for a simple pendulum of mass $m$ and length $l$.

b) Find the equation of motion for a physical pendulum of mass $m$ and moment of inertia $I_0$ through its center of motion around a particular axis if it is oscillating about a parallel axis $l$ away from its center of mass. From this equation, read off the frequency of small oscillations.

a) The Lagrangian is

$$ L = \left[ \frac{1}{2} ml^2 \dot{\theta}^2 \right] - \left[ mgl \left( 1 - \cos \theta \right) \right], $$

and so the equation of motion is

$$ \frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \Rightarrow -mgl \sin \theta = ml^2 \ddot{\theta} \Rightarrow \ddot{\theta} = -\frac{g}{l} \sin \theta. $$

b) The Lagrangian is

$$ L = \left[ \frac{1}{2} \left( I_0 + ml^2 \right) \dot{\theta}^2 \right] - \left[ mgl \left( 1 - \cos \theta \right) \right], $$

and so the equation of motion is

$$ -mgl \sin \theta = \left( I_0 + ml^2 \right) \ddot{\theta}. \Rightarrow \ddot{\theta} = -\frac{g/l}{1 + I_0/ml^2} \sin \theta. $$

And so for small oscillations, we see that the

$$ w = \sqrt{\frac{g/l}{1 + I_0/ml^2}}, $$

Example 23.20: An Escape Trajectory

A rocket is launched from the earth such that it ejects mass at a rate proportional to its mass with a fixed exhaust speed $w$. Find its speed as a function of distance from the center of the earth.

The so-called rocket equation is

$$ F_{ext} = m \frac{dv}{dt} - v_{rel} \frac{dm}{dt}, $$
where
\[
F_{\text{ext}} = -G \frac{M E m(t)}{r^2}
\]
\[
\frac{dm}{dt} = -\gamma m(t)
\]
\[
v_{\text{rel}} = -w \dot{r}.
\]

And so the equation of motion for the rocket becomes
\[
-G \frac{M_{E}}{r^2} = \ddot{r} - \gamma w.
\]

Using the trick
\[
\ddot{r} = \frac{1}{2} \frac{d}{dr} \dot{r}^2,
\]
to rewrite the equation of motion and integrating from \(R_0\) where \(v(R_0) = 0\) to \(R\) where \(v(R) = v\), we get that
\[
\frac{1}{2} \dot{v}^2 = \gamma w (R - R_E) - \frac{G M_{E}}{R R_E} (R - R_E)
\]
\[
v = \sqrt{2 (R - R_E) \left[ \gamma w - \frac{G M_{E}}{R R_E} \right]}.
\]

**Example 23.21: Green Function for a Damped Oscillator**

Suppose a lightly damped oscillator characterized by mass, natural frequency, and damping \(m, w_0,\) and \(\gamma,\) respectively. If it starts at rest and is subject to the time dependent force
\[
F(\tau) = \begin{cases} 
F_0 (1 + t/t_0) & -t_0 < t < 0 \\
F_0 (1 - t/t_0) & 0 < t < t_0 \\
0 & \text{otherwise}
\end{cases}
\]

use Green’s functions to obtain \(x(t)\) for

- a) \(-t_0 < t < 0\)
- b) \(0 < t < t_0\)
- c) \(t > t_0\)
  - a) For this question, we need only plug our force into the formula,
\[
x(t) = \int_{-\infty}^{\infty} F(\tau) G(t - \tau) d\tau.
\]

where the Green function is
\[
G(t - \tau) = \frac{\sin (w (t - \tau))}{m w} e^{-\gamma (t-\tau)^2/2} \theta (t - \tau).
\]
For the first time interval, this gives

\[ x(t) = \frac{F_0}{mw} \int_{t_0}^{t} \left(1 + \frac{t}{t_0}\right) \sin \left(w(t - \tau)\right) e^{-\gamma(t-\tau)/2} d\tau. \]

b) For the second time interval, the integral includes the preceding time interval and so the answer has the following two pieces.

\[ x(t) = \frac{F_0}{mw} \int_{-t_0}^{0} \left(1 + \frac{t}{t_0}\right) \sin \left(w(t - \tau)\right) e^{-\gamma(t-\tau)/2} d\tau + \frac{F_0}{mw} \int_{0}^{t} \left(1 - \frac{t}{t_0}\right) \sin \left(w(t - \tau)\right) e^{-\gamma(t-\tau)/2} d\tau. \]

c) For the final interval, the integral gives.

\[ x(t) = \frac{F_0}{mw} \int_{-t_0}^{0} \left(1 + \frac{t}{t_0}\right) \sin \left(w(t - \tau)\right) e^{-\gamma(t-\tau)/2} d\tau + \frac{F_0}{mw} \int_{0}^{t_0} \left(1 - \frac{t}{t_0}\right) \sin \left(w(t - \tau)\right) e^{-\gamma(t-\tau)/2} d\tau. \]

Example 23.22: Motion in a Gravitational Field

Suppose you are given a mass density

\[ \rho = \rho_0 \left(\frac{r}{r_0}\right)^{-\alpha} \]

for \( 0 < \alpha < 3 \).

a) Show that the gravitational potential for \( \alpha \neq 2 \) is

\[ \Phi(r) = \frac{4\pi G \rho_0 r_0^2}{(3 - \alpha)(2 - \alpha)} \left(\frac{r}{r_0}\right)^{2-\alpha} \]

b) Write the Lagrangian for a mass \( m \) that moves under the influence of this potential.

c) What can you say about the conserved quantities of this system?

a) \( \nabla^2 \Phi = \frac{1}{r^2} \partial_r \left[ r^2 \partial_r \left[ \frac{4\pi G \rho_0}{(3 - \alpha)(2 - \alpha)} \left(\frac{r}{r_0}\right)^{2-\alpha} \right]\right] \]

\[ = \frac{4\pi G \rho_0}{(3 - \alpha)(2 - \alpha)} \frac{1}{r_0^{1-\alpha}} \frac{1}{r_2} \partial_r \left[ r^2(2 - \alpha)r^{1-\alpha}\right] \]

\[ = \frac{4\pi G \rho_0}{(3 - \alpha)} \frac{1}{r_0^{1-\alpha}} \frac{1}{r^2(3 - \alpha)r^{2-\alpha}} = 4\pi G \rho_0 \left(\frac{r}{r_0}\right)^{-\alpha} = 4\pi G \rho(r) \]
b) Spherical coordinates is most appropriate, and so the Lagrangian is given as

\[ L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - \frac{4\pi G m \rho r_0^2}{(3 - \alpha)(2 - \alpha)} \left( \frac{r}{r_0} \right)^2. \]

c) Energy is conserved because the potential is constant in time. Angular momentum is conserved because there is no torque. There is no torque because \( \mathbf{r} \times \mathbf{F} = 0 \) due to the central nature of the force.

**Example 23.23: Coriolis Influenced Drop**

Suppose a ball is dropped from a vertical tower of height \( h \) at latitude \( \lambda \).

a) Using the solution for the ball’s motion in the absence of any effects due to the earth’s rotation, write down the approximate equations of motion for the ball in the presence of the Coriolis acceleration.

b) By solving these approximate equations of motion, calculate the deflection of the ball due to the Coriolis acceleration.

a) The naive answer to the question neglects all rotation of the earth, and is

\[ \ddot{z} = -g \quad \dot{z} = -gt \quad z = h - \frac{1}{2} gt^2. \]

Using this velocity helps us to compute the Coriolis acceleration

\[ \mathbf{a}_{\text{coriolis}} = -2\Omega \times \mathbf{v}_\text{rot}, \quad \Omega = \Omega \hat{z}, \quad \mathbf{v}_\text{rot} = -gt\hat{r}. \]

All of this together gives

\[ \mathbf{a}_{\text{coriolis}} = -2\Omega gt \cos \lambda \hat{x}. \]

b) Integrating this equation of motion gives us the deflection to be

\[ x(t) = -\frac{1}{3} \Omega \sqrt{\frac{8h^3}{g}} \cos \lambda. \]

**Example 23.24: Bead Sliding on a Rotating Hoop**

A vertical hoop of radius \( R \) rotates with constant angular speed \( \Omega \) about a vertical axis with a bead that is free to slide along it.

a) Find the equation of motion for the bead.

b) Identify equilibrium points and investigate the conditions for the stability of solutions at these points.
a) The Lagrangian is

\[ L = \left[ \frac{1}{2} m R^2 \left( \dot{\theta}^2 + \Omega^2 \sin^2 \theta \right) \right] - [mgR (1 - \cos \theta)] \]

\[ m R^2 \sin \theta \left( \Omega^2 \cos \theta - \frac{g}{R} \right) = m R^2 \ddot{\theta} \implies \ddot{\theta} = \sin \left( \Omega^2 \cos \theta - \frac{g}{R} \right) \]

b) If there is an equilibrium, \( \ddot{\theta} = 0 \), and so the equation of motion implies it must occur at

\[ \theta = 0, \pi, \arccos \left( \frac{\Omega_0^2}{\Omega^2} \right), \]

where \( \Omega_0 = \sqrt{g/R} \). To investigate the stability, we look at the effective potential

\[ U_{\text{eff}} = -m R^2 \Omega_0^2 \left( \frac{1}{2} \Omega^2 - \frac{1}{2} \Omega_0^2 \right) \sin^2 \theta + \cos \theta - 1 \]

which is plotted in Figure 23.5. Analytically, we can look at the second derivative

\[ U''_{\text{eff}} = -m R^2 \Omega_0^2 \left( \frac{2}{\Omega_0^2} \cos \theta - 1 \right) - \cos \theta \]

This is always negative for \( \theta = \pi \), and surprisingly turns negative for \( \theta = 0 \) if \( \Omega > \Omega_0 \). In this case, the bead spins out from the bottom to \( \theta = \arccos \left( \frac{\Omega_0^2}{\Omega^2} \right) \). This equilibrium only exists if \( \Omega > \Omega_0 \), and for this specific \( \theta \), the effective potential is

\[ U''_{\text{eff}} = -m R^2 \Omega_0^2 \left( \frac{\Omega_0^2}{\Omega^2} \left( 2 \cos^2 \theta - 1 \right) - \cos \theta \right) \]

which is positive for \( \Omega > \Omega_0 \) implying a stable equilibrium.

**Example 23.25: Corrected Orbits**

General relativity suggests that the potential energy of interactions is approximately

\[ U = -\frac{GM_1m}{r - r_g} \]

Consider \( m \) to be some small mass.

a) What is the orbital frequency for circular orbits at radius \( R \)?

b) Find the effective potential relevant for this scenario. Sketch it for two different values of angular momentum \( l \).

c) Find a relation between the radius \( R \) of a circular orbit and the angular momentum it must possess.

d) Show that circular orbits are stable for \( r > 3r_g \) and unstable for \( r < 3r_g \).
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a) 

\[ L = \left[ \frac{1}{2} m \ddot{r}^2 + \frac{1}{2}mr^2 \dot{\phi}^2 \right] - \left[ \frac{GM_1 m}{r - r_g} \right] \]

The centripetal force associated with this potential can be used in conjunction with Newton’s second law to find the orbital frequency of circular orbits.

\[ mrw^2 = \frac{GM_1 m^2}{(r - r_g)} \implies w = \sqrt{\frac{GM_1}{r (r - r_g)^2}}. \]

The angular Euler-Lagrange equation gives us that angular momentum is conserved.

\[ l = mr^2 \dot{\phi} \]

This can be used in the radial equation as follows.

\[ m\ddot{r} = -\frac{GM_1 m}{(r - r_g)^2} + mr^2 \ddot{\phi} = -\frac{GM_1 m}{(r - r_g)^2} + \frac{l^2}{mr^3} \]

\[ = -\nabla \left( -\frac{GM_1 m}{r - r_g} + \frac{l^2}{2mr^2} \right). \]

b) This gives the effective potential

\[ U_{\text{eff}} = -\frac{GM_1 m}{r - r_g} + \frac{l^2}{2mr^2}, \]

which is plotted in Figure 23.6.

c) One natural question that arises in the context of gravitational potentials is the existence of stable, bound orbits. We have the effective potential at hand, so we are in a position to study this. The answer will obviously depend on the angular momentum \( l \). First, observe that the existence of a stable and bound orbit for an orbit of a particular angular momentum necessarily implies that there is a region in the effective potential that is bowl-shaped, i.e. possessing a positive second derivative. If that is the case, it must be that there is a minimum in the potential which would correspond to the existence of a stable circular orbit. Conversely, the existence of a stable circular orbit also proves the existence of more general bound orbits. Therefore, to understand bound orbits in this potential, it is instructive to consider the special case of circular orbits. For circular orbits where \( r = R \), the radial equation simplifies to give

\[ \frac{l^2}{mR^3} = \frac{GM_1 m}{(R - r_g)^2}. \]
d) This can be used to simplify the second derivative of the potential,

\[ U''_{\text{eff}} = -\frac{2GM_1m}{(r - r_g)^3} + \frac{3l^2}{mr^4} \]

\[ = \frac{GM_1m}{(R - r_g)^2} \left( -\frac{2}{R - r_g} + \frac{3}{R} \right). \]

This is clearly negative for \( R \approx r_g \), and setting it equal to zero tells us where it transitions to being positive, which is at

\[ R = 3r_g. \]

And so there are stable orbits for \( R > 3r_g \) but none for \( R < 3r_g \). The orbit at \( R = 3r_g \) is called the **Innermost Stable Circular Orbit**, or the ISCO.

We can plug this into the simplified radial equation to get the value of the angular momentum at the ISCO in terms of \( r_g \),

\[ l^2_{\text{ISCO}} = \frac{27}{4} GM_1m^2r_g. \]

This expression is necessary to produce Figure 23.6.

### References