Unit 7: Momentum and position

After this unit, you should be able to

- Explain whether a wave function has definite momentum.
- From a superposition of eigenstates, compute the probability of a measurement outcome.
- Apply the Heisenberg Uncertainty Principle to determine the limits of what can be predicted about measurement outcomes.

Wave function of a particle with definite momentum

We can explain the electron double slit experiment (and many other similar experiments) if we assume that the wave function of a particle with momentum p[1] is:

$$\Psi(x,t) = Ae^{i(kx-\omega t)}, \tag{1}$$

where $k = p/\hbar$. We don't yet know how to determine ω (we will do that in Unit 10). As a simplification, if we suppose that the particle is in a large box of length L, then $A = \sqrt{\frac{1}{L}}$ from the normalization condition. So for the moment, let's just consider the wave function at a given time $t = 0$, so

$$\Psi(x) = Ae^{ikx}. \tag{2}$$

The wave function tells us the probability of measuring quantities. This is one of the very fundamentally different things about the quantum mechanical description as opposed to the classical mechanics description.

For the wave function in Eqn 2, if we measure the momentum, say by measuring the change in momentum the particle imparts on some other thing when it impacts it, we will always find that the particle has momentum p. On the other hand, if we measure the location of the particle, then we will find that it can be anywhere within that large box, since

$$\rho(x) = \Psi(x)\Psi^*(x) = |A|^2. \tag{3}$$

This is referred to as *uncertainty* in quantum mechanics. For this particular wave function, we can predict the outcome of a measurement of the momentum very precisely (it's always p), but we cannot predict the outcome of a measurement of the position very precisely. We would find it somewhere in the box, but it's completely random where we will find it!

[1] In quantum mechanics, we work with momentum rather than velocity. Part of the reason is that the math is easier this way. Another reason is that momentum is actually more fundamental to physics than velocity. For example, the more general formula for Newton's first law is $F = \frac{dp}{dt}$.

Eigenstates

The wave function in Eqn 2 is called an *eigenstate*[2] of momentum. This means that when we measure the momentum of the particle with that wave function, we can predict that we will obtain one particular value of the momentum, $\hbar k$. We will also sometimes say that such a wave function has **definite** momentum.

One can also have eigenstates of position, which are wave functions that are only non-zero in one location. The mathematics of this is a little bit beyond this course,[3] so we will not cover this. In later units, we will learn how to find *energy eigenstates*, which are very important in quantum mechanics.

Superposition of wave functions

Suppose we have a particle which we confine to a box of side L with wave function

$$\Psi(x) = A\left(e^{ik_1 x} + e^{ik_2 x}\right) \tag{4}$$

with $k_1 = \frac{2\pi}{L}$ and $k_2 = \frac{4\pi}{L}$. What will happen when we measure the momentum of a particle with this wave function? Since the wave function is an equal superposition of two wave functions with different momenta, we have an equal probability of measuring $\hbar k_1$ and $\hbar k_2$. This is not a momentum eigenstate since two different momenta could be measured.

In general, if the wave function is given by

$$\Psi(x) = A\left(ae^{ik_1 x} + be^{ik_2 x}\right), \tag{5}$$

we will obtain $\hbar k_1$ with probability $\frac{a^2}{a^2+b^2}$ and $\hbar k_2$ with probability $\frac{b^2}{a^2+b^2}$. The denominator $(a^2 + b^2)$ ensures that the probabilities add to 1 (normalization).

What about measuring the position of the particle? From Fig 1, we can see that as we add together more momentum eigenstates, the position is more likely to be measured near the middle of the box than near the edges. So this wave function has more uncertainty in momentum than the momentum eigenstate, but less uncertainty in position.

Heisenberg uncertainty principle

It turns out that there is a general relationship between the spread of the momentum values and the spread of position values, which is encoded in the Heisenberg uncertainty principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{6}$$

[2]"Eigen" is from German, where it means 'same.'

[3]Dirac delta functions

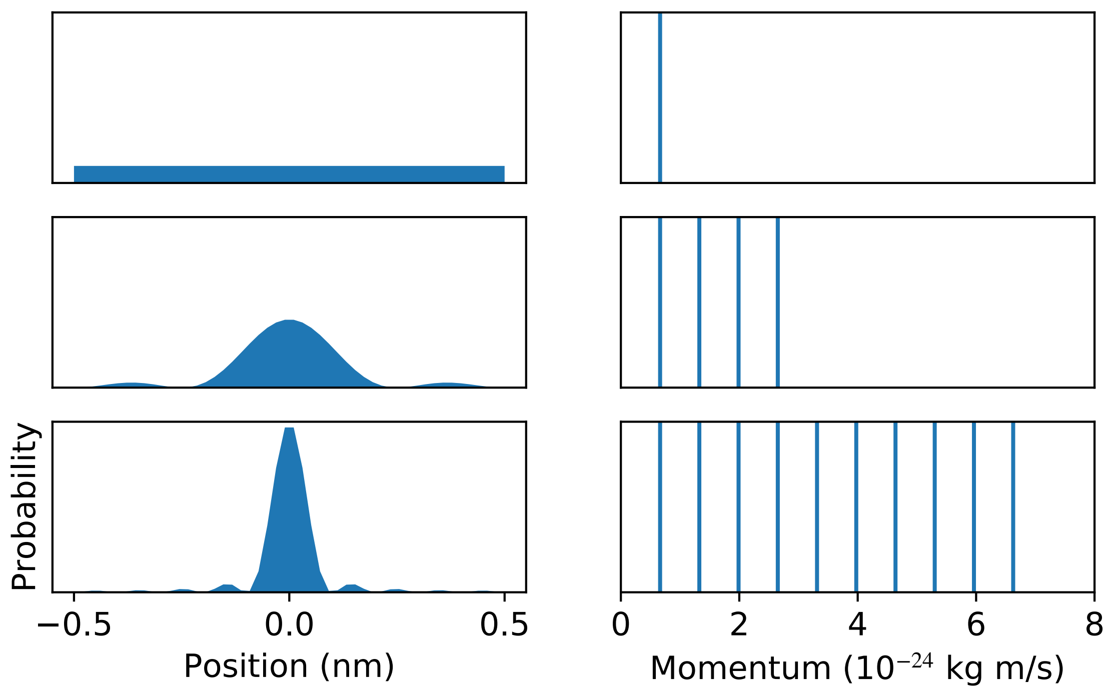

Figure 1: Making a localized wave function using a sum of momentum eigenstates. Each row is one wave function; the left is the probability density for position, while the right is the probability density for momentum. The momentum probability density is a bunch of spikes because it is a sum of momentum eigenstates. To make a wave function with high probability to be in one position, it's necessary to include many possible momenta.

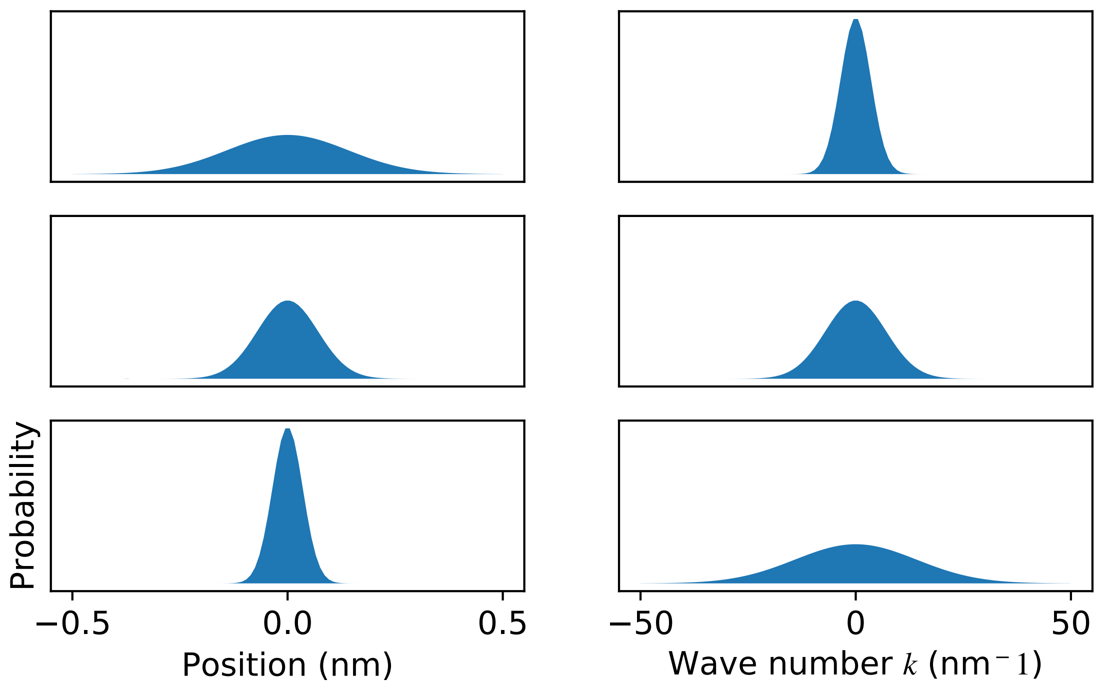

Figure 2: The position and momentum probabilities for various wave functions. Each row corresponds to one wave function with form proportional to $\exp(-x^2/2\sigma^2)$, and the probability of measuring position and momentum is given in each column.

This relationship puts fundamental physical limits on how well we can predict the outcome of an experiment.[4]

[4]If you are paying extremely close attention, you might notice that a particle in a box with wave function e^{ikx} seems to violate this. This is because we ignored boundary conditions. This is ok if the box is very large.