Chapter 3: Counting Statistics

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Statistical Laws governing Radioactive Decay

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Radioactive Disintegration – Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

☞ It consists of N trials.

☞ Each trial has a binary outcome: success or failure (decay or not).

☞ The probability of success (decay) is a constant from trial to trial – all atoms have an equal probability p of decay.

☞ The trials are independent.

In statistics, these four conditions characterize a Bernoulli process.

Binomial Distribution

Given, p, N and t, what is the probability of observing n disintegrations within a time t?

☞ The number of ways to chose n atoms from a total of N atoms in the sample is

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}$$

☞ So the probability of the n atoms chosen decayed during the time span t is

$$P_n = \binom{N}{n} p^n q^{N-n}$$

☞ The above equation describes the so-called Binomial distribution.

☞ What are the mean and standard deviation of a Binomial distribution?

Binomial Distribution

For a binomial distribution, the mean or the expectation of the number of disintegrations in time t is given by

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

The fluctuation in the number of disintegrations is given by the variance or the standard deviation of the number of disintegrations in time t is given by

$$\sigma^2 \equiv \sum_{n=0}^{N} (n-\mu)^2 \cdot P_n = Npq$$

and

$$\sigma \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

An Example Binomial Distribution

Example

More realistically, consider a ^{42}K source with an activity of 37 Bq (= 1 nCi). The source is placed in a counter, having an efficiency of 100%, and the numbers of counts in one-second intervals are registered.

(a) What is the mean disintegration rate?

(b) Calculate the standard deviation of the disintegration rate.

(c) What is the probability that exactly 40 counts will be observed in any second?

The decay constant for 42K is $\lambda = 0.0559 h^{-1} = 1.55 \times 10^{-5} s^{-1}$

Binomial Distribution

For a binomial distribution, the mean or the expectation of the number of disintegration in time t is given by

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$$\sigma^2 \equiv \sum_{n=0}^{N} (n-\mu)^2 \cdot P_n = Npq$$

and

$$\sigma \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

An Example Binomial Distribution

Solution

(a) The mean disintegration rate is the given activity, $r_d = 37\ s^{-1}$.

(b) The standard deviation of the disintegration rate is given by Eq. (11.18). We work with the time interval, $t = 1$ s. Since the decay constant is $\lambda = 0.0559\ h^{-1} = 1.55 \times 10^{-5}\ s^{-1}$, we have

the probability that an atom would survive a unit period of time

$$q = e^{-\lambda t} = e^{-1.55 \times 10^{-5} \times 1} = 0.9999845 \qquad (11.26)$$

and $p = 1 - q = 0.0000155$.† The number of atoms present is

the probability that an atom would decay within a unit period of time

$$N = \frac{r_d}{\lambda} = \frac{37\ s^{-1}}{1.55 \times 10^{-5}\ s^{-1}} = 2.39 \times 10^6. \qquad (11.27)$$

From Eq. (11.18), we obtain for the standard deviation of the disintegration rate

$$\sigma_{dr} = \frac{\sqrt{Npq}}{t} = \frac{\sqrt{2.39 \times 10^6 \times 0.0000155 \times 0.9999845}}{1\ s} = 6.09\ s^{-1}, \qquad (11.28)$$

which is about 16% of the mean disintegration rate.

(c) The probability of observing exactly $n = 40$ counts in 1 s is given by Eq. (11.13). However, the factors quickly become unwieldy when N is not small (e.g., $69! = 1.71 \times 10^{98}$). For large N and small n, as we have here, we can write for the binomial coefficient

$$\binom{N}{n} \equiv \frac{N(N-1)\cdots(N-n+1)}{n!} \cong \frac{N^n}{n!}, \qquad (11.29)$$

$$P_n = \binom{N}{n} p^n q^{N-n}$$

since each of the n factors in the numerator is negligibly different from N. Equation (11.13) then gives

$$P_{40} = \frac{(2.39 \times 10^6)^{40}}{40!}(0.0000155)^{40}(0.9999845)^{2.39\times 10^6 - 40} \qquad (11.30)$$

$$= \frac{(2.39)^{40}(10^{240})(0.0000155)^{40}(0.9999845)^{2.39\times 10^6}}{40!}, \qquad (11.31)$$

where $n = 40 \ll N$ has been dropped from the last exponent. The right-hand side can be conveniently evaluated with the help of logarithms. To reduce round-off errors, we use four decimal places:

$$\begin{aligned}
\log\,(2.39)^{40} &= 15.1359 \\
\log\,(10)^{240} &= 240.0000 \\
\log\,(0.0000155)^{40} &= -192.3867 \\
\log\,(0.9999845)^{2.39\times 10^6} &= -16.0886 \\
-\log 40! &= \underline{-47.9116} \\
\log P_{40} &= -1.251\ . \qquad (11.32)
\end{aligned}$$

Thus, $P_{40} = 10^{-1.251} = 0.0561$.

Radioactive Disintegration – Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- ☞ It consists of N trials.
- ☞ Each trial has a binary outcome: success or failure (decay or not).
- ☞ The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- ☞ The trials are independent.

In statistics, these four conditions characterize a Bernoulli process.

What happens if $N >> n \quad and \quad p \rightarrow 0$?

Poisson Process

The counting statistics related to nuclear decay processes is often more conveniently described by the Poisson distribution, is related to situations that involves a collection of multiple trials that satisfy the following conditions:

1. The number of trials, N, is very large, e.g. , N>>1.
2. Each trial is independent.
3. The probability that each single trial is successful is a constant and approaching zero, p<<1. So, the number of successful trials fluctuates around a finite number.

From the previous derivation, when N>>1, p<<1, a Binomial distribution can be approximated as

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of n:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation :

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$

Comparing Binomial and Poisson Distributions

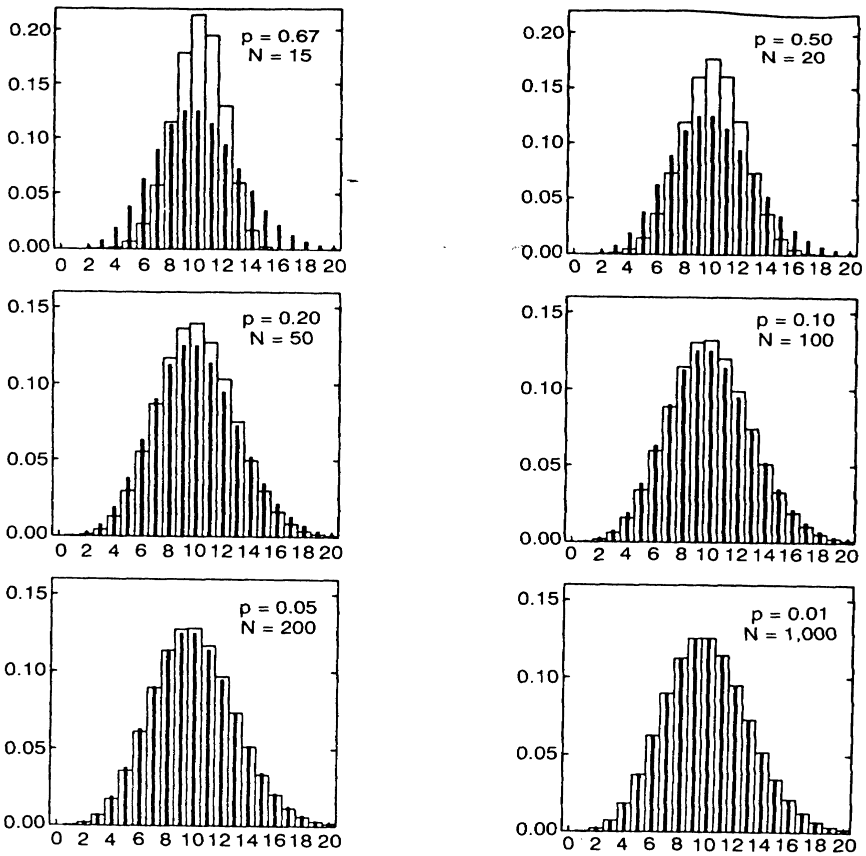


FIGURE 11.1. Comparison of binomial (histogram) and Poisson (solid bars) distributions, having the same mean, $\mu = 10$, but different values of the probability of success p and sample size N. The ordinate in each panel shows the probability P_n of exactly n successes, shown on the abscissa. With fixed μ, the Poisson distribution is the same throughout. (Courtesy James S. Bogard.)

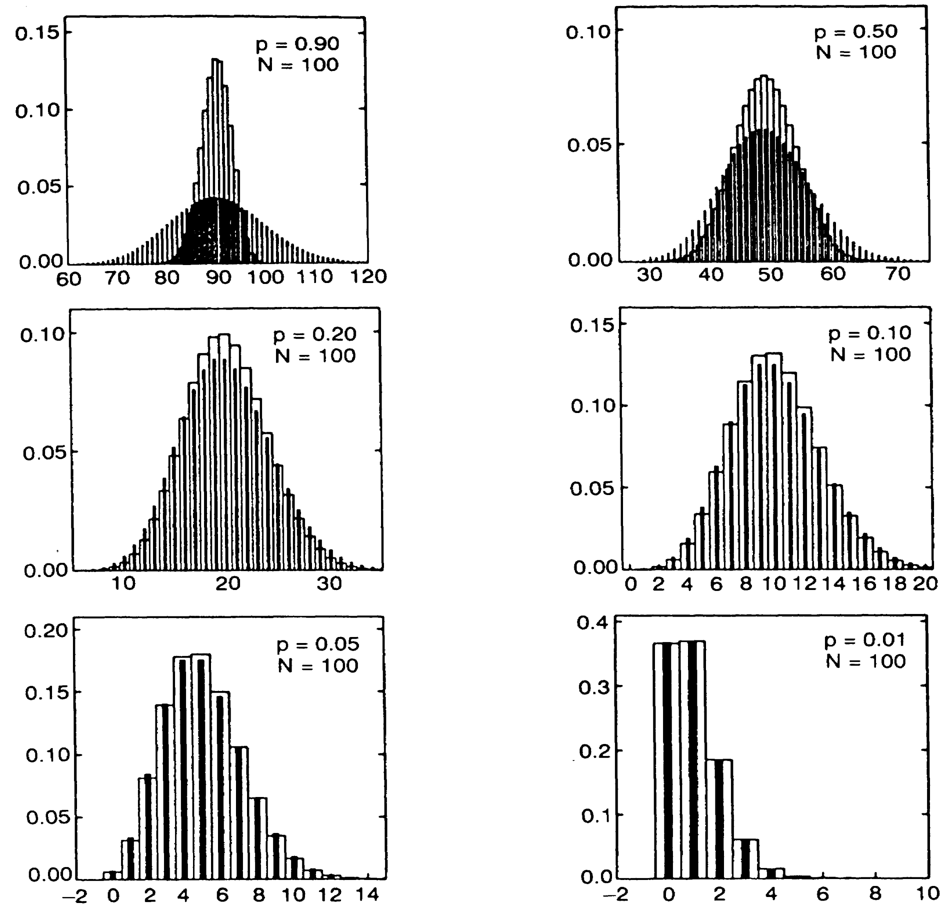


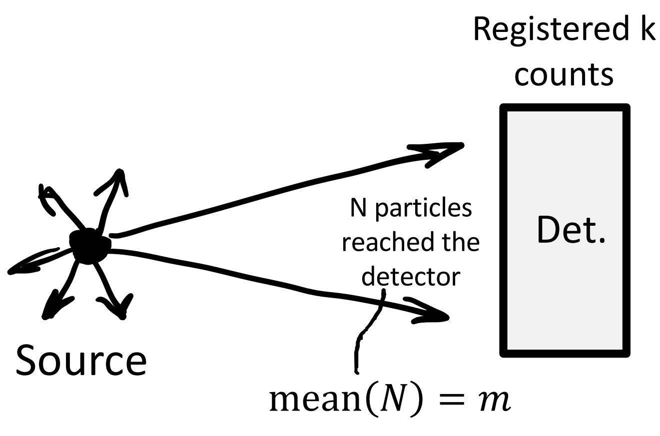
FIGURE 11.2. Comparison of binomial (histogram) and Poisson (solid bars) distributions for fixed N and different p. The ordinate shows P_n and the abscissa, n. The mean of the two distributions in a given panel is the same. (Courtesy James S. Bogard.)

Statistical Laws governing Radioactive Decay

Part II: An Example of Using Counting Statistics to Extract Critical Information regarding the Activity of a Radioactive Source

Binomial Distribution and Poisson Distribution

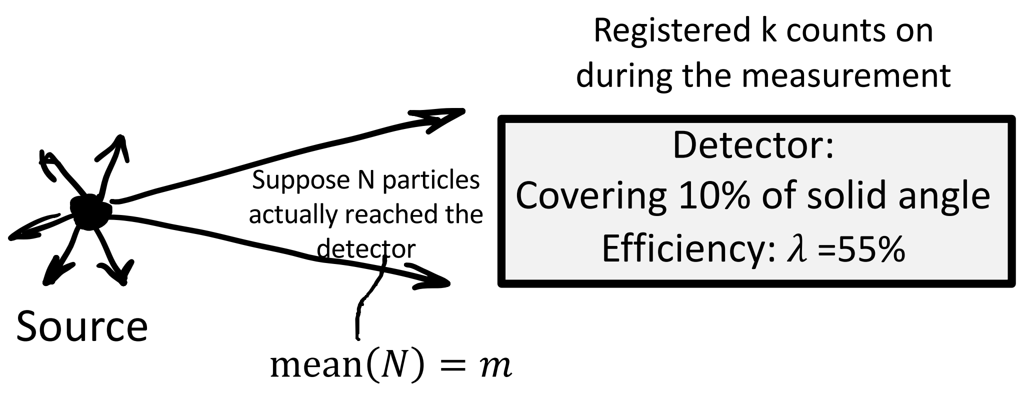
An example: Consider the following particle counting experiment.



1. The detector covers 1% solid angle.
2. Detection efficiency: λ =55%.
3. The measurement takes t = 1 s.
4. N particles reached the detector.
5. Detected k=0 count.

What do we learn from this experiment?

Binomial Distribution and Poisson Distribution



1. The detector covers 1% solid angle.
2. Detection efficiency: λ =55%.
3. The measurement takes t = 1 s.
4. *N* particles reached the detector.
5. Detected k=0 count.

Suppose there are, on average, *m* particles reaching the detector during the given time *t*. The probability $P(N|m)$ of having *N* particles reach the detector during the experiment would be given by the

$$\text{Poisson distribution: } P(N|m) = \frac{m^N}{N!} e^{-m}.$$

Once the N particles reached the detector, the number of particles detected would follow …

the Binomial distribution, so that the probability of detecting *k* particles is

$$P(k|N) = \binom{N}{k} \lambda^k (1-\lambda)^{N-k}.$$

Poisson Distribution – An example

Therefore, given the mean (average) number of particles reaching the detection during a period of t is m, the total probability of detecting k counts is

$$P(k) = \sum_{N=k}^{\infty} P(k|N)P(N|m)$$

$\lambda = \mathbf{0.55}$: detector efficiency

$$= \sum_{N=k}^{\infty} \frac{N!}{(N-k)!k!} \lambda^k (1-\lambda)^{N-k} \cdot \frac{m^N}{N!} \mathrm{e}^{-m}$$

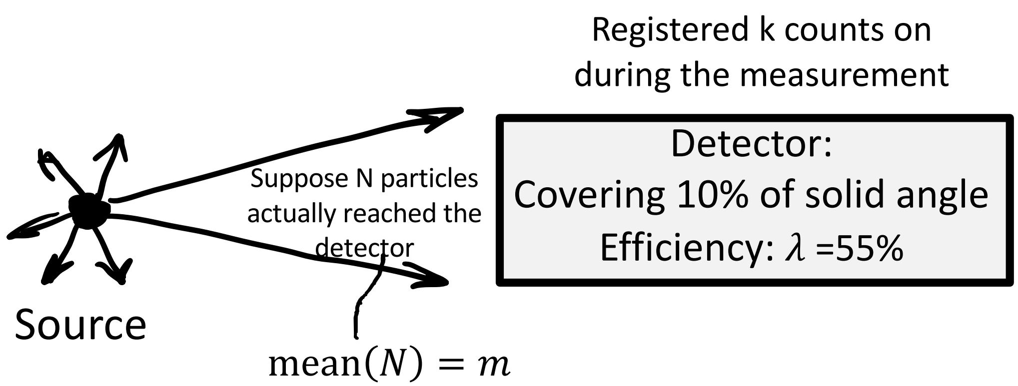
$$= \frac{(m\lambda)^k}{k!} \mathrm{e}^{-(m\lambda)}$$

If we would like to ensure 90% chance of detecting at least 1 particle, then we could let

$$1 - P(k=0) = 1 - \mathrm{e}^{-m\lambda} = 0.9,$$

the average (mean) number of particles reaching the detector during the measurement should be

$$m = 4.18.$$



1. The detector covers 1% solid angle.
2. Detection efficiency: λ =55%.
3. The measurement takes t = 1 s.
4. N particles reached the detector.
5. Detected k=0 count.

Finally, we can conclude that:

<u>Because we did not detect any count during the 1-second measurement, we have 90% confidence to claim that the source strength (average number of particles emitted per second) is</u>

$$A \leq \frac{4.18}{1\% \cdot 1\ s} = 418\ (particles\ per\ sec) = 418\ Bq$$

Binomial Distribution and Poisson Distribution

Binomial distribution

The probability of observing n successful trails out of a total of N independent trails:

$$P_n = \binom{N}{n} p^n q^{N-n}$$

mean of the observed number of successful trails :

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

Standard deviation:

$$Std(n) \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

Poisson distribution when N>>1, p<<1

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of n:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation :

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$

Statistical Laws governing Radioactive Decay

Part III: Application of the Gaussian Distribution in Counting Statistics

Binomial Distribution and Poisson Distribution

Binomial distribution

The probability of observing n successful trails out of a total of N independent trails:

$$P_n = \binom{N}{n} p^n q^{N-n}$$

mean of the observed number of successful trails:

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

Standard deviation:

$$Std(n) \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

Poisson distribution when N>>1, p<<1

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of n:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation:

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$

Gaussian distribution, If N is further increased, and p is further decreased

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Gaussian (Normal) Distribution

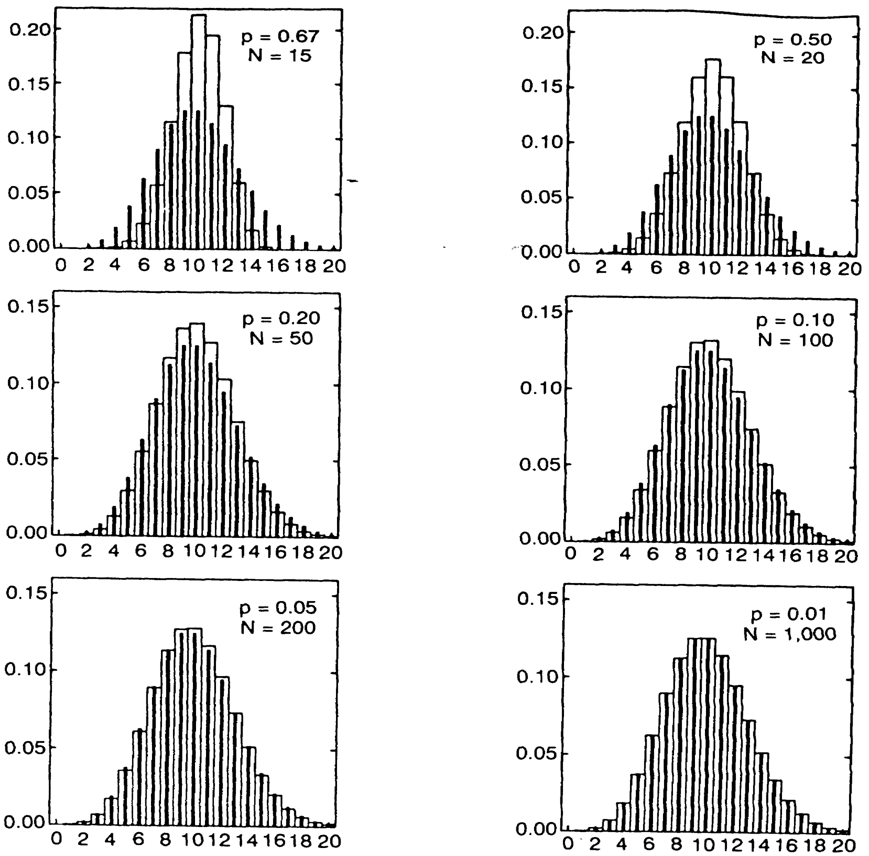
As p (the prob. of an atom decay within t) is getting even smaller and N is getting larger, both Binomial and Poisson distributions are approaching an extremely useful form of distribution – the Gaussian distribution.

Gaussian distribution is defined for a continuous variable x

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

but it is very useful for describing the counting fluctuation on discrete numbers.

Comparing Binomial and Poisson Distributions



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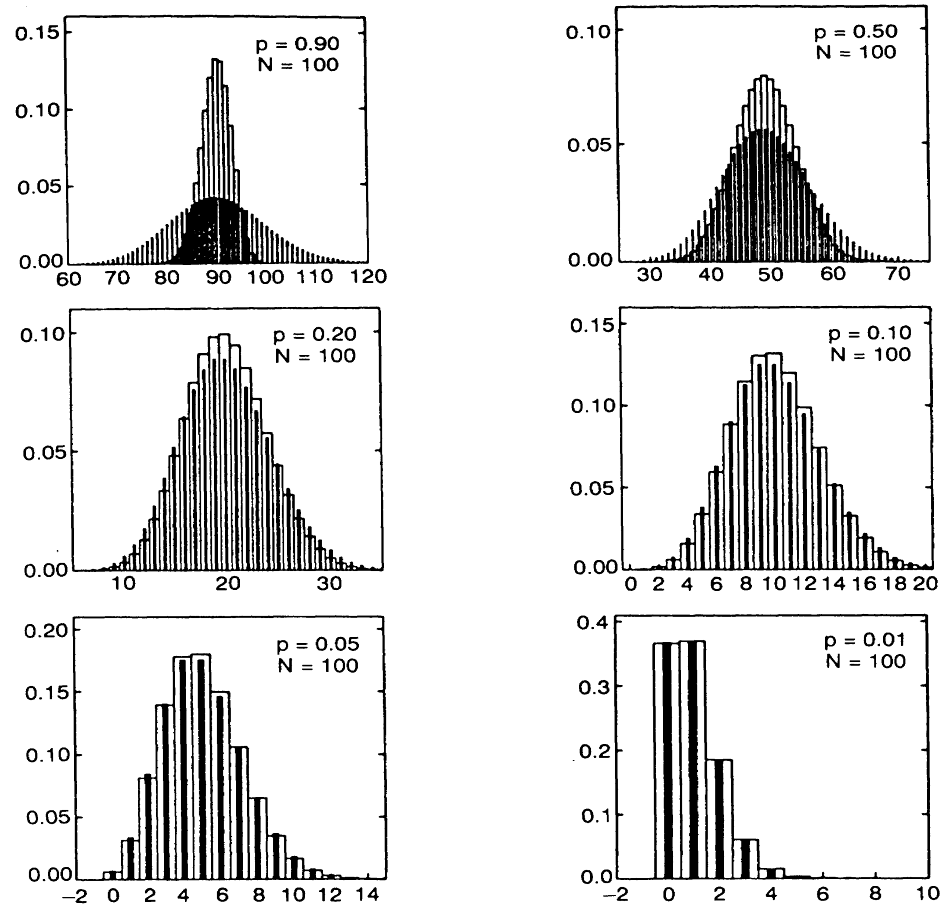


FIGURE 11.2. Comparison of binomial (histogram) and Poisson (solid bars) distributions for fixed N and different p. The ordinate shows P_n and the abscissa, n. The mean of the two distributions in a given panel is the same. (Courtesy James S. Bogard.)

Comparing Binomial and Gaussian (Normal) Distributions

Binomial and Poisson distributions practically match the normal distribution when μ >=30.

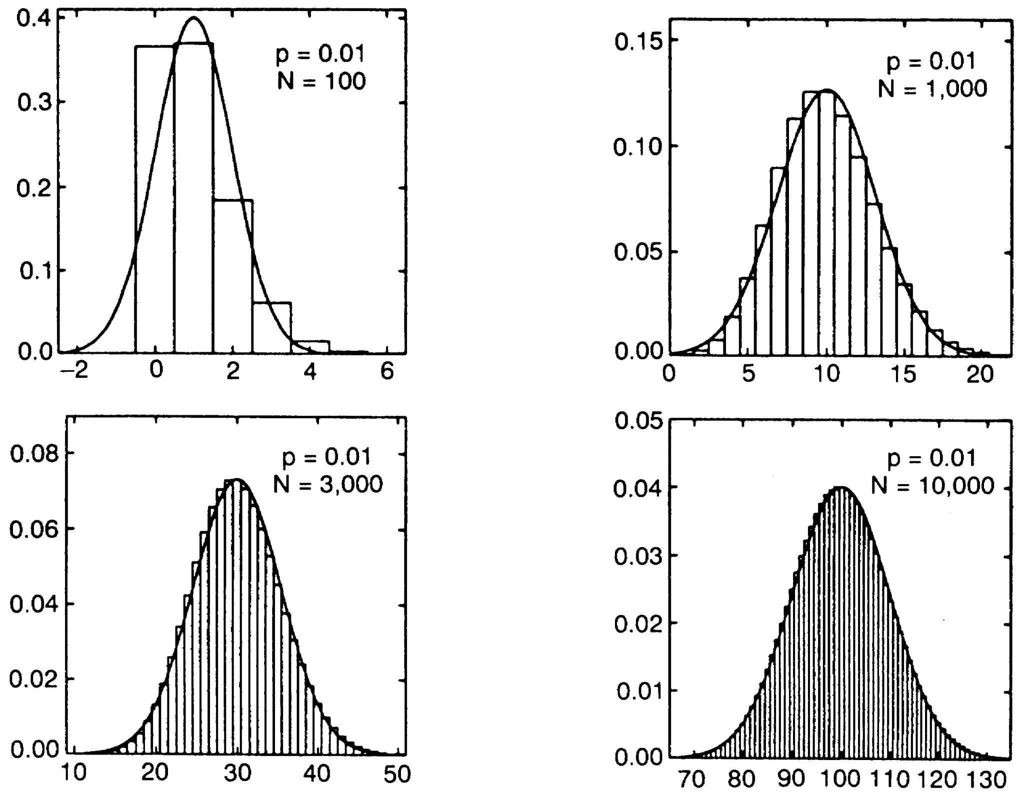


FIGURE 11.3. Comparison of binomial (histogram) and normal (solid line) distributions, having the same means and standard deviations. The ordinate in each panel gives the probability P_n for the former and the density $f(x)$ [Eq. (11.37)] for the latter, the abscissa giving n or x. (Courtesy James S. Bogard.)

The Gaussian (Normal) Distribution

For a variable, x, following the normal distribution, the probability that it takes a value between x and x is equal to the area under under the curve p(x) between these two values:

$$P(x_1 \leq x \leq x_2) = \frac{1}{\sqrt{2\pi\sigma}} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2}\, dx.$$

Many common manipulations when carried out on counting data that were originally Gaussian distributed will produce derived values that also follow Gaussian shape:

☞ Multiplying or dividing the data by a constant,

☞ Combining two Gaussian-distributed variables through addition, subtraction, or multiplication or,

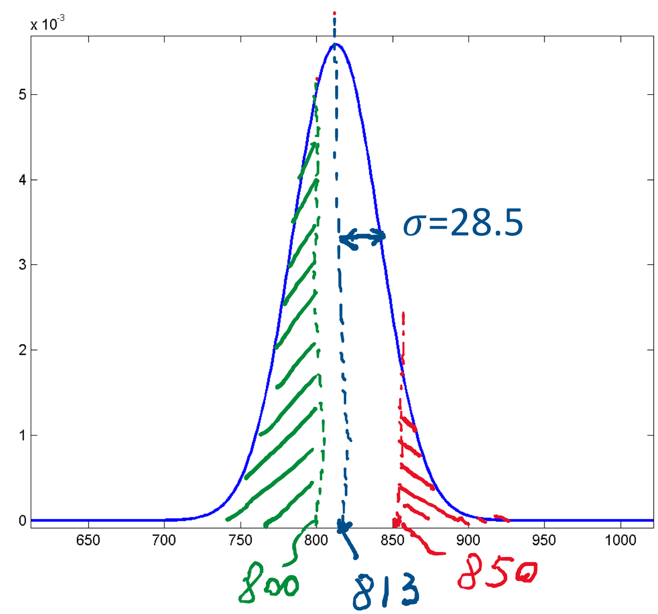
☞ Calculating the average of a series of independent measurements.

The Gaussian (Normal) Distribution

Example

Repeated counts are made in 1-min intervals with a long-lived radioactive source. The observed mean value of the number of counts is 813, with a standard deviation of 28.5 counts. (a) What is the probability of observing 800 or fewer counts in a given minute? (b) What is the probability of observing 850 or more counts in 1 min? (c) What is the probability of observing 800 to 850 counts in a minute? (d) What is the symmetric range of values about the mean number of counts within which 90% of the 1-min observations are expected to fall?

$$P(x_1 \le x \le x_2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2}\, dx.$$



Turner, pp. 316-317.

Error Propagation

Part I: The Quadratic Error Propagation Equation and Its Limitations

Error and Error Propagation

Two ways to express the error associated with a given measurement:

Probable error:

☞ The symmetric range about the mean, within which there is 50% chance that a measurement will fall.

☞ The width of the range depends on the distribution of the variable. For example, for Gaussian distributed error, the probable error is $\pm 0.675\,\sigma$.

Fractional standard deviation:

☞ The ratio of the standard deviation and the mean of the distribution of the random variable.

☞ For Poisson distributed random variable, the fractional standard deviation is simply

$$\frac{\sigma}{\mu} = \frac{1}{\sqrt{\mu}}$$

Error Propagation

In some situations, the variable of interest (Q) is not measured directly but derived as a function of more than one independent random variable whose values are directly measured. The error on the measured values is propagated into the uncertainty on the resultant quantity Q.

Suppose a quantity Q(x,y) that depends on two independent random variables x and y.

The sample mean and variance of variables x and y are derived as σ_x and σ_y, by repeating measurements.

The standard deviation of the indirect quantity Q is approximately given by

$$\sigma_Q^{\ 2} \cong \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^{\ 2} + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^{\ 2}$$

$$\sigma_Q^{\ 2} \cong \sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 \sigma_{x_i}^{\ 2}$$

Error and Error Propagation

A Taylor series of a real function of a single variable, f(x), around a point x_0 is given by

$$f(x_0+\Delta x)=f(x_0)+f_x(x_0)\Delta x+\frac{1}{2!}f_{xx}(x_0)(\Delta x)^2+\frac{1}{3!}f_{xxx}(x_0)(\Delta x)^3+......$$

where

$$f_{xx}(x_0)=\left[\frac{d}{dx}\frac{d}{dx}f(x)\right]\Bigg|_{x=x_0}$$

A Taylor series of a real function of two variables, f(x,y), is given by

$$\begin{aligned}
&f(x_0+\Delta x, y_0+\Delta y)=\\
&f(x_0,y_0)+\left[f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y\right]\\
&+\frac{1}{2!}\left[f_{xx}(x_0,y_0)(\Delta x)^2+2f_{xy}(x_0,y_0)\Delta x\Delta y+f_{yy}(x_0,y_0)(\Delta y)^2\right]\\
&+\frac{1}{3!}\left[f_{xxx}(x_0,y_0)(\Delta x)^3+3f_{xxy}(x_0,y_0)(\Delta x)^2(\Delta y)+3f_{xyy}(x_0,y_0)(\Delta x)(\Delta y)^2+f_{yyy}(x_0,y_0)(\Delta y)^3\right]+......
\end{aligned}$$

Error Propagation

We determine the standard deviation of a quantity $Q(x, y)$ that depends on two independent, random variables x and y. A sample of N measurements of the variables yields pairs of values, x_i and y_i, with $i = 1, 2, \ldots, N$. For the sample one can compute the means, $\bar{x}$ and $\bar{y}$; the standard deviations, σ_x and σ_y; and the values $Q_i = Q(x_i, y_i)$. We assume that the scatter of the x_i and y_i about their means is small. We can then write a power-series expansion for the Q_i about the point $(\bar{x}, \bar{y})$, keeping only the first powers. Thus,

$$Q_i = Q(x_i, y_i) \cong Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y}), \quad \text{(E.36)}$$

where the partial derivatives are evaluated at $x = \bar{x}$ and $y = \bar{y}$.

By definition, the mean of Q is

$$\bar{Q} = \frac{1}{N}\sum_{i=1}^{N} Q_i = \frac{1}{N}\sum_{i=1}^{N} Q(x_i, y_i)$$

and the variance of Q is

$$\sigma^2(Q) = \frac{1}{N}\sum_{i=1}^{N}(Q_i - \bar{Q})^2$$

Error Propagation

where the partial derivatives are evaluated at $x = \bar{x}$ and $y = \bar{y}$. The mean value of Q_i is simply

$$\bar{Q} \equiv \frac{1}{N}\sum_{i=i}^{N} Q_i = \frac{1}{N}\sum_{i=1}^{N} Q(x_i, y_i)$$

$$\cong \frac{1}{N}\sum_{i=1}^{N}\left[Q(\bar{x},\bar{y}) + \left.\frac{\partial Q}{\partial x}\right|_{(\bar{x},\bar{y})}(x_i - \bar{x}) + \left.\frac{\partial Q}{\partial y}\right|_{(\bar{x},\bar{y})}(y_i - \bar{y})\right] = Q(\bar{x},\bar{y}), \quad \text{E.36}$$

since the sums of the $x_i - \bar{x}$ and $y_i - \bar{y}$ over all i in Eq. (E.36) are zero, by definition of the mean values. Thus, the mean value of Q is the value of the function $Q(x, y)$ calculated at $x = \bar{x}$ and $y = \bar{y}$.

Error and Error Propagation

The variance of the Q_i is given by

$$\sigma_Q^2 = \frac{1}{N} \sum_{i=1}^{N} (Q_i - \overline{Q})^2. \qquad \text{(E.38)}$$

$$Q_i = Q(x_i, y_i) \cong Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y}), \qquad \text{(E.36)}$$

Applying Eq. (E.36) with $\overline{Q} = Q(\bar{x}, \bar{y})$, we find that

$$\sigma_Q^2 = \frac{1}{N} \sum_{i=1}^{N} \left[\frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y}) \right]^2 \qquad \text{(E.39)}$$

$$= \left(\frac{\partial Q}{\partial x}\right)^2 \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2$$

$$+ 2 \left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial Q}{\partial y}\right) \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y}). \qquad \text{(E.40)}$$

Variance of y, or $\sigma^2(y)$

Covariance, $Cov(x, y)$

Error and Error Propagation

The last term, called the *covariance* of x and y, vanishes for large N if the values of x and y are uncorrelated. (The factors $y_i - \bar{y}$ and $x_i - \bar{x}$ are then just as likely to be positive as negative, and the covariance also decreases as $1/N$). We are left with the first two terms, involving the variances of the x_i and y_i:

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^2. \qquad \text{(E.41)}$$

This is one form of the error propagation formula, which is easily generalized to a function Q of any number of independent random variables.

Assumptions ??

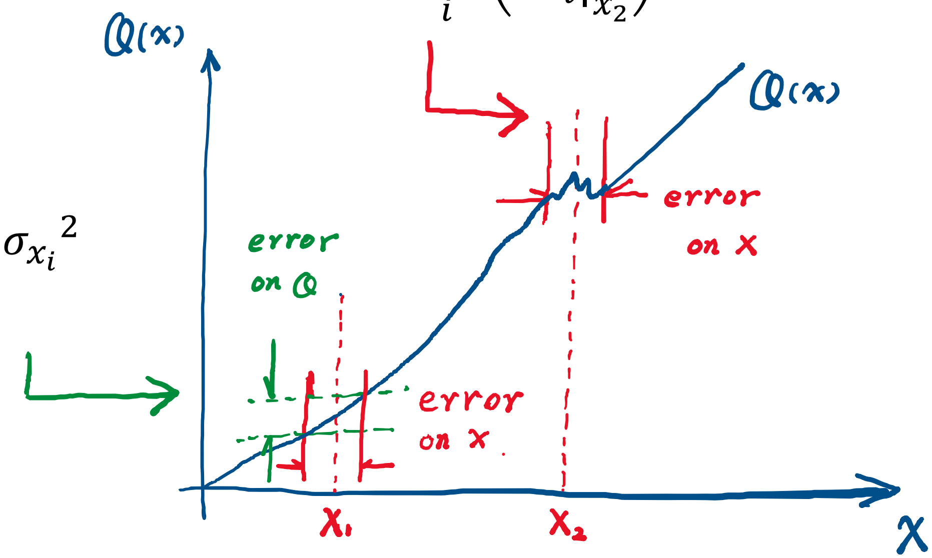
Error Propagation Formula

The error propagation formula is exact only when

- the two variables, x and y, are independent to each other,
- and when Q(x,y) could be approximated as a linear function of both x and y.

$$\sigma_Q{}^2 \cong \sum_i \left(\left.\frac{\partial Q}{\partial x_i}\right|_{x_2} \right)^2 \sigma_{x_i}{}^2$$

$$\sigma_Q{}^2 \cong \sum_i \left(\left.\frac{\partial Q}{\partial x_i}\right|_{x_1} \right)^2 \sigma_{x_i}{}^2$$



Error Propagation

Case 1: Sums or differences of counts – u is the sum or difference of two random numbers representing counts measured in two independent experiments.

$$u = x + y \quad \text{or} \quad u = x - y$$

$$\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}$$

$$\sigma_Q{}^2 \cong \sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 \sigma_{x_i}{}^2 ,$$

or

$$\sigma_Q \cong \sqrt{\sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 \sigma_{x_i}{}^2} .$$

Example: estimating the net counts from a sample.

$$\text{net counts} = \text{total counts} - \text{background counts}$$

or

$$u = x - y$$

Error Propagation

Case 2: Multiplication or division by a constant

$$u = Ax$$

⇩

$$\sigma_u = A\sigma_x$$

Example: estimating the count rate, $\text{counting rate} \equiv r = \frac{x}{t}$

Assuming that the error in the measuring time is negligible, we get

$$\sigma_r = \frac{\sigma_x}{t}$$

Error Propagation

Case 3: Multiplication or division of counts

$$u = xy, \quad \frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x$$

Using the equation

$$\sigma_Q{}^2 \cong \sum_i \left(\frac{\partial Q}{\partial x_i} \right)^2 \sigma_{x_i}{}^2$$

One gets

$$\sigma_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 = y^2 \cdot \sigma_x^2 + x^2 \cdot \sigma_y^2 \, .$$

Therefore,

$$\boxed{\left(\frac{\sigma_u}{u}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

Error Propagation

Part II: Error on the measured net count rate

Error Propagation in Net Count Rate Measurement

As an application of the error-propagation formula, Eq. (11.46), we find the standard deviation of the net count rate of a sample, obtained experimentally as the difference between gross and background count rates, r_g and r_b. As with gross counting, one also measures the number n_b of background counts in a time t_b. The net count rate ascribed to the sample is then the difference

$$r_n = r_g - r_b = \frac{n_g}{t_g} - \frac{n_b}{t_b}. \qquad (11.49)$$

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^2.$$

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n/\partial n_g = 1/t_g$ and $\partial r_n/\partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

Assuming no error on t

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \qquad (11.50)$$

where $\sigma_{nr}^2 \equiv \sigma^2(r_n)$, $\sigma_g^2 \equiv \sigma^2(n_g)$, $\sigma_b^2 \equiv \sigma^2(n_b)$,

and $\sigma_{gr}^2 = \sigma^2\left(\frac{n_g}{t_g}\right) = \frac{\sigma_g^2}{t_g^2}$, $\sigma_{br}^2 = \sigma^2\left(\frac{n_b}{t_b}\right) = \frac{\sigma_b^2}{t_b^2}$.

Turner, pp. 324.

Error Propagation in Net Count Rate Measurement

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n/\partial n_g = l/t_g$ and $\partial r_n/\partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \quad (11.50)$$

Here σ_g and σ_b are the standard deviations of the numbers of gross and background counts, and σ_{gr} and σ_{br} are the standard deviations of the gross and background count rates. Equation (11.50) expresses the well-known result for the standard deviation of the sum or difference of two Poisson or normally distributed random variables. Using n_g and n_b as the best estimates of the means of the gross and background distributions and assuming that the numbers of counts obey Poisson statistics, we have $\sigma_g^2 = n_g$ and $\sigma_b^2 = n_b$. Therefore, the last equation can be written

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}}, \quad (11.51)$$

Error Propagation in Net Count Rate Measurement

Example

A long-lived radioactive sample is placed in a counter for 10 min, and 1426 counts are registered. The sample is then removed, and 2561 background counts are observed in 90 min. (a) What is the net count rate of the sample and its standard deviation? (b) If the counter efficiency with the sample present is 28%, what is the activity of the sample and its standard deviation in Bq? (c) Without repeating the background measurement, how long would the sample have to be counted in order to obtain the net count rate to within ±5% of its true value with 95% confidence? (d) Would the time in (c) also be sufficient to ensure that the *activity* is known to within ±5% with 95% confidence?

Turner, pp. 324.

Error Propagation in Net Count Rate Measurement

(a) What is the net count rate of the sample and its standard deviation?

Solution

(a) We have $n_g = 1426$, $t_g = 10$ min, $n_b = 2561$, and $t_b = 90$ min. The gross and background count rates are $r_g = 1426/10 = 142.6$ cpm and $r_b = 2561/90 = 28.5$ cpm.

Therefore, the net count rate is $r_n = 142.6 - 28.5 = 114$ cpm. The standard deviation can be found from either of the expressions in (11.51). Using the first (which does not depend on the calculated values, r_g and r_b), we find

$$\sigma_{nr} = \sqrt{\frac{1426}{(10\text{ min})^2} + \frac{2561}{(90\text{ min})^2}} = 3.82\text{ min}^{-1} = 3.82\text{ cpm}. \quad (11.52)$$

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}},$$

Error Propagation in Net Count Rate Measurement

(b) If the counter efficiency with the sample present is 28%, what is the activity of the sample and its standard deviation in Bq?

Solution:

(b) Since the counter efficiency is $\epsilon = 0.28$, the inferred activity of the sample is $A = r_n/\epsilon = (114\ \text{min}^{-1})/0.28 = 407\ \text{dpm} = 6.78\ \text{Bq}$. The standard deviation of the activity is $\sigma_{nr}/\epsilon = (3.82\ \text{min}^{-1})/0.28 = 13.6\ \text{dpm} = 0.227\ \text{Bq}$.

$$u = Ax$$

$$\Downarrow$$

$$\sigma_u = A\sigma_x$$

Error Propagation in Net Count Rate Measurement

(c) Without repeating the background measurement, how long would the sample have to be counted in order to obtain the net count rate to within ±5% of its true value with 95% confidence? (d) Would the time in (c) also be sufficient to ensure that the *activity* is known to within ±5% with 95% confidence?

Solution:

(c) A 5% uncertainty in the net count rate is $0.05r_n = 0.05 \times 114 = 5.70$ cpm. For the true net count rate to be within this range of the mean at the 95% confidence level means that 5.70 cpm $= 1.96\sigma_{nr}$ (Table 11.2), or that $\sigma_{nr} = 2.91$ cpm. Using the second expression in (11.51) with the background rate as before (since we do not yet know the new value of n_g), we write

$$u = Ax$$

$$\sigma_{nr} = 2.91\ \text{min}^{-1} = \sqrt{\frac{142.6\ \text{min}^{-1}}{t_g} + \frac{28.5\ \text{min}^{-1}}{90\ \text{min}}}. \quad (11.53)$$

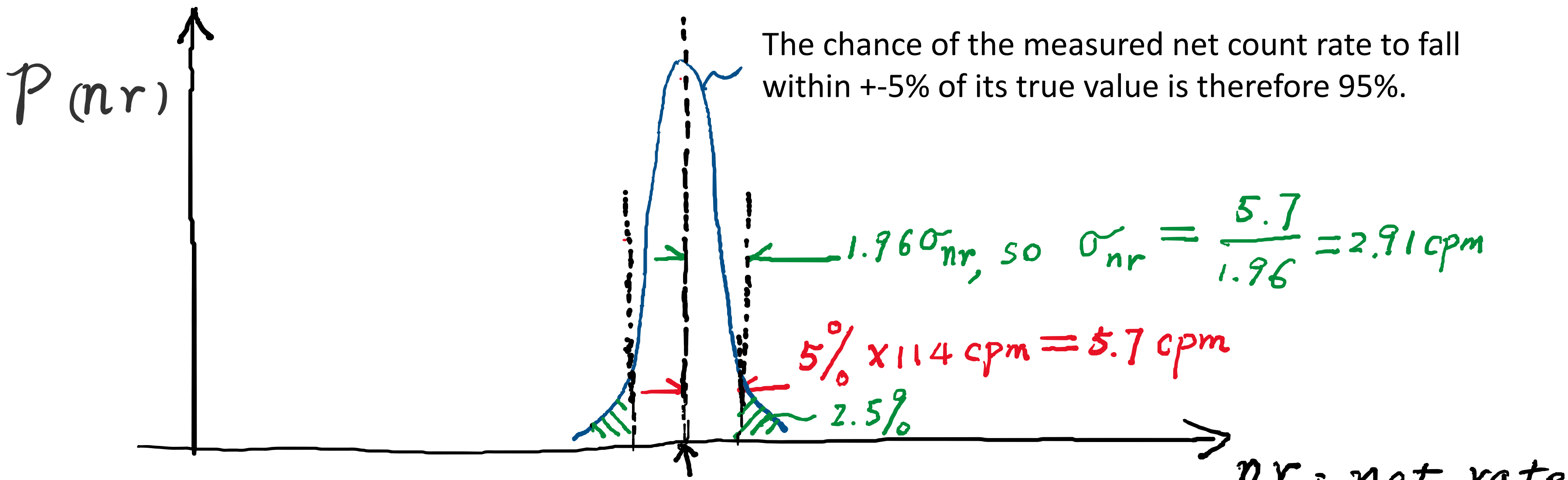
$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}},$$

$$\sigma_u = A\sigma_x$$

Solving, we find that $t_g = 17.5$ min.

(d) Yes. The relative uncertainties remain the same and scale according to the efficiency. If the efficiency were larger and the counting times remained the same, then a larger number of counts and less statistical uncertainty would result.

Error Propagation in Net Count Rate Measurement



The chance of the measured net count rate to fall within +-5% of its true value is therefore 95%.

measured net count rate: 114(cpm)

If we assume that the measured net count rate of 114 cpm is close enough to the true net count rate, then to ensure there is 95% chance that the measured net count rate would fall within +-5% of its true value, we need

$$114(cpm)\times 5\% = 1.96\times\sigma_{nr}.$$

Remember that

$$\sigma_{\mathrm{nr}} = \sqrt{\frac{n_{\mathrm{g}}}{t_{\mathrm{g}}^2} + \frac{n_{\mathrm{b}}}{t_{\mathrm{b}}^2}} = \sqrt{\frac{r_{\mathrm{g}}}{t_{\mathrm{g}}} + \frac{r_{\mathrm{b}}}{t_{\mathrm{b}}}},$$

then

$$5.71(cpm) = 1.96\cdot\sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}} = 1.96\sqrt{\frac{r_b + 114(cpm)}{t_g} + \frac{r_b}{t_b}}, \qquad \boldsymbol{so}\ \ t_g = 17.5\ (min)$$

Error Propagation

Part III: Optimization of experimental design and data processing

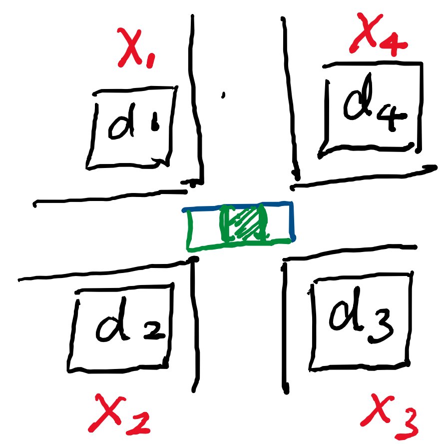
Error Propagation

Case 5: Combination of independent measurements with unequal errors

If N independent measurements of the same quantity have been carried out and not all the measurements have the same precision, what is the best way to estimate the best estimate of the mean value of the quantity to be measured?

The best estimate of the quantity, <x>, can be achieved by the weighted average

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$



How to assign the weighting factors a_i's ?

Error Propagation

Let each individual measurement x_i be given a weighting factor a_i and the best value $\langle x \rangle$ computed from the linear combination

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i} \tag{3.45}$$

We now seek a criterion by which the weighting factors a_i should be chosen in order to minimize the expected error in $\langle x \rangle$.

For brevity, we write

$$\alpha \equiv \sum_{i=1}^{N} a_i$$

so that

$$\langle x \rangle = \frac{1}{\alpha} \sum_{i=1}^{N} a_i x_i$$

Now apply the error propagation formula [Eq. (3.37)] to this case:

$$\sigma_{\langle x \rangle}^2 = \sum_{i=1}^{N} \left(\frac{\partial \langle x \rangle}{\partial x_i} \right)^2 \sigma_{x_i}^2$$

Error Propagation

Now apply the error propagation formula [Eq. (3.37)] to this case:

$$\sigma_{\langle x\rangle}^2 = \sum_{i=1}^{N} \left(\frac{\partial\langle x\rangle}{\partial x_i}\right)^2 \sigma_{x_i}^2$$

$$\langle x\rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

$$= \sum_{i=1}^{N} \left(\frac{a_i}{\alpha}\right)^2 \sigma_{x_i}^2$$

$$= \frac{1}{\alpha^2} \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2$$

$$\sigma_{\langle x\rangle}^2 = \frac{\beta}{\alpha^2} \quad (3.46)$$

where

$$\alpha \equiv \sum_{i=1}^{N} a_i \qquad \beta \equiv \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2$$

In order to minimize $\sigma_{\langle x\rangle}$, we must minimize $\sigma_{\langle x\rangle}^2$ from Eq. (3.46) with respect to a typical weighting factor a_j:

$$0 = \frac{\partial \sigma_{\langle x\rangle}^2}{\partial a_j} = \frac{\alpha^2 \frac{\partial\beta}{\partial a_j} - 2\alpha\beta \frac{\partial\alpha}{\partial a_j}}{\alpha^4} \quad (3.47)$$

Error Propagation

$$0 = \frac{\partial \sigma^2_{\langle x \rangle}}{\partial a_j} = \frac{\alpha^2 \dfrac{\partial \beta}{\partial a_j} - 2\alpha\beta \dfrac{\partial \alpha}{\partial a_j}}{\alpha^4} \tag{3.47}$$

Note that

$$\alpha \equiv \sum_{i=1}^{N} a_i \qquad \frac{\partial \alpha}{\partial a_j} = 1 \qquad \frac{\partial \beta}{\partial a_j} = 2a_j \sigma^2_{x_j} \qquad \beta \equiv \sum_{i=1}^{N} a_i^2 \sigma^2_{x_i}$$

Putting these results into Eq. (3.47), we obtain

$$\frac{1}{\alpha^4}(2\alpha^2 a_j \sigma^2_{x_j} - 2\alpha\beta) = 0$$

and solving for a_j, we find

$$a_j = \frac{\beta}{\alpha} \cdot \frac{1}{\sigma^2_{x_j}} \tag{3.48}$$

If we choose to normalize the weighting coefficients,

$$\sum_{i=1}^{N} a_i \equiv \alpha = 1$$

$$a_j = \frac{\beta}{\sigma^2_{x_j}}$$

Error Propagation

$$a_j = \frac{\beta}{\sigma_{x_j}^2}$$

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

Putting this into the definition of β, we obtain

$$\beta = \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2 = \sum_{i=1}^{N} \left(\frac{\beta}{\sigma_{x_i}^2} \right)^2 \sigma_{x_i}^2$$

or

$$\beta = \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1} \tag{3.49}$$

Therefore, the proper choice for the normalized weighting coefficient for x_j, is

$$\boxed{a_j = \frac{1}{\sigma_{x_j}^2} \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1}} \tag{3.50}$$

We therefore see that *each data point should be weighted inversely as the square of its own error.*

Error Propagation

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

Therefore, the proper choice for the normalized weighting coefficient for x_j, is

$$a_j = \frac{1}{\sigma_{x_j}^2} \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1} \tag{3.50}$$

We therefore see that *each data point should be weighted inversely as the square of its own error.*

$$\frac{1}{\sigma_{x_i}^2} \sim \text{creditability of the measurement } x_i.$$

$$a_i = \frac{1}{\sigma_{x_i}^2} \Big/ \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right) \sim \text{relative creditability of the measurement } x_i.$$

Error Propagation

Assuming that this optimal weighting is followed, what will be the resultant (minimum) error in $\langle x \rangle$? Because we have chosen $\alpha = 1$ for normalization, Eq. (3.46) becomes

$$\sigma_{\langle x \rangle}^2 = \beta$$

In the case of optimal weighting, β is given by Eq. (3.49). Therefore,

$$\boxed{\frac{1}{\sigma_{\langle x \rangle}^2} = \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2}} \tag{3.51}$$

From Eq. (3.51), the expected standard deviation $\sigma_{\langle x \rangle}$ can be calculated from the standard deviations σ_{x_i} associated with each individual measurement.

Error Propagation

Case 5: Combination of independent measurements with unequal errors (continued)

The proper choice for the normalized weighting factors for x_i is

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

$$a_j = \frac{1}{\sigma_{x_j}^2}\left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2}\right)^{-1}$$

And the error (variance) on the weighted average is

$$\sigma_{\langle x \rangle}^2 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2}\right)^{-1}$$

Optimization of Counting Experiments

Case 6: Measuring the net count rate from a long-lived radioisotope.

$S \equiv$ counting rate due to the source alone without background

$B \equiv$ counting rate due to background

The measurement of S is normally carried out by counting the source plus background (at an average rate of $S + B$) for a time T_{S+B} and then counting background alone for a time T_B. The net rate due to the source alone is then

$$S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B} \qquad \text{(}{}^{\text{!}}\text{2)}$$

where N_1 and N_2 are the total counts in each measurement.

If the total measurement $T=T_{S+B}+T_B$ is fixed, how to minimize the statistical error on the measured net count rate?

Error Propagation in Net Count Rate Measurement

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n / \partial n_g = l/t_g$ and $\partial r_n / \partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \tag{11.50}$$

Here σ_g and σ_b are the standard deviations of the numbers of gross and background counts, and σ_{gr} and σ_{br} are the standard deviations of the gross and background count rates. Equation (11.50) expresses the well-known result for the standard deviation of the sum or difference of two Poisson or normally distributed random variables. Using n_g and n_b as the best estimates of the means of the gross and background distributions and assuming that the numbers of counts obey Poisson statistics, we have $\sigma_g^2 = n_g$ and $\sigma_b^2 = n_b$. Therefore, the last equation can be written

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}}, \tag{11.51}$$

Optimization of Counting Experiments

Applying the results of error propagation analysis to Eq. (3.52), we obtain

$$\sigma_S = \left[\left(\frac{\sigma_{N_1}}{T_{S+B}}\right)^2 + \left(\frac{\sigma_{N_2}}{T_B}\right)^2\right]^{1/2}$$

$$\sigma_S = \left(\frac{N_1}{T_{S+B}^2} + \frac{N_2}{T_B^2}\right)^{1/2}$$

$$\sigma_S = \left(\frac{S+B}{T_{S+B}} + \frac{B}{T_B}\right)^{1/2}$$

N_1: measured counts during the source+background measurement.

N_2: measured counts during the background-only measurement.

S: measured count-rate during the source+background measurement.

B: measured count-rate during the background - only measurement.

If we now assume that a fixed total time $T = T_{S+B} + T_B$ is available to carry out both measurements, the above uncertainty can be minimized by optimally choosing the fraction of T allocated to T_{S+B} (or T_B). We square Eq. (3.53) and differentiate

$$2\sigma_S d\sigma_S = -\frac{S+B}{T_{S+B}^2} dT_{S+B} - \frac{B}{T_B^2} dT_B$$

and set $d\sigma_S = 0$ to find the optimum condition. Also, because T is a constant, $dT_{S+B} + dT_B = 0$. The optimum division of time is then obtained by meeting the condition

$$\left.\frac{T_{S+B}}{T_B}\right|_{\text{opt}} = \sqrt{\frac{S+B}{B}} \tag{3.54}$$

False Positive Rate and Minimum Significant Net Count Rate – An Example

Example

A sample, counted for 10 min, registers 530 gross counts. A 30-min background reading gives 1500 counts. (a) Does the sample have activity? (b) Without changing the counting times, what minimum number of gross counts can be used as a decision level such that the risk of making a type-I error is no greater than 0.050?

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Solution

(a) The numbers of gross and background counts are $n_g = 530$ and $n_b = 1500$; the respective counting times are $t_g = 10$ min and $t_b = 30$ min. The gross and background count rates are $r_g = n_g/t_g = 53$ cpm and $r_b = n_b/t_b = 50$ cpm, giving a net count rate $r_n = r_g - r_b = 3$ cpm. The question of whether activity is present cannot be answered in an absolute sense from these measurements. The observed net rate could occur randomly with or without activity in the sample. We can, however, compute the *probability* that the result would occur randomly when we assume that the sample has no activity. To do this, we compare the net count rate with its estimated standard deviation σ_{nr}, given by Eq. (11.51):

$$\sigma_{nr} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}} = \sqrt{\frac{53}{10} + \frac{50}{30}} = 2.64 \text{ cpm}. \qquad (11.64)$$

The observed net rate differs from 0 by $3/2.64 = 1.14$ standard deviations. As found in Table 11.1, the area under the standard normal curve to the right of this value is 0.127. Assuming that the activity A is zero, as shown in Fig. 11.4, we conclude that an observation giving a net count rate greater than the observed $r_n = 1.14\sigma_{nr} = 3$ cpm would occur randomly with a probability of 0.127. This single set of measurements, gross and background, is thus consistent with the conclusion that the sample likely contains little or no activity. However, one does not know where the bell-shaped curve in Fig. 11.4 should be centered. Based on this single measurement, the most likely place is $r_n = 3$ cpm, with the sample activity corresponding to that value of the net count rate.

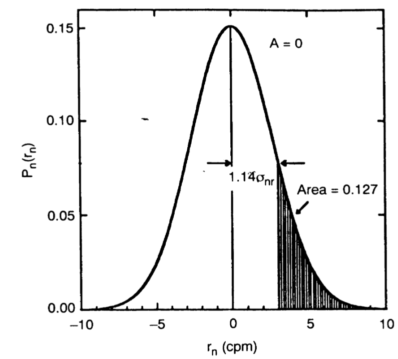


FIGURE 11.4. Probability density $P_n(r_n)$ for measurement of net count rate r_n when no activity is present. See example in text. (Courtesy James S. Bogard.)

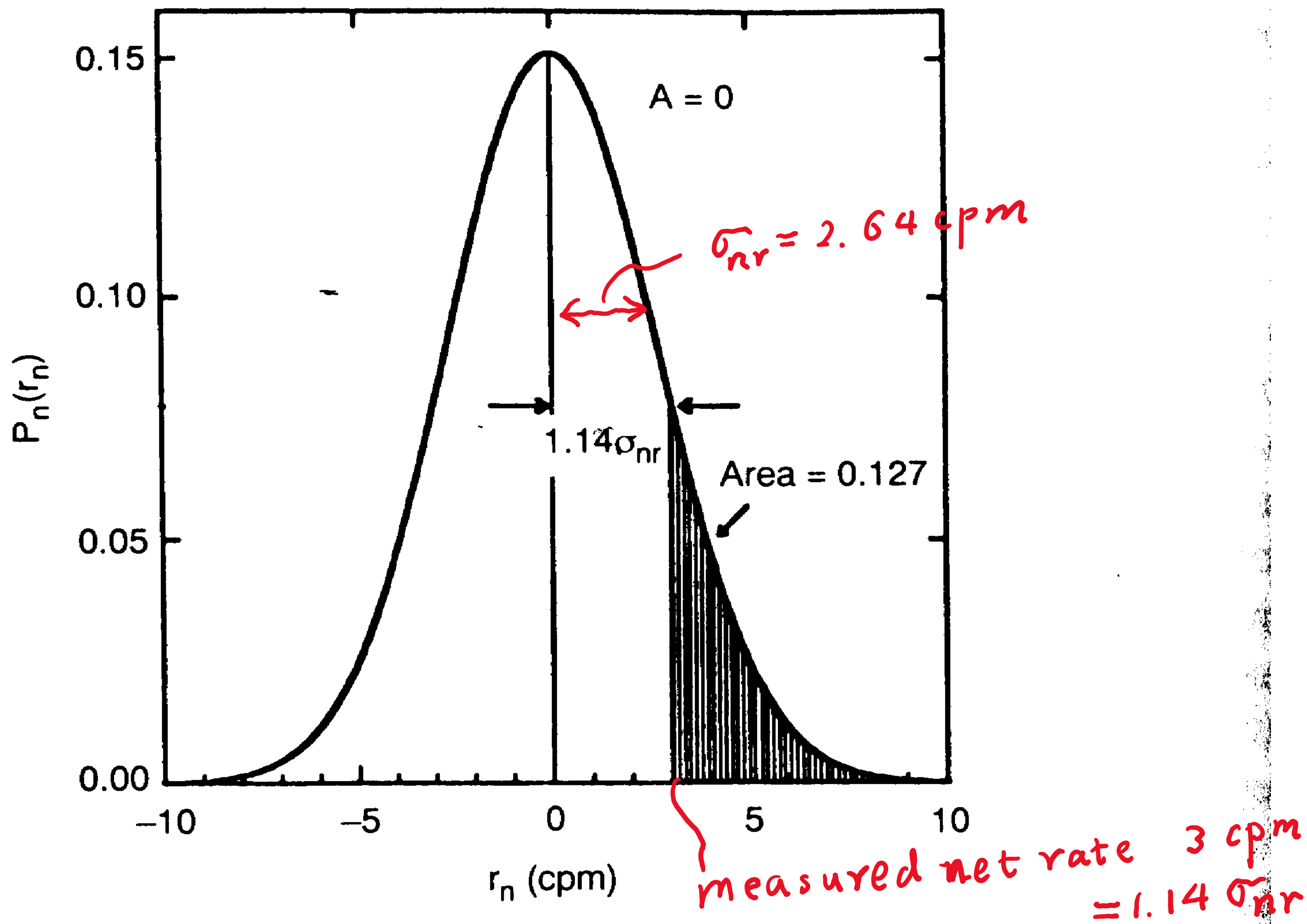


FIGURE 11.4. Probability density $P_n(r_n)$ for measurement of net count rate r_n when no activity is present. See example in text. (Courtesy James S. Bogard.)

Limits of Detectability

Part I: Detection limit, false-positive error, and false-negative error

Limits of Detectability

For a counting system, it is useful to set a detection limit. That is the amount of activity can be detected reliably.

The basic procedure could be

(1) Setting a certain confidence level – the probability that a decision (on whether or not a source is present) is correct.

(2) Define a quantity based on which the decision can be made. In the source counting case, it is the net count per unit time

$$n_S = n_g - n_b$$

where

n_S : net counts

n_g : gross counts

n_b : background counts

(3) Finding a critical level, L_c. If n_s exceeds L_c, we assume source activity is present, otherwise we assume that the source does not contain activity.

False Positive and False Negative Errors

Due to the statistical fluctuation on the counts measured within a given time t, there will be

(1) many instances in which a positive n_S is above the critical level even for samples with no activity, which leads to the false positive.

(2) and similarly, measured counts is lower than the critical level even when the source contains non-zero activity, which leads to the false negative.

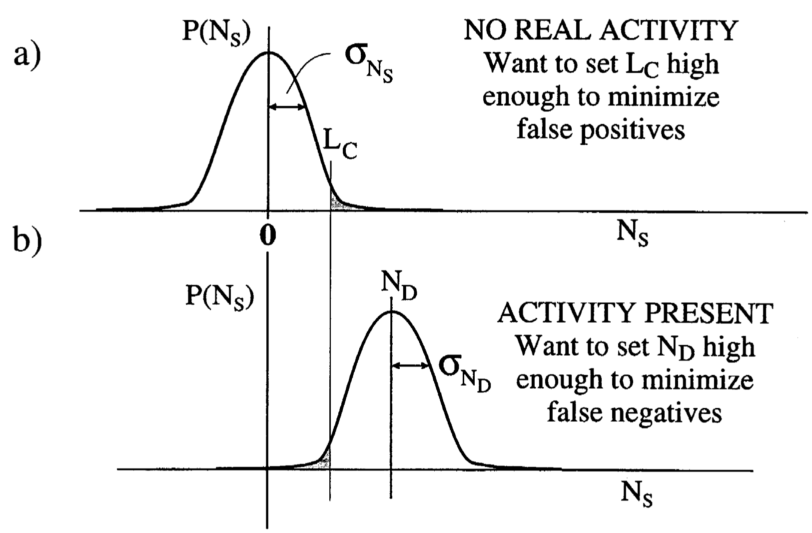


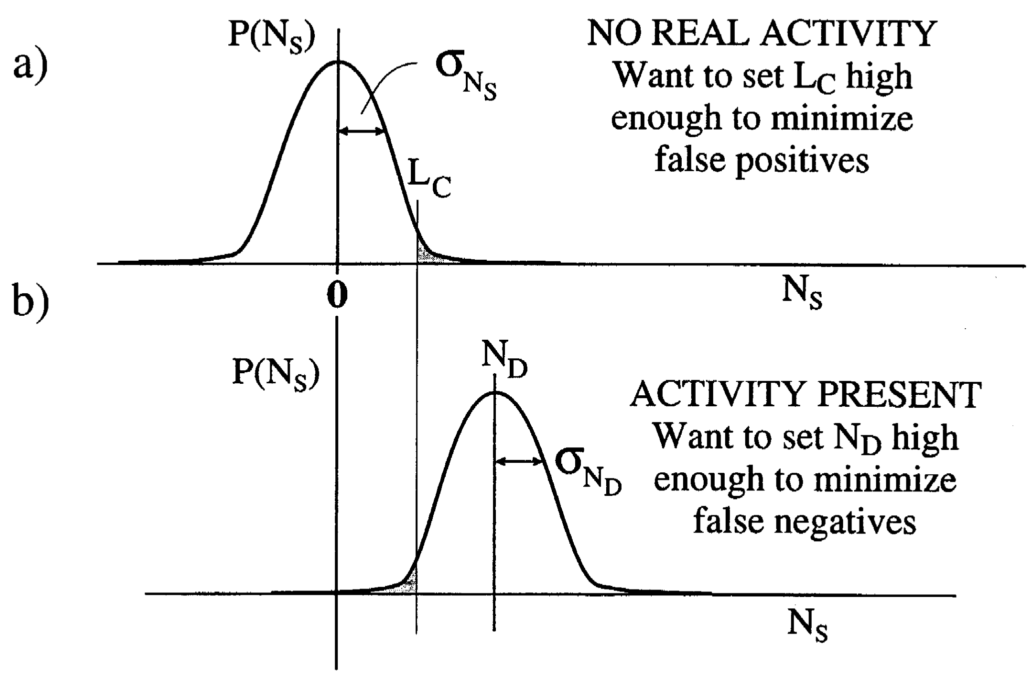
Figure 3.14 The distributions expected for the net counts N_S for the cases of (*a*) no activity present, and (*b*) a real activity present. L_C represents the critical level or “trigger point” of the counting system.

False Positive and False Negative Errors

Type I error (false positive) and Type II error (false negative) are two types of errors that carry different implications.

False positive ←→ minimum significant measured activity

False negative ←→ minimum detectable true activity



Limits of Detectability

Part II: False Positive Error and Minimum Significant Net Count Rate

False Positive Rate and Minimum Significant Net Count Rate – An Example

Example

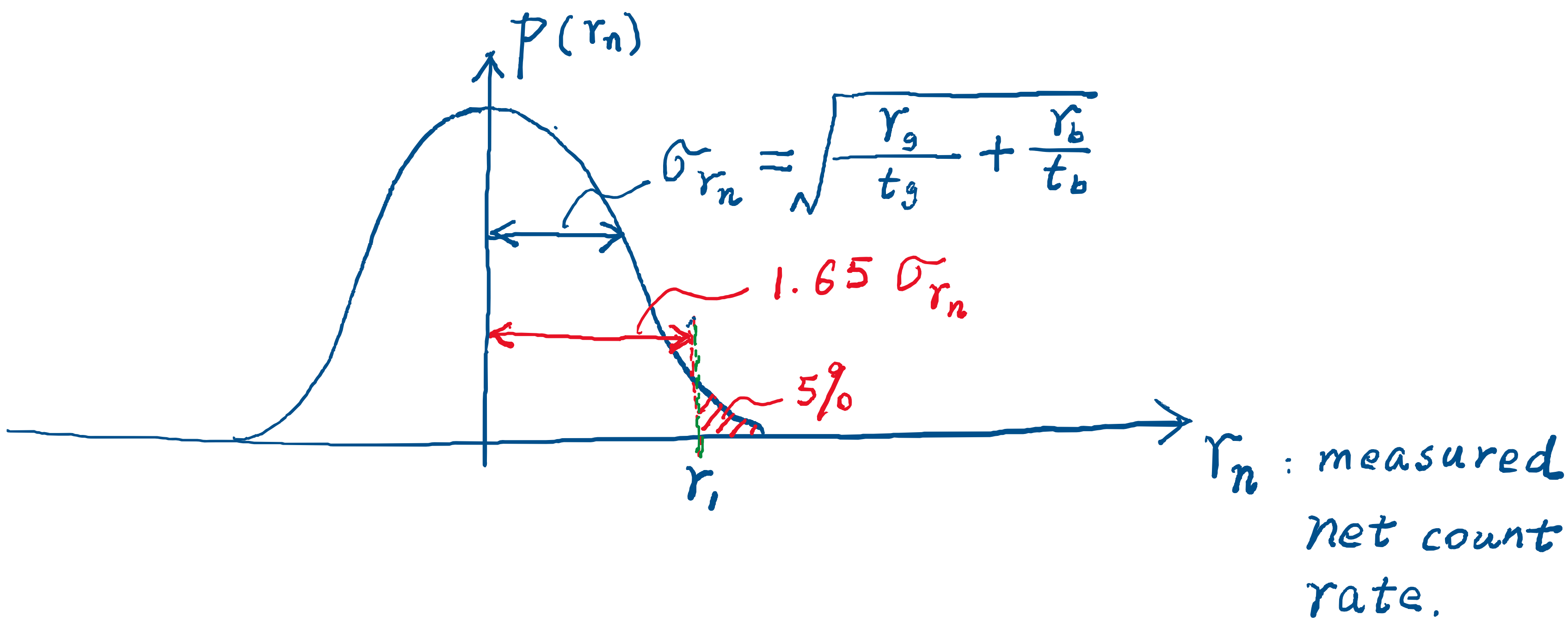
A sample, counted for 10 min, registers 530 gross counts. A 30-min background reading gives 1500 counts. (a) Does the sample have activity? (b) Without changing the counting times, what minimum number of gross counts can be used as a decision level such that the risk of making a type-I error is no greater than 0.050?

Turner, pp. 315-316

False Positive Rate and Minimum Significant Net Count Rate – An Example

Example

A sample, counted for 10 min, registers 530 gross counts. A 30-min background reading gives 1500 counts. (a) Does the sample have activity? (b) Without changing the counting times, what minimum number of gross counts can be used as a decision level such that the risk of making a type-I error is no greater than 0.050?



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False Positive Rate and Minimum Significant Net Count Rate – An Example

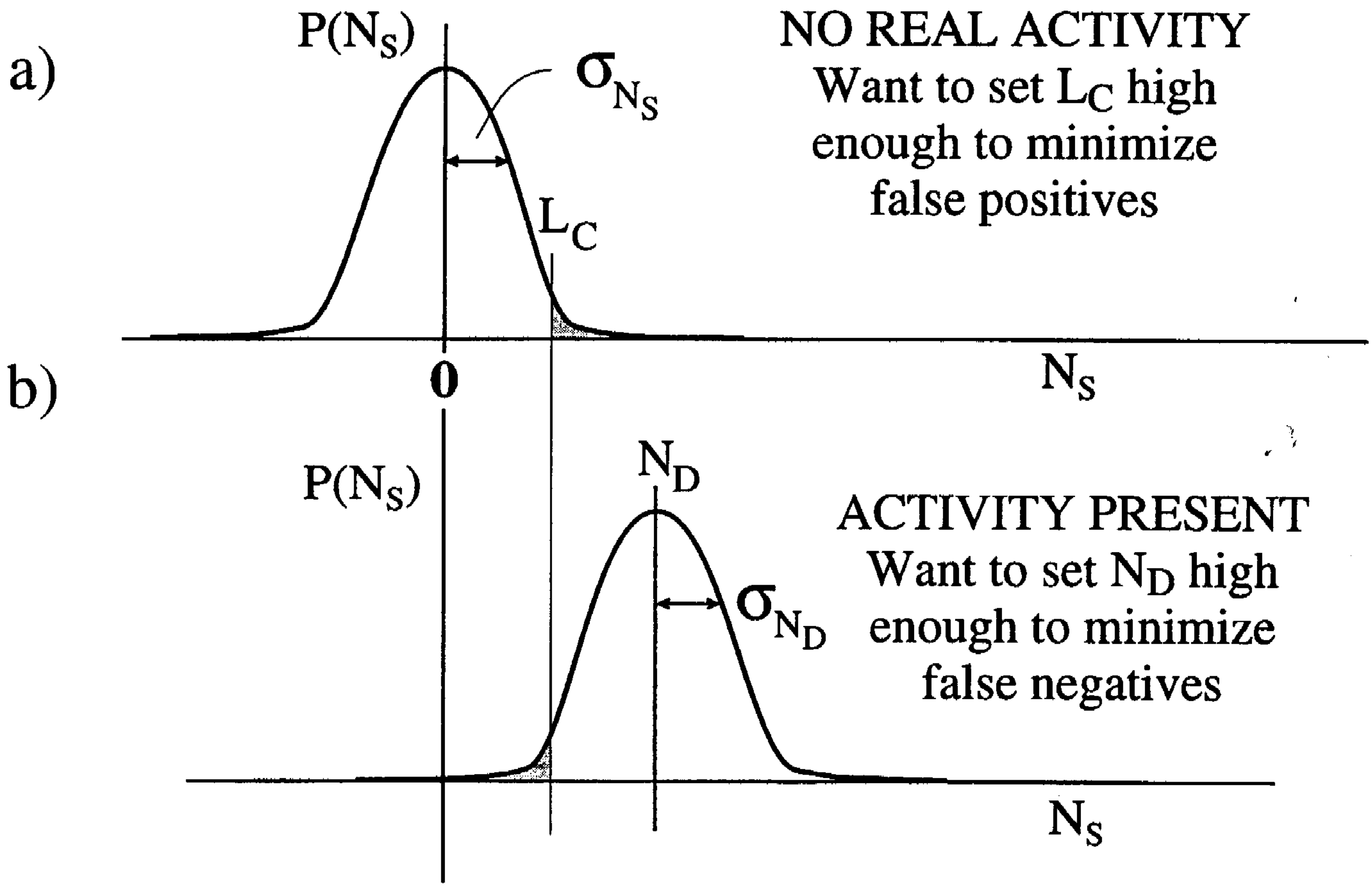
$$r_1 = 1.65\sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}} = 1.65\sqrt{\frac{r_1 + 50}{10} + \frac{50}{30}}, \tag{11.65}$$

where the substitution $r_g = r_1 + r_b$ has been made. This equation is quadratic in r_1. After some manipulation, one finds that

$$r_1^2 - 0.272r_1 - 18.2 = 0. \tag{11.66}$$

The solution is $r_1 = 4.40$ cpm. The corresponding gross count rate is $r_g = r_1 + r_b =$ 4.40 + 50 = 54.4 cpm, and so the critical number of gross counts is $n_g = r_g t_g =$ (54.4 min^{-1}) × (10 min) = 544. Thus, a sample giving $n_g > 544$ (i.e., a minimum of 545 gross counts) can be reported as having significant activity, with a probability no greater than 0.05 of making a type-I error.

Minimum Significant Net Count Rate



Minimum Significant Net Count Rate

Minimum significant measured net count rate (r_1) – the minimum measured net count rate that enables one to confirm the presence of activity while ensuring the probability of making a false positive error to be less than a given threshold α.

To derive the minimum significant measured net count rate (r_1), we write

$$r_1 = k_\alpha \sqrt{\sigma_{gr}^2 + \sigma_{br}^2} = k_\alpha \sqrt{\frac{r_1 + r_b}{t_g} + \frac{r_b}{t_b}}.$$

α : maximum probability for false positive error k_α : number of standard deviations of the net count rate that gives a one-tail area (under a Gaussian distribution) equal to α r_1 : the minimum significant measured net count rate

Minimum Significant Net Count Rate

To derive the minimum significant measured net count rate (r_1), we write

$$r_1 = k_\alpha \sqrt{\sigma_{gr}^2 + \sigma_{br}^2} = k_\alpha \sqrt{\frac{r_1 + r_b}{t_g} + \frac{r_b}{t_b}}.$$

Solving for r_1, we get the minimum significant measured net count rate (r_1) as

$$r_1 = \frac{k_\alpha^2}{2t_g} + \frac{k_\alpha}{2} \sqrt{\frac{k_\alpha^2}{t_g^2} + 4r_b \left(\frac{t_g + t_b}{t_g t_b} \right)}.$$

Minimum Significant Count Difference

When the gross and background only counting times are equal (t), we can derive the minimum significant count difference, Δ_1 as

The minimum difference in the counts measured in both measurements (gross and background) that ensures the probability of having Type I error to be smaller than the threshold α.

$$\Delta_1 = r_1 t = \frac{1}{2} k_\alpha^2 + \frac{1}{2} k_\alpha \sqrt{k_\alpha^2 + 8n_b},$$

$$= k_\alpha \sqrt{2n_b} \left(\frac{k_\alpha}{\sqrt{8n_b}} + \sqrt{1 + \frac{k_\alpha^2}{8n_b}} \right).$$

In many instances, we have $k_\alpha / \sqrt{n_b} << 1.$

Then

$$\Delta_1 \cong k_\alpha \sqrt{2n_b},$$

False Positive Rate and Minimum Significant Measured Count Difference

Often, the background can be measured accurately. The expected number of background counts B in time t is known.

In such case, if there is no source activity, the standard deviation of the net count is equal to sqrt(B). It follows that the minimum significant net count difference is

$$\Delta_1 = k_\alpha \sqrt{B} \qquad \text{(Background accurately known).}$$

$$\Delta_1 \cong k_\alpha \sqrt{2n_b}, \qquad k_\alpha / \sqrt{n_b} << 1.$$

The minimum significant net count difference is lowered by a factor of 1.414 when the background is well known.

Minimum Significant Measured Activity

Consider that the measurements were done with a detector of efficiency ε, then the minimum significant measured activity is

$$A_I = \frac{\Delta_1}{\epsilon t}.$$

If the measured net activity $A > A_I$, we state that the source contains activity, with the probability of false positive is $< \alpha$.

$$\Delta_1 = r_1 t = \tfrac{1}{2} k_\alpha^2 + \tfrac{1}{2} k_\alpha \sqrt{k_\alpha^2 + 8n_b},$$

$$= k_\alpha \sqrt{2n_b} \left(\frac{k_\alpha}{\sqrt{8n_b}} + \sqrt{1 + \frac{k_\alpha^2}{8n_b}} \right).$$

Minimum Significant Measured Activity

Example

A 10-min background measurement with a certain counter yields 410 counts. A sample is to be measured for activity by taking a gross count for 10 min. The maximum acceptable risk for making a type-I error is 0.05. The counter efficiency is such that 3.5 disintegrations in a sample result, on average, in one net count.

(a) Calculate the minimum significant net count difference and the minimum significant measured activity in Bq.

(b) How much error is made in (a) by using the approximate formula (11.72) in place of (11.69)?

(c) What is the decision level for type-I errors in terms of the number of gross counts in 10 min?

Minimum Significant Measured Activity

Example

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(a) Calculate the minimum significant net count difference and the minimum significant measured activity in Bq.

$$\Delta_1 = r_1 t = \tfrac{1}{2} k_\alpha^2 + \tfrac{1}{2} k_\alpha \sqrt{k_\alpha^2 + 8n_b},$$

Solution

(a) With equal counting times, $t_g = t_b = t = 10$ min, one can use Eq. (11.69) in place of the general expression (11.68). For $\alpha = 0.05$, $k_\alpha = 1.65$. With $n_b = 410$, we obtain

$$\Delta_1 = \tfrac{1}{2}(1.65)^2 + \tfrac{1}{2}(1.65)\sqrt{(1.65)^2 + 8(410)} = 48.6 = 49 \qquad (11.74)$$

for the minimum significant count difference in 10 min (rounded upward to the nearest integer). The counter efficiency is $\epsilon = 1/3.5 = 0.286$ dpm/cpm. It follows from Eq. (11.71) that the minimum significant measured activity is $A_I = 48.6/(0.286 \times 10 \text{ min}) = 17.0$ dpm $= 0.283$ Bq.

$$A_I = \frac{\Delta_1}{\epsilon t}.$$

Minimum Significant Measured Activity

(b) How much error is made in (a) by using the approximate formula (11.72) in place of (11.69)?

→ (c) What is the decision level for type-I errors in terms of the number of gross counts in 10 min?

Solution

(c) The decision level for gross counts in 10 min is $n_1 = n_b + \Delta_1 = 459$.

$$\Delta_1 = r_1 t = \frac{1}{2} k_\alpha^2 + \frac{1}{2} k_\alpha \sqrt{k_\alpha^2 + 8n_b},$$

Minimum Significant Measured Activity

The value $n_1 = 459$ in the last example can serve as a decision level for screening samples for the presence of activity by gross counting for 10 min. A sample showing $n_g < 459$ counts can be reported as having less than the "minimum significant measured activity," $A_I = 0.283$ Bq. A sample showing $n_g \geq 459$ counts can be reported as having an activity $(n_g - n_b)/\epsilon t = (n_g - 410)/2.86$ dpm.

Or "having no reportable activity."