Radioactivity

- Radioactivity is defined as the spontaneous nuclear transformation that results in the formation of new elements.
- Radioactivity and radioactive properties of nuclide are determined by nuclear considerations and independent of chemical and physical states of the radioisotope.
- The probability of radioactive transformation depends primarily on two factors:
  - Nuclear stability as related to the neutron-to-proton ratio.
  - The mass-energy relationship among the parent nucleus, daughter nucleus and the emitted particles.

Alpha Decay

Key concepts
- Coulomb barrier and energy release through alpha decay.
- Energy spectrum of alpha particles.
- Major health hazards related to alpha emission
An alpha particle is a highly energetic helium nucleus consisting of two neutrons and 2 protons.

- It is normally emitted from isotopes when the neutron-to-proton ratio is too low—called the alpha decay.
- Atomic number and atomic mass number are conserved in alpha decays.
Alpha Decay

With only a few exceptions (Samarium-147), naturally occurring alpha decay are found only among elements of atomic number greater than 82 because of the following reasons:

- Electrostatic repulsive force in heavy nuclei increases much more rapidly with the increasing atomic number than the cohesive nuclear force. The magnitude of the electrostatic repulsive force may closely approach or even exceed that of the nuclear force.

- Emitted alpha particles must have sufficiently high kinetic energy to overcome the potential barrier resultant from the strong nuclear force.

Energy Release from Alpha Decay

An example: Alpha decay of $^{226}\text{Ra}$

$$^{226}\text{Ra} \rightarrow ^{222}\text{Rn} + {^{4}}\text{He}.$$  

The energy $Q$ released in the decay arises from a net loss in the masses $M_{\text{Rn}}$, $M_{^{222}\text{Rn}}$, and $M_{^{4}\text{He}}$ of the radium, radon, and helium nuclei:

$$Q = M_{\text{Rn}} - M_{^{222}\text{Rn}} - M_{^{4}\text{He}}.$$  

The energy release can be found using the data shown in the table previously used for deriving binding energy

$$Q = 23.69 \rightarrow 16.39 - 2.42 = 4.88 \text{ MeV}.$$  

Understanding the Mass Defect and Nuclear Binding Energy

An example: Alpha decay of $^{226}\text{Ra}$

The energy release can be found using the data shown in the table previously used for deriving binding energy

$$Q = 23.69 \rightarrow 16.39 - 2.42 = 4.88 \text{ MeV}.$$  

Energy Release in Alpha Emission

A more accurate version

The required kinetic energy has to come from the decrease in mass following the decay process.

The relationship between mass and energy associated with an alpha emission is given as

$$M_{f} = M_{p} + M_{a} + 2M_{e} + Q.$$  

where $M_{p}$, $M_{a}$, and $M_{e}$ are respectively equal to the masses of the parent, the alpha, the emitted alpha particle, and the two orbital electrons that are lost during the transition to the lower atomic numbered daughter, while $Q$ is the total energy release associated with the radioactive transformation.
Energy Release from Alpha Decay

An example: Alpha decay of $^{226}\text{Ra}$

$$^{226}\text{Ra} \rightarrow ^{222}\text{Rn} + ^{4}\text{He}.$$  

The same example, when considering the daughter atom to have two less electrons,

$$M_p = M_d + M_e + 2M_n + Q.$$  

Here, $M_p$ is the mass of the parent, and $M_d$ is the mass of the product. $M_e$ is the mass of the electron, and $Q$ is the energy released in the reaction. For the $^{226}\text{Ra}$ example above:

$$Q = 7.94 \times 10^{-7} - 7.94 \times 10^{-8} - 4.78 \times 10^{-8} = 0.00005.$$  

What is the energy of the alpha particle?

**Note:** $M_p, M_d$: masses of the parent and daughter atoms

Naturally Occurring Radioactivity

Common characteristics of radioactive series:

- The first member of each series is very long-lived – $^{232}\text{Th}$: 1.39 x 10^{10} years, $^{238}\text{U}$: 4.51 x 10^9 years and $^{235}\text{U}$: 7.13 x 10^8 years.

- All three naturally occurring series each has a gaseous member.

- $^{222}\text{Rn}$ appears in uranium series and is called Radon

- $^{222}\text{Rn}$ appears in thorium series and is called Thoron

- $^{222}\text{Rn}$ appears in actinium series and is called Actinon

Artificially created radioactive series, such as the neptunium series has no gaseous member.

- The end product of all three naturally occurring radioactive series is lead.

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Series</th>
<th>Lifetime</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{222}\text{Rn}$</td>
<td>Uranium</td>
<td>3.81 days</td>
</tr>
<tr>
<td>$^{222}\text{Rn}$</td>
<td>Thorium</td>
<td>56 seconds</td>
</tr>
<tr>
<td>$^{222}\text{Rn}$</td>
<td>Actinium</td>
<td>4 seconds</td>
</tr>
</tbody>
</table>

The health concerns of these isotopes are determined by two factors:

- The rate of production from their parent nuclides.

- The probability of decay before get airborne.

The contributions from the daughters of $^{220}\text{Rn}$ and $^{222}\text{Rn}$ to internal exposure are usually negligible compared with that from $^{222}\text{Rn}$.  

Energy Spectra of Alpha Particles

Radon $^{222}\text{Rn}$ will decay either by alpha emission, or by $\gamma$-ray emission. With the $\gamma$-ray emission $(0.08\%\text{ of decay})$, the $\alpha$-particle has an energy of about 4.78 MeV. When there is no $\gamma$-ray emission in this case, the $\alpha$-particle has the full energy of 4.78 MeV, and we can look at the energy of the recoil nucleus from a simple consideration of conservation of energy and momentum:

$$E = \frac{m^2}{\sqrt{2}}.$$  

$E$ is the energy of the alpha particle, and $m$ is the mass of the recoil nucleus.

$$E = \frac{m^2}{\sqrt{2}} = 4.78$$

$E_{\text{max}} = 0.0069 \text{ MeV}$

Measured energy spectrum of alpha particles emitted from the decay of $^{238}\text{Pu}$.
Airborne radon itself poses little health hazard. It is not retained in significant amounts by the body.

The health hazard is closely related to the short-lived daughters of radon.

Beta Emission Processes

- Three favors of beta decay
- Energy spectrum of beta particles through beta decay
- Other processes involving beta emission, internal conversion, photoelectric effect and Auger electrons.
- Major health hazards related to beta emission
Chapter 1: Radioactivity

**Beta Emission**

- Beta particle is an ordinary electron. Many atomic and nuclear processes result in the emission of beta particles.
- One of the most common source of beta particles is the beta decay of nuclides, in which

\[ ^{A}Z_{X} \rightarrow ^{A+1}Z_{Y} + ^{0}_{-1}\beta + \bar{\nu} \]

**Beta decay**

\[ ^{A}Z_{X} \rightarrow ^{A+1}Z_{Y} + ^{0}_{-1}\beta + \bar{\nu} \]

**Beta-plus decay**

\[ ^{A}Z_{X} \rightarrow ^{A-1}Z_{Y} + ^{0}_{+1}\beta + \nu \]

**Electron capture**

\[ ^{A}Z_{X} + e^{-} \rightarrow ^{A}Z_{Y} + \nu \]

\[ p + \beta^{-} \rightarrow n + \nu \]

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**Energy Release of Beta Decay**

The energy release in a beta decay is given as

\[ Q = M_{p} - (M_{d} + M_{e}) \]

- The energy release is once again given by the conversion of a fraction of the mass into energy. Note that atomic electron bonding energy is neglected.
- For a beta decay to be possible, the energy release has to be positive.

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**Typical Energy Spectrum of Beta Particles**

The energy release is shared by all three daughter products. Due to the relatively large mass of the daughter nucleus, it attains only a small fraction of the energy. Therefore, the kinetic energy of the beta particle is

\[ E_{\beta} \approx Q - E_{\pi} \]

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**Positron Emission**

- A positron is the anti-particle of electrons, which carries the same mass as an electron but is positively charged.
- Positrons are normally generated by those nuclides having a relatively low neutron-to-proton ratio.
- A typical example of positron emitter is

\[ ^{22}_{11}\text{Na} \rightarrow ^{22}_{10}\text{Ne} + ^{0}_{+1}\beta + \nu \]

FIGURE 3.11. Decay scheme of \([\text{Na}].\)

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Energy Release Through Positron Decay

The energy release $Q$ associated with the positron emission process is given by

$$Q = M_p - M_e - M_\nu - M_\nu = M_p - (M_e + 2M_\nu)$$

where the atomic electron binding energy is ignored.

Example: Radioactive Decay

Nuclide A decays into nuclide B by $\beta^+$ emission (24%) or by electron capture (76%). The major radiation, energies (MeV), and frequencies per disintegration are, in the notation of Appendix D:

- $\beta^+$: 1.62 max (16%), 0.98 max (8%)
- $\gamma$: 1.51 (6%), 0.64 (55%), 0.614 (44%, $\nu$)

Daughter X rays

- $\nu$: 0.614

(a) Draw the nuclear decay scheme, labeling type of decay, percentages, and energies.

(b) What leads to the emission of the daughter X rays?

Gamma Ray Emission following Beta Decay

- Beta emissions are normally associated with complicated decay schemes and the emission of other particles such as gamma rays.

- There exist the so-called “pure beta emitters”, such as $^3$H, $^{14}$C, $^{32}$P, and $^{90}$Sr, which have no accompanying gamma rays.
Internal Conversion

An excited nucleus

De-excite through the emission of a gamma ray

\( \text{Gamma Ray Emission} \)

The excitation energy is transferred directly to an orbital electron, causing it to be ejected from the atom

\( \text{Internal Conversion} \)

Conversion electron with an energy

\[ E_\beta = E_{ex} - E_b \]

IC Coefficient (or Branching Ratio) = \( \frac{N_c}{N_b} \)

Auger Electrons

- The excitation energy of the atom may be transferred to one of the outer electrons, causing it to be ejected from the atom.

- Auger electrons are roughly the analogue of internal conversion electrons when the excitation energy originates in the atom rather than in the nucleus.

\[ E_{a.e.} = (E_{k_e} - E_{k_i}) - E_{e_i} \]

Different Types of Radiation from Hg-203 Beta Decay

- Energy of the beta particles?
- Relative frequency per decay?

\[ \text{IC Coefficient (or Branching Ratio)} = \frac{N_c}{N_b} \]
Chapter 1: Radioactivity

Atomic Radiation from Excited Atoms

Characteristic X-ray vs Auger Electron

The relative probability of the emission of characteristic radiation to the emission of an Auger electron is called the fluorescent yield, \( \omega_x \):

\[
\omega_x = \frac{\text{Number K x ray photons emitted}}{\text{Number K shell vacancies}} \quad (5-12)
\]

Values for \( \omega_x \) are given in Table 5-1. We see that for large \( Z \) values fluorescent radiation is favored, while for low values of \( Z \) Auger electrons tend to be produced.

From this table we see that if a nucleus with \( Z = 40 \) had a K shell hole, then on the average 0.74 fluorescent photons and 0.26 Auger electrons would be emitted.

<table>
<thead>
<tr>
<th>TABLE 5-1</th>
<th>Fluorescent Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>( \omega_x )</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>0.05</td>
</tr>
<tr>
<td>20</td>
<td>0.04</td>
</tr>
<tr>
<td>25</td>
<td>0.08</td>
</tr>
<tr>
<td>30</td>
<td>0.10</td>
</tr>
</tbody>
</table>

From Evans (11).

Understanding the Radiation from Cs-137

Decay scheme:

\[
^{137}_{55}\text{Cs} \rightarrow ^{137}_{56}\text{Ba} + ^0_\beta + ^0\bar{\nu}.
\]

Typical Decay Products from Unstable Radioisotopes

Alpha decay

Beta decay

Alpha particles

Daughter nuclei

Beta decay

Beta-plus decay

Beta particles

Positrons

Emission of characteristic X-ray

Annihilation gamma rays

Orbital electron capture (I.C.)

Excited Daughter Nuclei

Gamma ray emission

Internal conversion (I.C.)

Gamma rays

Bremsstrahlung X-rays

IC electrons

Excited atoms

Bremsstrahlung X-rays

Auger electrons

Characteristic X-rays

Understanding the Radiation from Cs-137

What will happen to the excited Ba-137 nucleus?

1. Beta particles for sure, what else?

2. Gamma-rays

3. Internal conversion electrons

4. Emission of characteristic X-ray

5. Auger electrons

6. Bremsstrahlung X-rays
If you are holding a Cs-137 source, what are the radiations that your hand/body is exposed to?

\[ ^{137}\text{Cs} ightarrow ^{137}\text{Ba} + e^- + \gamma. \]

**Radioactivity from Direct Excitation**

**X-ray Emission**

**X-ray Generation – X-ray Tube**

- Target nucleus positive charge (Z\(\cdot p^+\)) attracts incident e\(^-\)
- Deceleration of an incident e\(^-\) occurs in the proximity of the target atom nucleus
- Energy lost by e\(^-\) is gained by the EM photon (x-ray) generated
  - The impact parameter distance, the closest approach to the nucleus by the e\(^-\) determines the amount of E loss
  - The Coulomb force of attraction varies strongly with distance \(\propto \frac{1}{r^2}\); ↓ distance → ↑ deceleration and E loss → ↑ photon E
  - Direct impact on the nucleus determines the maximum x-ray E (E_{max})
**Chapter 3: Radioactivity**

### X-ray Generation – Characteristic X-rays

![X-ray Generation Diagram](Image)

**Figure 5.6**
Relative intensity of x-ray photons. (Adapted from Webster, 1994. This material is used by permission of John Wiley & Sons, Inc.)

- **Bremstralung (x-rays within anode)**
- **Leaving anode**
- **Characteristic radiation**
- **External filtering to reduce low energy photons**
- **Beam hardening**

**Characteristics of X-rays**
- Superimposed multiple flat spectrum with decreasing cutoff energy
- Low energy X-rays suffer attenuation inside the anode
- Further attenuation by the source package

### Neutron Sources

**Neutron Sources – Spontaneous Fission**

- **Cf-252 neutron source** can be made extremely compact

An engineer tests the prototype Towed Neutron Detector, a device that detects landmines. The neutron source of the landmine detector holds a tiny amount of californium-252. (Photo credit: Pacific Northwest National Lab)

<table>
<thead>
<tr>
<th>Term</th>
<th>Energy Range</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ultracold</td>
<td>&lt;2 × 10^-6 eV</td>
<td>6 m/s</td>
</tr>
<tr>
<td>Very cold</td>
<td>2 × 10^-6 eV to 5 × 10^-4 eV</td>
<td>100 m/s</td>
</tr>
<tr>
<td>Cold neutrons</td>
<td>5 × 10^-4 eV to 0.025 eV</td>
<td>—</td>
</tr>
<tr>
<td>Thermal</td>
<td>0.025 eV</td>
<td>2200 m/s</td>
</tr>
<tr>
<td>Epithermal</td>
<td>1 eV – 1 keV</td>
<td>4.4 × 10^3 m/s</td>
</tr>
<tr>
<td>Cadmium</td>
<td>&lt;0.4 eV</td>
<td>8800 m/s</td>
</tr>
<tr>
<td>Epicadmium</td>
<td>&gt;0.6 eV</td>
<td>1.1 × 10^3 m/s</td>
</tr>
<tr>
<td>Slow</td>
<td>&lt;1 to 10 eV</td>
<td>1.4 × 10^3 m/s</td>
</tr>
<tr>
<td>Resonance</td>
<td>1 to 300 eV</td>
<td>2.4 × 10^3 m/s</td>
</tr>
<tr>
<td>Intermediate</td>
<td>1 keV to 0.1 MeV</td>
<td>4.4 × 10^3 m/s</td>
</tr>
<tr>
<td>Fast</td>
<td>&gt;0.1 MeV</td>
<td>1.4 × 10^3 m/s</td>
</tr>
<tr>
<td>Ultra fast</td>
<td>&gt;20 MeV</td>
<td>—</td>
</tr>
<tr>
<td>Fissile</td>
<td>100 keV to 15 MeV</td>
<td>—</td>
</tr>
</tbody>
</table>

*In pile neutron physics usually refers to neutrons which are strongly captured in the resonance of U-238, and of a few commonly used elements, e.g., Bi, Au.
*Most probable energy: 0.6 MeV; Average energy: 2.0 MeV.
*Mesovital distribution of 20°C extends to about 0.1 eV.
**Neutron Sources – Spontaneous Fission**

Spontaneous fission of tarsuranic heavy nuclides, such as $^{252}$Cf, produces several fast neutrons, in addition to heavy fission products, prompt fission gamma rays and beta and gamma ray activities.

- Half-life: 2.65 years
- Neutron yield: 0.116 n/s per Bq, or $2.3 \times 10^6$ n/s per mg
- Neutron energy peaking at 0.5 MeV and extends beyond 10 MeV.

**Neutron Sources – Radioisotope ($\alpha$,n) Sources**

Energetic alpha particles can induce ($\alpha$,n) reaction in certain target materials.

$$^4\alpha + ^{12}Be = ^{12}C + ^1n$$  

Q-value: 5.71 MeV

The source is normally prepared in the form of alloy [MBe$_x$], where M is alpha-emitting radioisotopes.

**TABLE 9.2. ($\alpha$,n) Neutron Sources**

<table>
<thead>
<tr>
<th>Source</th>
<th>Average Neutron Energy (MeV)</th>
<th>Half-Life</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{239}$PuBe</td>
<td>4.2</td>
<td>138 d</td>
</tr>
<tr>
<td>$^{239}$PuB</td>
<td>2.5</td>
<td>138 d</td>
</tr>
<tr>
<td>$^{229}$RaBe</td>
<td>3.9</td>
<td>1600 y</td>
</tr>
<tr>
<td>$^{228}$RaB</td>
<td>3.0</td>
<td>1600 y</td>
</tr>
<tr>
<td>$^{235}$PuBe</td>
<td>4.5</td>
<td>24100 y</td>
</tr>
</tbody>
</table>

James Turner, Atoms, Radiation and Radiation Protection, p210-p211.
Neutron Sources – Radioisotope ($\alpha$,n) Sources

Practical considerations for choosing appropriate $\alpha$ emitter.

- Radioisotope ($\alpha$,n) sources are normally associated with other significant background radiations, especially when $^{226}$Ra and $^{227}$Ac are used.

- Choice has to be made between specific activity of the alpha emitter (and therefore neutron yield), source life-time and the availability of the isotope.

Some radioisotope gamma ray emitters can also be used to produce neutrons when combined with an appropriate target material.

- $^9$Be + $\gamma$ $\rightarrow$ $^9$Be + $\nu$, $Q$-value: $-1.666$ MeV

- $^2$H + $\gamma$ $\rightarrow$ $^2$H + $\nu$, $Q$-value: $-2.226$ MeV

- A gamma ray photon with an energy greater than the negative of the $Q$-value is required.

- Some practical gamma ray emitter include: $^{226}$Ra, $^{124}$Sb, $^{72}$Ga, $^{140}$La and $^{24}$Na.

Typical structure of photon neutron sources
Neutron Sources – Photo-neutron Sources

Calculated neutron energy spectra

Neutrons Generated by Accelerated Charged Particles

- Neutrons can be produced by nuclear reaction between accelerated charged particles.

  The D-D reaction: $^1_2H + ^1_2H \rightarrow ^3_2He + ^0_1n$, Q-value: 3.26MeV, $E_n=2.5MeV$
  The D-T reaction: $^1_2H + ^3_2H \rightarrow ^4_2He + ^0_1n$, Q-value: 17.6MeV, $E_n=14.1MeV$

- Due to the Coulomb barrier between the incident deuteron and the light target nucleus, a relatively small accelerating potential is required (about 100 to 300kV) to induce the reaction.

- The neutrons produced by a given nuclear reaction (D-D or D-T) have roughly the same energies.
### Neutrons Generated by Accelerated Charged Particles

**TABLE 9.1. Reactions Used to Produce Monoenergetic Neutrons with Accelerated Protons (p) and Deuterons (d)**

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Q Value (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^3\text{H}(d,n)^4\text{He})</td>
<td>17.6</td>
</tr>
<tr>
<td>(^3\text{H}(d,n)^4\text{He})</td>
<td>3.27</td>
</tr>
<tr>
<td>(^1\text{C}(d,n)^1\text{N})</td>
<td>-0.281</td>
</tr>
<tr>
<td>(^3\text{H}(p,n)^4\text{He})</td>
<td>-0.764</td>
</tr>
<tr>
<td>(^7\text{Li}(p,n)^7\text{Be})</td>
<td>-1.65</td>
</tr>
</tbody>
</table>

James Tuner, Atoms, radiation and Radiation Protection, p210-p211.

### A Typical D-T Neutron Generator

This compact and simple device can generate 5e11 neutrons per second by accelerating deuterium or tritium (depending on the desired neutron spectrum) into a deuterated or tritiated neutron production target.

http://btl.lbl.gov/neutrongamma.html

### Large Sized Neutron Sources

**Nuclear fission reactors**

Nuclear fission which takes place within a reactor produces very large quantities of neutrons and can be used for a variety of purposes including power generation and experiments.

**Nuclear fusion systems**

Nuclear fusion, the combining of the heavy isotopes of hydrogen, also has the potential to produce large quantities of neutrons. Small scale fusion systems exist for research purposes at many universities and laboratories around the world. A small number of large scale nuclear fusion systems also exist including the National Ignition Facility in the USA, JET in the UK, and the recently started ITER experiment in France.

**High energy particle accelerators**

A spallation source is a high-flux source in which protons that have been accelerated to high energies hit a target material, prompting the emission of neutrons.

Chapter 1.2: Transformation Kinetics
In many situations, the parent nuclides produce one or more radioactive offspring in a chain. In such cases, it is important to consider the radioactivity from both the parent and the daughter nuclides as a function of time.

- Due to their short half lives, $^{90}\text{Kr}$ and $^{90}\text{Rb}$ will be completely transformed, resulting in a rapid building up of $^{90}\text{Sr}$.
- $^{90}\text{Y}$ has a much shorter half-life compared to $^{90}\text{Sr}$. After a certain period of time, the instantaneous amount of $^{90}\text{Sr}$ transformed per unit time will be equal to that of $^{90}\text{Y}$.
- In this case, $^{90}\text{Y}$ is said to be in a secular equilibrium.

**Specific Activity (SA)**

Specific activity of a sample is defined as its activity per unit mass, given in units of Bq/g or Ci/g.

Specific activity for pure radioisotopes is defined as the number of Becquerels per unit mass.

\[
\text{Specific Activity} = \frac{\lambda \times 6.03 \times 10^{23} \text{ (atoms/mole)} \times A \text{ (g/mole)}}{\text{Number of atoms per unit mass} \times \text{The probability of an atom decaying within a unit time span}}
\]

SA can be related to the half-life ($T$) of the radionuclide by

\[
\text{SA} = 4.18 \times 10^{13} \frac{\text{Bq}}{\text{g}} \times \frac{\lambda}{A \cdot T}
\]

**Serial Transformation**

An example:

A solution of Hg(NO$_3$)$_2$ tagged with $^{208}\text{Hg}$ has a specific activity of $1.5 \times 10^9$ Bq/mL (4 $\mu$Ci/mL). If the concentration of mercury in the solution is 5 mg/mL.

(a) what is the specific activity of the mercury?

(b) what fraction of the mercury in the Hg(NO$_3$)$_2$ is $^{208}\text{Hg}$?

(c) what is the specific activity of the Hg(NO$_3$)$_2$?
Specific Activity (Continued)

(a) what is the specific activity of the mercury?

Solution:

$$SA(Hg) = \frac{activity\ from\ Hg\ per\ mL}{weight\ of\ Hg\ per\ mL} = \frac{1.5 \times 10^6\ Bq/mL}{5\ mg\ Hg/mL} = 3 \times 10^5\ \frac{Bq}{mg\ Hg}.$$  

and the specific activity of $^{203}$Hg is calculated from

$$SA = \frac{4.18 \times 10^{15}}{A - T} \frac{Bq}{g} = \frac{4.18 \times 10^{15}}{203 - 46.5d \cdot 24h \cdot d \cdot 3600s/h} \frac{Bq}{g} = 5.2 \times 10^{11} \frac{Bq}{g}.$$  

Specific Activity (Continued)

(b) what fraction of the mercury in the Hg(NO$_3$)$_2$ is $^{203}$Hg?

Solution:

The weight-fraction of mercury that is tagged is given by

$$\frac{SA(Hg)}{SA(^{203}Hg)}$$

and the specific activity of $^{203}$Hg is calculated from

$$SA = \frac{4.18 \times 10^{15}}{A - T} \frac{Bq}{g} = \frac{4.18 \times 10^{15}}{203 - 46.5d \cdot 24h \cdot d \cdot 3600s/h} \frac{Bq}{g} = 5.2 \times 10^{11} \frac{Bq}{g}.$$  

The weight fraction of $^{203}$Hg, therefore, is

$$\frac{SA(Hg)}{SA(^{203}Hg)} = \frac{0.3 \times 10^6\ Bq/g\ Hg}{5.2 \times 10^{11}\ Bq/g\ Hg} = 5.8 \times 10^{-5} \frac{^{203}Hg}{Hg}.$$  

Specific Activity (Continued)

(c) what is the specific activity of the Hg(NO$_3$)$_2$?

Solution:

Since an infinitesimally small fraction of the mercury is tagged with $^{203}$Hg, it may be assumed that the formula weight of the tagged Hg (NO$_3$)$_2$ is $324.63$ and that the concentration of Hg (NO$_3$)$_2$ is

$$\frac{324.63\ mg\ Hg\ (NO_3)_2}{200.61\ mg\ Hg} \times \frac{5\ mg\ Hg}{mL} = 8.1\ mg\ Hg\ (NO_3)_2/mL.$$  

The specific activity of the Hg(NO$_3$)$_2$:

$$1.5 \times 10^6\ Bq/mL \times \frac{8.1\ mg\ Hg\ (NO_3)_2}{mL} = 1.9 \times 10^4\ \frac{Bq}{mg\ Hg\ (NO_3)_2}.$$  

Specific Activity (Continued)

(d) what fraction of the mercury in the Hg(NO$_3$)$_2$ is $^{203}$Hg?

Solution:

The weight-fraction of mercury that is tagged is given by

$$\frac{SA(Hg)}{SA(^{203}Hg)}$$

and the specific activity of $^{203}$Hg is calculated from

$$SA = \frac{4.18 \times 10^{15}}{A - T} \frac{Bq}{g} = \frac{4.18 \times 10^{15}}{203 - 46.5d \cdot 24h \cdot d \cdot 3600s/h} \frac{Bq}{g} = 5.2 \times 10^{11} \frac{Bq}{g}.$$  

The weight fraction of $^{203}$Hg, therefore, is

$$\frac{SA(Hg)}{SA(^{203}Hg)} = \frac{0.3 \times 10^6\ Bq/g\ Hg}{5.2 \times 10^{11}\ Bq/g\ Hg} = 5.8 \times 10^{-5} \frac{^{203}Hg}{Hg}.$$  

General Case

Consider a more general case, in which (a) the half-life of the parent can be of any conceivable value and (b) no restrictions are applied on the relative half-lives of both the parent and the daughter.

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C,$$

The number of atoms of the parent A and the daughter B at any given time are therefore related by

$$N_B = \frac{\lambda_A N_A}{\lambda_B - \lambda_A} (e^{-\lambda_A t} - e^{-\lambda_B t}).$$
Proof of the Previous Serial Decay Equation

From Cember, p123-124

of the daughter, it follows that secular equilibrium is a special case of a more general situation in which the half-life of the parent may be of any conceivable magnitude but greater than that of the daughter. For this general case, where the parent activity is not relatively constant,

\[ A \rightarrow A_p \rightarrow A_d \]

the time rate of change of the number of atoms of species \( B \) is given by the differential equation

\[ \frac{dN_B}{dt} = \lambda_{A,B} N_A - \lambda_{B,N} N_B. \]  \hspace{1cm} (4.42)

In this equation, \( \lambda_{A,B} \) is the rate of transformation of species \( A \) and is exactly equal to the rate of formations of species \( B \), the rate of transformation of isotope \( B \) is \( \lambda_{B,N} N_B \), and the difference between these two rates at any time is the instantaneous rate of growth of species \( B \) at that time.

According to Eq. (4.18), the value of \( \lambda_{B,N} \) in Eq. (4.42) may be written as

\[ \lambda_{B,N} = \frac{N_B}{t}. \]  \hspace{1cm} (4.43)

Equation (4.42) may be rewritten, after substituting the expression above for \( N_B \) and transposing \( \lambda_{B,N} N_B \), as

\[ \frac{dN_A}{dt} \lambda_{B,N} = \lambda_{A,B} N_A e^{-\lambda_{A,B} t}. \]  \hspace{1cm} (4.44)

Proof of The Serial Decay Equation (Continued)

\[ N_{B} e^{-\lambda_{A,B} t} = \frac{\lambda_{A,B} N_A e^{\lambda_{A,B} t} - \lambda_{B,N} N_B}{\lambda_{B,N} - \lambda_{A,B}} + C. \]  \hspace{1cm} (4.48)

If the integrand in Eq. (4.48) is multiplied by the integrating factor \( \lambda_{B,N} - \lambda_{A,B} \), then Eq. (4.48) is in the form

\[ \int e^{-\lambda_{A,B} t} \frac{dN_B}{dt} e^{\lambda_{A,B} t} = e^{\lambda_{A,B} t} + C. \]  \hspace{1cm} (4.49)

and the solution is

\[ N_{B} e^{\lambda_{A,B} t} = \frac{1}{\lambda_{B,N} - \lambda_{A,B}} \lambda_{A,B} N_A e^{\lambda_{A,B} t} + C. \]  \hspace{1cm} (4.50)

The constant \( C \) may be evaluated by applying the boundary conditions

\[ N_B = 0 \quad \text{when} \quad t = 0 \]

\[ 0 = \frac{1}{\lambda_{B,N} - \lambda_{A,B}} \lambda_{A,B} R_{N_a} + C \]

\[ C = -\frac{\lambda_{A,B} R_{N_a}}{\lambda_{B,N} - \lambda_{A,B}}. \]  \hspace{1cm} (4.51)

If the value for \( C \), from Eq. (4.51), is substituted into Eq. (4.50), the solution for \( N_B \)

\[ N_B = \frac{\lambda_{A,B} R_{N_a}}{\lambda_{B,N} - \lambda_{A,B}} \left( e^{-\lambda_{A,B} t} - e^{-\lambda_{B,N} t} \right). \]  \hspace{1cm} (4.52)
Secular Equilibrium: $T_A >> T_B$ ($\lambda_A << \lambda_B$) and $t > 7T_B$

For the following serial transformation:

$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C,$

where $\lambda_A << \lambda_B$ and $T_A >> T_B$, B is said to be in secular equilibrium. For example

$^{90}\text{Kr} \xrightarrow{\frac{1}{2} \text{min}} ^{90}\text{Rb} \xrightarrow{2.74 \text{ min}} ^{90}\text{Sr} \xrightarrow{28.8 \text{ years}} ^{90}\text{Y} \xrightarrow{64.2 \text{ h}} ^{90}\text{Zr}.$

Activity Peaking Times Under General Case

From this relationship,

\[ Q_B = Q_A \]

one can see that $e^{-\lambda_B t}$ decreases and $Q_B$ approaches $Q_A$.

At equilibrium, we have

1. As the time goes by, $e^{-\lambda_B t}$ decreases and $Q_B$ approaches $Q_A$.

2. Since A has a relatively long half life, $Q_A$ may be considered as a constant. So the total activity converges to a constant.
Chapter 1: Radioactivity

Transient Equilibrium: $T_A \geq T_B \left(\lambda_A \leq \lambda_B\right)$ and $t > t_{md}$

A and B are in Transient Equilibrium

General case

$$\lambda_B N_B = \frac{\lambda_B A N_{B0}}{\lambda_B - \lambda_A} \left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

$$Q_B = \frac{\lambda_B}{\lambda_B - \lambda_A} Q_A$$

Summary of Serial Transformations

$$A \rightarrow \lambda_A \rightarrow B \rightarrow \lambda_A \rightarrow C$$

General case

Secular Equilibrium

Transient Equilibrium

No Equilibrium

$T_A > T_B$

$T_B > T_A$

$T_A \geq T_B \left(t > t_{md}\right)$

$T_A < T_B$

$Q_A = \frac{e^{-\lambda_A t}}{\lambda_A - \lambda_B}$

$Q_B = \frac{e^{-\lambda_B t}}{\lambda_B - \lambda_A}$

Secular Equilibrium: $T_A >> T_B \left(\lambda_A << \lambda_B\right)$

From "Radiation Protection and Dosimetry", by Michael Stabin.

We can continue on with species D, E, F and so on, but the relationships among the species obviously become more complicated and are difficult to categorize. If Species A is very long-lived, however, relative to other members of the chain, after a long time (even to any half-lives of the longest-lived progeny species), all the members of the chain will be in secular equilibrium and decaying with the half-life of Species A, and all having the same activity as Species A. An important example is the $\text{$^{226}$Ra}$ decay series (Figure 5.11).
Chapter 2: Interaction of Radiation with Matter

Key Aspects of Beta Interactions

Collisional interactions of beta particles with matter.
Specific energy loss of beta particles.
Dependence of the specific energy loss on the effective Z of absorbing material and the energy of the beta particles.
Mass stopping, what and why?
Radiative energy loss of beta particles.
Relative importance of collisional and radiative energy loss.
Fraction of energy loss due to Bremsstrahlung process and its implementation to shielding design for beta particles.
Range of beta particles.
Backscattering of beta particles.
Mechanisms of Energy Loss by Electrons

**Ionization and excitation:**

Beta particles may interact with orbital electrons through the electric fields surrounding these charged particles, which leads to excitation and ionization.

Ionization process can be modeled as an inelastic collision, the energy loss by the electron and the kinetic energy carried by the ejected electron is related by

\[ E_k = E_{\text{loss}} - \phi \]

where \( \phi \) is the ionization potential of the absorbing medium.

Specific Energy Loss of Beta Particles

**Specific energy loss:** the linear rate of energy loss by an electron through excitation and ionization, which is given by

\[
\frac{dE}{dx} = 2e\gamma^4 N Z \left( 3 \times 10^6 \right) f \left[ \frac{E_{\text{E}}E_{\beta}}{E_{\text{E}}(1 - \beta)} - 1 \right] \text{MeV/cm}
\]

where
- \( \gamma \) = charge on the electron, \( 1.6 \times 10^{-19} \text{C} \),
- \( N \) = number of absorber atoms per cm\(^3\),
- \( Z \) = atomic number of the absorber,
- \( N = 3.80 \times 10^{-6} \text{atom/cm}^3 \) for air at 10\(^3\) and 76\(^{0}\)cm Hg,
- \( E_{\text{E}} \) = energy equivalent of electron mass, \( 0.55 \text{MeV} \),
- \( E_{\beta} \) = kinetic energy of the beta particles, MeV,
- \( f \) = mass attenuation and mass energy of absorbing atoms, MeV,
- \( f = 8.6 \times 10^{-11} \text{for air, for other substances, } f = 3.15 \times 10^{-2} \).

Mechanisms of Energy Loss

Energy expenditure for creating ion pairs in media:

The average energy needed for creating an ion pair is normally 2 to 3 times greater than the corresponding electron binding energy in the absorbing medium.

<table>
<thead>
<tr>
<th>Gas</th>
<th>Ionization Potential</th>
<th>Mass Energy Expenditure per Ion Pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>H(_2)</td>
<td>11.6 eV</td>
<td>24.6 eV</td>
</tr>
<tr>
<td>H(_2)O</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
<tr>
<td>CO(_2)</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
<tr>
<td>Ar</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
<tr>
<td>Ne</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
<tr>
<td>Kr</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
<tr>
<td>Cl(_2)</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
<tr>
<td>CN(_2)</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
<tr>
<td>CN(_2)</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
<tr>
<td>N(_2)</td>
<td>13.6 eV</td>
<td>34.6 eV</td>
</tr>
</tbody>
</table>

The deviation between the ionization energy and the average energy required to create an ion pair is due to the excitation of the atoms, which does not lead to ionization.

**Specific Ionization**

In the context of radiation protection and health physics, it is normally important to specify the effect of the energy deposition by a beta particle in terms of the number of ion pairs created by the particle after traveling through a unit path length – the specific ionization.

\[ S.I. = \frac{dE/dx}{w} \text{ eV/\mu p} \]

where \( w \) is the average energy expenditure required to create a ion pair.

---

Cember, Introduction to Health Physics, Fourth Edition

NPREG 441 Principles of Radiation Protection, Spring 2022
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Range of beta particles.
Backscattering of beta particles.

Mass Stopping Power

It is also common to specify the energy loss of beta particles in a medium in terms of mass stopping power, which given by

\[ S = \frac{\text{specific energy loss (MeV/cm)}}{\text{density (g/cm}^3)} = \frac{dE}{dx} (\text{MeV} \cdot \text{cm}^2/\text{g}) \]

where \( \rho \) is the density of the absorbing medium.

Why mass stopping power?

• Mass stopping power does not differ greatly for materials with similar atomic compositions.
Mass stopping power for water can be scaled by density and used for tissue, plastics, hydrocarbons, and other materials that consist primarily of light elements.

In health physics, it is sometimes important to show the mass stopping power of different absorbers relative to that of air – the relative mass stopping power

\[ \rho_a = \frac{S_{a,\text{absorber}}}{S_{a,\text{air}}} \approx \frac{S_{\text{water}}}{3.67} \frac{\text{MeV}}{\text{g/cm}^2} \]

Remarks on the Mass Stopping Power

Mass Stopping Power for Compounds

H. Stopping Power in Compounds

The mass stopping power is the mass stopping power per unit mass. In the case of the mass stopping power of the material, the mass stopping power can be approximated by the mass stopping power of the constituent elements.

Mass stopping power of a compound is

\[ \left( \frac{dE}{dx} \right)_{\text{compound}} = \sum f_i \left( \frac{dE}{dx} \right)_{\text{element}_i} \]

where \( f_i \) are the mass fractions of the elements. 

References:

From Introduction to Radiological Physics and Radiation Dosimetry, by Frank Attix.
Chapter 2: Interaction of Radiation with Matter – Interaction of Beta Particles

Restricted Mass Stopping Power

1. Restricted Stopping Power

The mass collision stopping power ($d\Sigma /dx$) expresses the average rate of energy loss by a charged particle in all hard, as well as soft, collisions. The 8-rays resulting from hard collisions may be energetic enough to carry kinetic energy a significant distance away from the track of the primary particle. More importantly, if one is calculating the dose to a small object or tissue, it is convenient to consider the interactions of charged particles as being composed of hard collisions and 8-rays (as will be discussed in Section V.A). The use of the mass collision stopping power will overestimate the dose, unless the escaping 8-rays are replaced (i.e., unbacked-up 8-ray CPE exists).

The restricted stopping power is that fraction of the collision stopping power that includes all the soft collisions plus those hard collisions resulting in 8-rays with energies less than a cutoff value. The mass restricted stopping power in kg cm$^{-2}$/erg, will be symbolized here as ($d\Sigma /dx$)R.

Radiative Energy Loss of Beta Particles – Bremsstrahlung

Bremsstrahlung occurs when a beta particle is deflected or accelerated in the forced field of nucleus.

Key Aspects of Beta Interactions

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Backscattering of beta particles.

Radiative Energy Loss of Beta Particles – Bremsstrahlung

Part of the energy possessed by the beta particle is emitted in the form of photons.
The rate of energy loss is proportional to the square of the instantaneous acceleration experienced by the beta particle.

$$\left(\frac{dE}{dx}\right)_B = \frac{NEZ(Z+1)e^4}{137\gamma^2 E^2} \left(4 \ln\left(\frac{2E}{m_0 c^2}\right) - \frac{4}{3}\right)$$

$E$: kinetic energy of the beta particle,
$N$: number of absorber atoms per cm$^3$,
$Z$: atomic number of the absorber,
$m_0$: mass of an electron.
Backscattering

The fact that electrons often undergo large-angle deflections along their tracks leads to the phenomenon of backscattering. An electron entering one surface of an absorber may undergo subsequent deflections so that it re-emerges from the surface through which it entered. These backscattered electrons do not deposit all their energy in the absorbing medium and therefore can have a significant effect on the response of detectors designed to measure the energy of externally incident electrons. Electrons that backscatter in the detector “entrance window” or dead layer will escape detection entirely.

Knoll, Radiation Detection and Measurements, p47.

Monte Carlo Simulation of Electron Paths. This simulation is of 15 KeV electrons in fayalite (Fe2SiO4). Distances are given in nanometers (1000 nm = 1 µm). Paths of backscattered electrons are in red; those of absorbed electrons in blue. One should remember that this slice through a three-dimensional volume. This model was run using the Casino software described at http://www.gel.usherbrooke.ca/casino/what.html.

Knoll, Radiation Detection and Measurements, p49.

Figure 2.17 Fraction \( \eta \) of normally incident electrons that are backscattered from thick slabs of various materials, as a function of incident energy \( E \). (From Tabata et al.27)
Key Aspects of Beta Interactions

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Backscattering of beta particles.
Range of beta particles.

Tracks of Beta Particles in Absorbing Medium

Since beta particles have the same mass as the orbital electrons, they are easily scattered during collision and therefore follow tortuous paths in absorbing medium.
The electrons are “wondering” more significantly near the end of their tracks.
Energy-loss interactions are more sparsely distributed at the beginning of the track.

Range of Beta Particles

The range of beta particles is defined as the absorber thickness required to ensure that almost no beta particle can penetrate the entire thickness.

2.2 Interaction of Heavy Charged Particles
Chapter 2: Interaction of Radiation with Matter – Interaction of Heavy Charged Particles

Key Things to Remember

Interaction mechanisms

Bethe formula for linear stopping power

First collision energy transfer

Restricted stopping power and linear energy transfer

Stopping time and range of heavy charged particles

Energy Loss Mechanisms

For heavy charged particles, the maximum energy that can be transferred in a single collision is given by the conservation of energy and momentum:

\[
\frac{1}{2}Mv^2 = \frac{1}{2}Mv_f^2 + \frac{1}{2}mv_f^2
\]

\[
MV = MV_f + mv_f,
\]

where \(M\) and \(m\) are the mass of the heavy charged particle and the electron, \(V\) is the initial velocity of the charged particle, \(V_f\) and \(v_f\) are the velocities of both particles after the collision.

The maximum energy transfer is therefore given by

\[
Q_{\text{max}} = \frac{1}{2} MV^2 - \frac{1}{2} MV_f^2 = \frac{4mME}{(M + m)^2}
\]

Maximum Energy Loss by a Single Collision

For a more general case, which includes the relativistic effect, the maximum energy transferred by a single collision is

\[
Q_{\text{max}} = \frac{2\gamma^2 mV^2}{1 + 2\gamma m/M + m^2/M^2}
\]

where \(\gamma = 1/\sqrt{1 - \beta^2}\), \(\beta = V/c\), and \(c\) is the speed of light.
Maximum Energy Loss by a Single Collision

<table>
<thead>
<tr>
<th>Proton Kinetic Energy $E$ (MeV)</th>
<th>$Q_{\text{max}}$ (MeV)</th>
<th>Maximum Percentage Energy Transfer $100Q_{\text{max}}/E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.00022</td>
<td>0.22</td>
</tr>
<tr>
<td>1</td>
<td>0.0022</td>
<td>0.22</td>
</tr>
<tr>
<td>10</td>
<td>0.0219</td>
<td>0.22</td>
</tr>
<tr>
<td>100</td>
<td>0.229</td>
<td>0.23</td>
</tr>
<tr>
<td>$10^4$</td>
<td>3.33</td>
<td>0.53</td>
</tr>
<tr>
<td>$10^6$</td>
<td>$1.06 \times 10^4$</td>
<td>10.6</td>
</tr>
<tr>
<td>$10^8$</td>
<td>$5.38 \times 10^5$</td>
<td>53.8</td>
</tr>
<tr>
<td>$10^6$</td>
<td>$9.21 \times 10^6$</td>
<td>92.1</td>
</tr>
</tbody>
</table>

Linear Stopping Power of a Medium for Heavy Charged Particles (revisited)

The linear stopping power of a medium is given by the Bethe formula,

$$\frac{dE}{dx} = \frac{4\pi k^2 z^2 e^4 n}{mc^2 \beta^2} \left[ \ln \frac{2mc^2 \beta^2}{I(1 - \beta^2)} - \beta^2 \right].$$

$k = 8.99 \times 10^9$ N m$^2$ C$^{-2}$ (Appendix C),

- $z$ = atomic number of the heavy particle,
- $e$ = magnitude of the electron charge,
- $n$ = number of electrons per unit volume in the medium,
- $m$ = electron rest mass,
- $c$ = speed of light in m sec$^{-1}$,
- $\beta = v/c$ = speed of the particle relative to $c$,
- $I$ = mean excitation energy of the medium.

Linear Stopping Power – A Semiclassical Treatment

Consider the following diagram:

Step 1: Deriving the energy transfer from the heavy charged particle to a free electron nearby.

and assuming the electron is stationary during the collision...
Chapter 2: Interaction of Radiation with Matter – Interaction of Heavy Charged Particles

Deriving the Linear Stopping Power for Heavy Charged Particles – A Semiclassical Treatment

The total momentum imparted to the electron is given by

\[ p = \int_{-\infty}^{\infty} F_r \, dt = \int_{-\infty}^{\infty} \cos \theta \, dt = k_0 e \int_{-\infty}^{\infty} \cos \theta \, dt. \]

**Coulomb force**

\[ F = \frac{k_0 e}{r^2} \]  \hspace{1cm} (5.11)

To carry out the integration, we let \( t = 0 \) represent the time at which the heavy charged particle crosses the \( Y \)-axis in Fig. 5.4. Since \( \cos \theta = b/r \) and the integral is symmetric in time, we write

\[ \int_{-\infty}^{\infty} \cos \theta \, dt = 2 \int_{0}^{\infty} \frac{dt}{\sqrt{b^2 + V^2 t^2}} = 2 \int_{0}^{\infty} \frac{dt}{\sqrt{b^2 + V^2 t^2}} = \frac{2 \pi}{V b} \]  \hspace{1cm} (5.12)

Therefore, the total linear rate of energy-loss is given by

\[ \frac{dE}{dx} = 2\pi n \int_{Q_{\text{min}}}^{Q_{\text{max}}} \frac{Q_f \, db}{b} = 4\pi \frac{k_0^2 e^4 n}{mV^2} \ln \frac{b_{\text{max}}}{b_{\text{min}}}. \]  \hspace{1cm} (5.15)

---

Linear Stopping Power – A Semiclassical Treatment

Combining this result with (5.11) gives, for the momentum transferred to the electron in the collision, \( \text{2)} \)

\[ p = \frac{2k_0 e}{V b}. \]  \hspace{1cm} (5.13)

The energy transferred is

\[ Q = \frac{p^2}{2m} = \frac{2k_0^2 e^2 V^2}{mV^2 b^2}. \]

---

Linear Stopping Power – A Semiclassical Treatment

The maximum value of the impact parameter can be estimated from the physical principle that a quantum transition is likely only when the passage of the charged particle is rapid compared with the period of motion of the atomic electron. We denote the latter time by \( 1/\omega \), where \( \omega \) is the orbital frequency. The duration of the collision is of the order of \( b/V \). Thus, the important impact parameters are restricted to values approximately given by

\[ b \sim \frac{1}{V} \text{ or } b_{\text{max}} \sim \frac{V}{f}. \]

---

Step 2: Integrate the energy transfer to all the electrons surrounding the path of the heavy charged particle.

Fig. 5.3 Annular cylinder of length \( dx \) centered about path of heavy charged particle. See text.
Linear Stopping Power – A Semiclassical Treatment

For the minimum impact parameter, the analysis implies that the particles’ positions remain separated by a distance \( b_{\text{min}} \), at least as large as their de Broglie wavelengths during the collision. This condition is more restrictive for the less massive electron than for the heavy particle. In the rest frame of the latter, the electron has a de Broglie wavelength \( \lambda = \hbar/mV \), since it moves approximately with speed \( V \) relative to the heavy particle. Accordingly, we choose

\[
\frac{\hbar}{mV} \quad (5.17)
\]

Example

Calculate the maximum and minimum impact parameters for electronic collisions for an 8-MeV proton. To estimate the orbital frequency \( f \), assume that it is about the same as that of the electron in the ground state of the He\(^+\) ion.

\[
b_{\text{min}} \sim \frac{\hbar}{mV}.
\]

\[
b_{\text{max}} \sim \frac{V}{f}.
\]

\[\frac{V}{f} < 1 \quad \text{or} \quad b_{\text{max}} \sim \frac{V}{f}.
\]

Linear Stopping Power – A Semiclassical Treatment

\[
\frac{dE}{dx} = 2\pi \int_{Q_{\text{min}}}^{Q_{\text{max}}} Q \, db = \frac{4\pi k_e^2 e^4 n}{mV^2} \left( \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{db}{b} \right) \ln \frac{b_{\text{max}}}{b_{\text{min}}}.
\]

(5.15)

Combining the relations (5.15), (5.16), and (5.17) gives the semiclassical formula for stopping power,

\[
\frac{dE}{dx} = \frac{4\pi k_e^2 e^4 n}{mV^2} \ln \frac{mV^2}{\hbar f}.
\]

(5.18)
Linear Stopping Power – A Semiclassical Treatment

\[ b_{\text{min}} \sim \frac{h}{mV} \]

The minimum impact parameter is, from Eq. (5.17),

\[ b_{\text{min}} \sim \frac{h}{mV} = 6.63 \times 10^{-34} \text{ m} \times 9.11 \times 10^{-31} \text{ kg} \times 3.92 \times 10^7 \text{ m/s} \]
\[ = 1.86 \times 10^{-11} \text{ m} = 0.19 \AA. \quad (5.22) \]

Linear Stopping Power for Heavy Charged Particles

The linear stopping power of a medium is given by the Bethe formula,

\[ \frac{dE}{dx} = \frac{4\pi e^2 z^2 e^4 n}{m^2 v^2} \left[ \ln \frac{2mc^2\beta^2}{I(1-\beta^2)} - \beta^2 \right]. \]

\[ \frac{dE}{dx} = \frac{4\pi e^2 z^2 e^4 n}{m^2 v^2} \ln \frac{m^2 V^2}{I\hbar f} \]

\( k_0 = 8.99 \times 10^8 \text{ N m}^2 \text{ C}^{-2} \) (Appendix C),
\( z = \) atomic number of the heavy particle,
\( e = \) magnitude of the electron charge,
\( n = \) number of electrons per unit volume in the medium,
\( m = \) electron rest mass,
\( c = \) speed of light in vacuo,
\( \beta = \frac{V}{c} = \) speed of the particle relative to \( c \),
\( I = \) mean excitation energy of the medium.

Phenomena Associated with Charged Particles
Key Things to Remember

Interaction mechanisms

Bethe formula for linear stopping power

First collision energy transfer

Restricted stopping power and linear energy transfer

Stopping time and range of heavy charged particles

Restricted Stopping Power

In hard collisions, the scattered electron – or delta ray – can receive significant amounts of energy.

The delta ray can carry this energy a significant distance from the initial interaction site.

Need to separate this from the energy loss that is deposited locally.

The Rationale behind Restricted Stopping Power

If we are interested in microscopic events, in which incident particles deposit energy in local regions with finite sizes...

Since the predominate way for heavy charged particles to lose energy is to transferring its energies to energetic delta-rays...

If the range of the delta-ray is large compare to the dimension of the region-of-interest (ROI), it is likely that the energy carried by these delta-rays will not be fully deposited in the ROI.

To account for this effect, we will consider those delta-rays that carry energy less than a threshold, this give rise to the Restrict Stopping Power.

The value for the threshold is typically determined by the dimension of the ROI associated with the given application.
Chapter 2: Interaction of Radiation with Matter – Interaction of Heavy Charged Particles

Linear Energy Transfer (LET)

used with exactly the same meaning. In 1980, the ICRU defined LET$_\Delta$ as the restricted stopping power for energy losses not exceeding $\Delta$:

$$\text{LET}_\Delta = \left( \frac{dE}{dx} \right)_\Delta.$$  

(7.3)

with the symbol LET$_\infty$ denoting the usual (unrestricted) stopping power.

In 1998, the ICRU introduced the following new definition, also called “linear energy transfer, or restricted linear electronic stopping power, $L_\Delta$”:

$$L_\Delta = -\frac{dE}{dx}.$$ 

Restricted Stopping Power

Example

Use Table 7.1 to determine LET$_{1 \text{ keV}}$ and LET$_{5 \text{ keV}}$ for 1-MeV protons in water.

Solution

Note that Eq. (7.3) for LET involves stopping power rather than mass stopping power. Since $\rho = 1$ for water, the numbers in Table 7.1 also give $(-dE/dx)_\Delta$ in MeV cm$^2$/g. We find LET$_{1 \text{ keV}} = 238 \text{ MeV cm}^2$/g given directly in the table. Linear interpolation gives LET$_{5 \text{ keV}} = 252 \text{ MeV cm}^2$/g.

Table 7.1 Restricted Mass Stopping Power of Water, $(-dE/dx)_\Delta$, in MeV cm$^2$/g, for Protons

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>$(-dE/dx)_{1 \text{ keV}}$</th>
<th>$(-dE/dx)_{5 \text{ keV}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>910</td>
<td>910</td>
</tr>
<tr>
<td>0.10</td>
<td>711</td>
<td>910</td>
</tr>
<tr>
<td>0.50</td>
<td>399</td>
<td>438</td>
</tr>
<tr>
<td>1.00</td>
<td>146</td>
<td>236</td>
</tr>
<tr>
<td>10.0</td>
<td>26.8</td>
<td>33.5</td>
</tr>
<tr>
<td>100.0</td>
<td>3.92</td>
<td>5.97</td>
</tr>
</tbody>
</table>

Energy Loss Mechanisms

FIGURE 5.1. (Top) Alpha-particle autoradiograph of rat brain after inhalation of $^{212}$Po. Biological preparation by K. Masuz and N. Kamei. (Bottom) Beta-particle autoradiograph of isolated rat brain nucleus. The $^{32}$P-methylamine incorporated in the nucleus is located at the track origin of the electron emitted by the tracer element. Biological preparation by M. Wisniewski and F. Mandel. (Courtesy: B. Rechenmann and L. Whites.)

Key Things to Remember

Interaction mechanisms.

Bethe formula for linear stopping power

First collision energy transfer

Restricted stopping power and linear energy transfer

Stopping time and range of heavy charged particles
Range for Heavy Charged Particles

The mean range of any heavy charged particle with an initial kinetic energy $T$ can be related to the linear stopping power as the following:

$$ R(T) = \int_0^T \left( \frac{-dE}{dx} \right) dx $$

where

$$ \frac{-dE}{dx} = \frac{4\pi\varepsilon_0\gamma^2}{m_e^2} \ln \left( \frac{2m_e\beta^2}{(1-\beta^2)} \right) $$

where reciprocal of the linear stopping power gives the distance traveled per unit energy loss.

Substitute the expression for the linear stopping power into the above relationship, we have

$$ R(\beta) = \frac{M}{\gamma(1-\beta^2)} \beta^4 $$

where $\gamma = \frac{E'}{E}$, the energy of the moving particle $E'=Mc^2/(1-\beta^2)$, and the energy of the moving particle $E$.

We can substitute $T = Mc^2/(1-\beta^2)$, which reduces the alpha particle count to exactly one-half of its value in the absence of the absorber. The mean range $R_0$ and extrapolated range $R_e$ are indicated.

There are two related definitions of the range of heavy charged particles:

1. **Mean range**: the absorber thickness that reduces the alpha particle count to exactly one-half of its value in the absence of the absorber.

2. **Extrapolated range**: extrapolating the linear portion of the end of the transmission curve to zero.

The mean range of an alpha particle transmission experiment $I$ is the detected number of alpha particles for an absorber thickness $\delta$, whereas $I_0$ is the number detected without the absorber. The mean range $R_0$ and extrapolated range $R_e$ are indicated.

**Stopping Time for Heavy Charged Particles**

The formula for the stopping power can be used to calculate the rate at which a heavy charged particle slows down.

$$ -\frac{dE}{dt} = \left( -\frac{dE}{dx} \right) \frac{dx}{dt} = V \left( -\frac{dE}{dx} \right) $$

where

$$ V = \frac{dx}{dt} $$

is the velocity of the particle, and

$$ \frac{dx}{dt} = \frac{4\pi\varepsilon_0\gamma^2}{m_e^2} \ln \left( \frac{2m_e\beta^2}{(1-\beta^2)} \right) $$

So the time for a heavy charged particle with an initial kinetic energy $T$ to be fully stopped can be given by

$$ \tau(T) = \int_0^T \left( \frac{dE}{dE} \right)^{-1} dE $$

A rough estimate of the stopping time is given by

$$ \tau = \frac{T}{(dE/dx)_{|E=T}} = \frac{T}{V \cdot (dE/dx)_{|E=T}} $$

Note that since the stopping power is higher at lower particle energies, the actual slowing down time is shorter than the one estimated with the above equation.

**Stopping Time for Heavy Charged Particles**

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where

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Note that since the stopping power is higher at lower particle energies, the actual slowing down time is shorter than the one estimated with the above equation.
Limitation of the Bethe Formula
Since almost all analytical descriptions of the behavior of heavy charged particles are based on the Bethe formula, it is important to realize the limitation of this formula.

\[ \frac{dE}{dx} = 4 \times 10^{-3} e^2 \epsilon n \left[ \ln \frac{2mc^2 \beta^2}{\gamma(1 - \beta^2)} - \beta \right] \]

- The Bethe formula is valid for high energies as long as the inequality \( \gamma \sqrt{m/M} < 1 \) holds.
- At low energy, it fails because the term \( \ln \left[ \frac{2mc^2 \beta^2}{\gamma(1 - \beta^2)} \right] \) becomes negative giving a negative value for the stopping power.
- It does not account for the fact that at low energies, a charged particle may capture electrons as it moves, this will reduce its net charge and reduce the stopping power of the medium.
- The dependence of the Bethe formula on \( z^2 \) implies that a pair of particles, with the same amount of mass but opposite charge, have the same stopping power and range. Departures from this predication has been measured and theoretically predicted.

2.3 Interaction of Photons

Interactions of Photons with Matter

Reading Material:
- Chapter 5 in <<Introduction to Health Physics>>, Third edition, by Cumber.
- Chapter 8 in <<Atoms, Radiation, and Radiation Protection>>, by James E Turner.

Photoelectric Effect
In photoelectric process, an incident photon transfers its energy to an orbital electron, causing it to be ejected from the atom.

\[ E_e = h \nu - E_a \]

\( h \) is the Planck’s constant
\( \nu \) is the frequency of the photon

- Photoelectric interaction is with the atom in a whole and can not take place with free electrons.
- Photoelectric effect creates a vacancy in one of the electron shells, which leaves the atom at an excited state.
Photoelectric Effect Cross Section

Probability of photoelectric absorption per atom is

$$\sigma \propto \frac{Z^4}{(\hbar v)^3} \text{ low energy}$$

$$\sigma \propto \frac{Z^5}{(\hbar v)^6} \text{ high energy}$$

The interaction cross section for photoelectric effect depends strongly on $Z$.

Photoelectric effect is favored at lower photon energies. It is the major interaction process for photons at low hundred keV energy range.

Photoelectric Effect – Absorption Edges

- Requires sufficient photon energy for P.E. interaction.
- Interaction probability decreases dramatically with increasing energy.
- P.E. interaction is significant only for low energy photons, when the photon energy is close to the binding energies of the target atoms.

Relaxation Processes after Photoelectric Interaction

- The excited atoms will de-excite through one of the following processes:

  - Generation of characteristic X-rays
  - Generation of Auger electrons


Auger Electrons

The relative probability of the emission of characteristic radiation to the emission of an Auger electron is called the fluorescent yield, $\omega_k$:

$$\omega_k = \frac{\text{Number K x-ray photons emitted}}{\text{Number K shell vacancies}}$$

Values for $\omega_k$ are given in Table 3-1. We see that for large $Z$ values fluorescent radiation is favored, while for low values of $Z$ Auger electrons tend to be produced.

From this table we see that if a nucleus with $Z = 40$ had a K shell hole, then on the average 0.74 fluorescent photons and 0.26 Auger electrons would be emitted.

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\omega_k$</th>
<th>$Z$</th>
<th>$\omega_k$</th>
<th>$Z$</th>
<th>$\omega_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
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<td>0.4</td>
<td>40</td>
<td>0.74</td>
<td>70</td>
<td>0.92</td>
</tr>
<tr>
<td>20</td>
<td>0.19</td>
<td>45</td>
<td>0.80</td>
<td>75</td>
<td>0.95</td>
</tr>
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<td>80</td>
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<tr>
<td>50</td>
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<td>55</td>
<td>0.85</td>
<td>85</td>
<td>0.95</td>
</tr>
<tr>
<td>55</td>
<td>0.83</td>
<td>60</td>
<td>0.90</td>
<td>90</td>
<td>0.97</td>
</tr>
</tbody>
</table>

From Exams (EI)
Compton Scattering

In Compton scattering, the incident gamma ray photon is deflected by an orbital electron in the absorbing material.

Part of the energy carried by the incident photon is transferred to the target electron in the atom, causing it to be ejected from the atom.

Energy Transfer in Compton Scattering

If we assume that the electron is free and at rest, the scattered gamma ray has an energy

\[ h\nu' = \frac{h\nu}{1 + \frac{h\nu}{m_e c^2}(1 - \cos \theta)} \]

\( h\nu \) initial photon energy, \( \nu \) photon frequency
\( m_e \) mass of electron
\( c \) speed of light
\( \theta \) scattering angle

and the photon transfers part of its energy to the electron (assumed to be at rest before the collision), which is known as the recoil electron. Its energy is simply

\[ E_{\text{recoil}} = h\nu - h\nu' = \frac{h\nu}{1 + \frac{h\nu}{m_e c^2}(1 - \cos \theta)} \]

assuming the binding energy of the electron is negligible.

In the simplified elastic scattering case, there is an one-to-one relationship between scattering angle and energy loss!!

Basic Kinematics in Compton Scattering

The energy transfer in Compton scattering may be derived as the following:

• Assuming that the electron binding energy is small compared with the energy of the incident photon – elastic scattering.

• Write out the conservation of energy and momentum:

\[ h\nu + m_e c^2 = h\nu' + E' \]

Conservation of energy

\[ \frac{h\nu}{c} \cos \theta + P' \cos \varphi \]

Conservation of momentum

\[ \frac{h\nu}{c} \sin \theta = P' \sin \varphi \]

\( \theta \) scattering angle
\( \varphi \) energy transfer angle
\( h\nu \) incident photon energy
\( m_e c^2 \) binding energy of electron
\( P' \) momentum of recoiling electron


Energy Transfer in Compton Scattering

The scattering angles of the photon and the recoil electron are related by

\[ \cot \frac{\theta}{2} = \left( 1 + \frac{h\nu}{m_e c^2} \right) \tan \varphi \]

• The electron recoil angle is confined to the forward direction (0 ≤ \( \varphi \) ≤ 90°).

• The scattering angle of the photon can take any value between 0 and 180°.
Energy Transfer in Compton Scattering

The maximum energy carried by the recoil electron is obtained by setting $\theta$ to 180°,

$$E_{\text{max}} = \frac{2h\nu}{2 + mc^2/h\nu}$$

The maximum energy transfer is exemplified by the Compton edge in measured gamma ray energy spectra.

Example

In the previous example a 1.332-MeV photon from $^{60}$Co was scattered by an electron at an angle of 140°. Calculate the energy acquired by the recoil electron. What is the recoil angle of the electron? What is the maximum fraction of its energy that this photon could lose in a single Compton scattering?

$$\cos \frac{\theta}{2} = \left(1 + \frac{h\nu}{mc^2}\right) \tan \phi$$

Small angle scattering:

Energy carried by the scattered gamma ray depends strongly on scattering angle

Large angle scattering:

Energy carried by the scattered gamma ray depends only weakly on scattering angle

Initial photon energy

Compton scattered gamma energy

scattering angle

Figure from Page 320, Radiation Detection and Measurements, Third Edition, G. F. Knoll.
Derivation of the Relationship Between Scattering Angle and Energy Loss

The relation between energy the scattering angle and energy transfer are derived based on the conservation of energy and momentum:

\[ \overrightarrow{P_{hv}} + \overrightarrow{P_e} = \overrightarrow{P_{hv}'} + \overrightarrow{P_e'} \]

\[ E_{hv} + E_e = E_{hv'} + E_e' \]

Are those terms truly zero?

Compton Scattering with Non-stationary Electrons – Doppler Broadening

It is so far assumed that (a) the electron is free and stationary and (b) the incident photon is unpolarized.

When an incident photon is reflected by a non-stationary electron, for example an bond electron, an extra uncertainty is added to the energy of the scattered photon. This extra uncertainty is called Doppler broadening:

\[ h'v = \frac{h v}{m_e c^2 (1 - \cos(\theta))} \pm \sigma(h'v) \]

The one-to-one relationship between scattering angle and energy loss holds only when incident photon energy is far greater than the bonding energy of the electron...

Angular Distribution of the Scattered Gamma Rays

The differential scattering cross section of per electron is given by the Klein-Nishina formula:

\[ \frac{d\sigma}{d\Omega}(\theta) = \frac{1}{1 + \alpha(1 - \cos \theta)} \left( 1 + \alpha^2(1 - \cos \theta) \right)^{-\frac{5}{2}} \]

where

\[ a = \frac{hv}{m_e c^2} \quad \text{and} \quad r_e = \frac{ke^2}{m_e c^2} \]

is the classic electron radius (2.818 \times 10^{-13} \text{ m})
Angular Distribution of the Scattered Gamma Rays

The differential scattering cross section per electron for a photon scattered into a unit solid angle around a given scattering angle \( \theta \) can be expressed in the form,

\[
\frac{d\sigma}{d\Omega} = 2\pi \sin \theta \frac{d\theta}{
\left( \frac{1}{1 + \cos \theta} \right)^2 \left( \frac{1}{2} + \frac{a^2 (1 - \cos \theta)^2}{(1 + \cos \theta)^2 (1 + a (1 - \cos \theta))} \right) (m^2 sr^{-1})}
\]

where \( a = \frac{\hbar}{m_e c^2} \) and \( r_e \) is the classical electron radius.

Since

\[
d\Omega = 2\pi \sin \theta \, d\theta,
\]

then the Compton scattering cross section per electron is given by

\[
\sigma = 2\pi \int \frac{d\sigma}{d\Omega} \sin \theta \cdot d\theta \quad (m^2)
\]

Note that the Compton scattering cross section per electron is given in unit of \( m^2 \).

Total Compton Collision Cross Section for an Electron

Compton Collision Cross Section is defined as the total cross section per electron for Compton scattering. It can be derived by integrating the differential cross section, \( \frac{d\sigma}{d\Omega} \), over \( 4\pi \) solid angle.

Energy Distribution of Compton Recoil Electrons

Given the Klein-Nishina formula, how do we derive the energy spectrum of recoil electrons? In other words, how do we derive the probability of a gamma-ray undergoing a Compton scattering and transferring an energy falling into an energy window, \( E_{\text{rec}} \in [E' - \frac{1}{2} \Delta E, E' + \frac{1}{2} \Delta E] \)?

Incident photons with higher energies tend to scatter with smaller angles (forward scattering).

Incident photons with lower energy (a few hundred keV) have greater chance of undergoing large angle scattering (back scattering).
Energy Distribution of Compton Recoil Electrons

Klein-Nishina formula can be used to derive the energy spectrum of recoil electrons as the following:

$$\frac{d\sigma}{dE_{\text{recoil}}} = \frac{1}{2\pi} \frac{1}{(1 - \cos \theta)^2} \left( \frac{1 + n^2(1 - \cos \theta)^2}{1 + n^2(1 - \cos \theta)^2} \right) \times \left[ \frac{m_e^2}{k^2 \sin \theta} \right]$$

where

$$E_{\text{recoil}} = E' - \Delta E = E' - \frac{h\nu}{1 - \cos \theta}$$

Then $d\sigma/dE_{\text{recoil}}$ could be written as an explicit function of $E_{\text{recoil}}$.

$$\frac{d\sigma}{dE_{\text{recoil}}} (\theta) \rightarrow \frac{d\sigma}{dE_{\text{recoil}}} (E'_{\text{recoil}})$$
Chapter 2: Interaction of Radiation with Matter

Interaction of Photons with Matter

Energy Distribution of Compton Recoil Electrons

Remember that the maximum amount of energy that a photon can transfer to an electron in a single Compton scattering is given by:

\[ E_{\text{max}} = \frac{2h \nu}{2m_e \nu / h \nu} \]

The energy distribution of the recoil electrons derived using the Klein-Nishina formula is closely related to the energy spectrum measured with "small" detectors (in particular, the so-called Compton continuum).

Average fraction of energy transfer to the recoil electron through a single Compton Collision

Average recoil electron energy, \( E_{\text{avg, recoil}} \), is of special interest for dosimetry since it is an approximation of the radiation dose delivered by each photon through a single Compton scattering interaction.

The average fraction of energy transfer to the recoil electron through a single Compton scattering is given by:

\[ \frac{E_{\text{avg, recoil}}}{h \nu} = \int \frac{d \sigma}{d E_{\text{recoil}}} \frac{E_{\text{recoil}}}{h \nu} \left[ \frac{d \sigma}{d E_{\text{recoil}}} \right] \cdot d E_{\text{recoil}} \]

where \( \sigma \) is the Compton scattering cross section per electron and is given by

\[ \sigma = 2 \pi \int \frac{d \sigma}{d \Omega} \sin \theta \cdot d \theta \ (\text{m}^2) \]

Total Compton Collision Cross Section for an Electron

Compton Collision Cross Section is defined as the total cross section per electron for Compton scattering. It can be derived by integrating the differential cross section, \( d \Omega \), over \( 4 \pi \) solid angle.

Since

\[ d \Omega = 2 \pi \sin \theta \cdot d \theta, \]

the total Compton scattering cross section per electron is given by

\[ \sigma = 2 \pi \int \frac{d \sigma}{d \Omega} \sin \theta \cdot d \theta \ (\text{m}^2) \]

Note that the Compton scattering cross section per electron is given in unit of \( \text{m}^2 \).

Linear Attenuation Coefficient through Compton Scattering

The differential Compton cross section given by the Klein-Nishina Formula can also be related to another important parameter for gamma ray dosimetry — the linear attenuation coefficient.

Note that \( \sigma \) is the Compton scattering cross section per electron and is given by

\[ \sigma = 2 \pi \int \frac{d \sigma}{d \Omega} \sin \theta \cdot d \theta \ (\text{m}^2) \]

Linear attenuation coefficient through Compton scattering: the probability of a photon interacting with the absorber through Compton scattering while traversing a unit distance.

\[ \sigma_{\text{linear}} = N Z \sigma \ (\text{m}^{-1}) \]

Where \( N \) is the electron density of the absorber materials (number of electrons per \( \text{m}^3 \))
Pair Production

**Definition:**
Pair production refers to the creation of an electron-positron pair by an incident gamma ray in the vicinity of a nucleus.

**Characteristics**
- The minimum energy required is
  \[ E_x \geq 2m_e c^2 + \frac{2m_e^2 c^2}{m_{\text{nucleus}}} \approx 2m_e c^2 = 1.022 \text{MeV} \]
- The process is more probable with a **heavy nucleus** and incident **gamma rays with higher energies**.
- The positrons emitted will soon **annihilate** with ordinary electrons near by and produces two 511keV gamma rays.

Photonuclear Reaction

- A photon can be absorbed by an atomic nucleus and knock out a nucleon. This process is called **photonuclear reaction**. For example,
  \[
  ^{3}\text{Be} + h\nu \rightarrow ^{3}\text{Be} + ^{0}\text{n}, \quad Q \text{ value} \approx -1.666 \text{MeV} \\
  ^{1}\text{H} + h\nu \rightarrow ^{0}\text{H} + ^{0}\text{n}, \quad Q \text{ value} \approx -2.226 \text{MeV}
  \]
- The photon must possess enough energy to overcome the nuclear binding energy, which is generally several MeV.
- The threshold, or the minimum photon energy required, for \((\gamma, p)\) reaction is generally higher than that for \((\gamma, n)\) reactions. Since the repulsive Coulomb barrier that a proton must overcome to escape from the nucleus.
- Other nuclear reactions are also possible, such as \((\gamma, 2n), (\gamma, np), (\gamma, \alpha)\) and photon induced fission reaction.

Interaction of Photons in Matter

A few things to follow up from the discussions on interactions of photons with matter:
- **Photon attenuation coefficients.**
- **Energy transfer and energy absorption coefficients.**
Energy-Transfer Coefficient

The total mass energy transfer coefficient is given by

\[ \mu_{\text{tr}} = \frac{1}{\rho} \left( \frac{\delta}{\rho} + \frac{\sigma E_{\text{tr}}}{\rho h\nu} \right) + \frac{\kappa}{\rho} \left( 1 - \frac{2mc^2}{\rho h\nu} \right) \]

The fraction of energy that is carried away by characteristic x-rays following the photoelectric effect is

\[ \frac{\mu_{\text{p}}}{\rho} = \frac{1}{\rho} \left( 1 - \frac{\delta}{\rho} \right) + \frac{\sigma E_{\text{p}}}{\rho h\nu} + \frac{\kappa}{\rho} \left( 1 - \frac{2mc^2}{\rho h\nu} \right) \]

The fraction of energy that is transferred to recoil electron through Compton scattering.

Energy Transfer by a Gamma Ray Beam

Consider the fraction of energy that may be carried away by the subsequent bremsstrahlung photons, one can define the mass energy absorption coefficient as

\[ \mu_{\text{a}} = \frac{\mu_{\text{tr}} \rho}{\rho} (1 - g) \]

where \( g \) is the average fraction of energy of the initial kinetic energy transferred to electrons that is subsequently emitted as bremsstrahlung photons.

Energy Loss by Bremsstrahlung

For beta particles to stop in a given medium, the fraction of energy loss by bremsstrahlung process is approximately given by

\[ f_B = 3.5 \times 10^{-4} Z E_m \]

where \( f_B \) = the fraction of the incident beta energy converted into photons, 
\( Z \) = atomic number of the absorber, 
\( E_m \) = maximum energy of the beta particle, MeV.
Comparison Between Linear Attenuation Coefficient and Energy Absorption Coefficient

The fraction of photons removed from the beam after traveling through a unit distance.

The fraction of energy carried by the photons being absorbed in material.

Comparison Between Linear Attenuation Coefficient and Energy Absorption Coefficient

Calculation of Energy Transfer and Energy Absorption

For simplicity, we consider an idealized case, in which

- Photons are assumed to be monoenergetic and in broad parallel beam.
- Multiple Compton scattering of photons is negligible.
- Virtually all fluorescence and bremsstrahlung photons escape from the absorber.
- All secondary electrons (Auger electrons, photoelectrons and Compton electrons) generated are stopped in the slab.

Under these conditions, the transmitted energy intensity (the amount of energy transmitted through a unit area within each second) can be given by

$$\dot{\Psi} = \dot{\Psi}_0 e^{-\mu_{\text{abs}} x}$$

Interactions of Neutrons with Matter

Reading Material:

Elastic Scattering of Neutrons

Kinematics of neutron scattering:
- Energy transfer as a function of scattering angle.
- Angular distribution of scattered neutrons.
- Energy spectrum of scattered neutrons.
- Average logarithm energy decrement of a neutron in multiple scattering.

The maximum energy that a neutron of mass $M$ and kinetic energy $E_n$ can transfer to a nucleus of mass $m$ in a single elastic collision given by

$$E_{\text{max}} = E_n \frac{4Mm}{(M + m)^2}$$

### Table 9.4. Maximum Fraction of Energy Lost, $Q_{\text{el}}/E_n$ from Eq. (9.3), by Neutron in Single Elastic Collision with Various Nuclei

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$Q_{\text{el}}/E_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
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<tr>
<td>H</td>
<td>0.889</td>
</tr>
<tr>
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<td>Sn</td>
<td>0.033</td>
</tr>
<tr>
<td>U</td>
<td>0.017</td>
</tr>
</tbody>
</table>

Elastic Scattering of Neutrons

The elastic scattering plays an important role in neutron energy measurements. For example, a proton-neutron telescope illustrated below can be used to accurately measure the spectrum of neutrons in a collimated beam.

![Elastic Scattering of Neutrons](image)

**FIGURE 9.36.** Arrangement of proton-recoil telescope for measuring spectrum of neutron beam.

*Figure from Atoms, Radiation, and Radiation Protection, James E. Turner, p281*

As we will see later, over 85% of the “first-collision” dose in soft tissue (composed of H, C, O and N) arises from n-p scattering for neutron energy below 10MeV.

The recoil nuclei are essentially ionizing particles traveling in media and losing their energy through ionization and excitation.
Elastic Scattering of Neutrons

Kinematics of neutron scattering:

- Energy transfer as a function of scattering angle.
- Angular distribution of scattered neutrons.
- Energy spectrum of scattered neutrons.
- Average logarithm energy decrement of a neutron in multiple scattering.

Angular Distributions of the Scattered Neutrons

- For neutron energies up to 10 MeV, it is experimentally observed that the scattering of neutrons is isotropic in the center-of-mass coordinate system. The neutron and the recoil nuclei are scattered with equal probability in any direction in this 3-D coordinate system.

\[ d\Omega = \frac{2\pi \sin \theta \, r \, d\theta}{r^2} \]

Angular Distributions of the Scattered Neutrons

- Since the scattering in CM system is isotropic, the prob. of scatter into angular interval \( d\phi \) can be written as

\[ P(\phi) \, d\phi = \frac{2 \pi \sin \theta \, d\phi}{4\pi} \]

Elastic Scattering of Neutrons

- Energy transfer as a function of scattering angle.
- Angular distribution of scattered neutrons.
- Energy spectrum of scattered neutrons.
- Average logarithm energy decrement of a neutron in multiple scattering.
Angular Distributions of the Scattered Neutrons

Before collision, in laboratory system

\[ \vec{v}_0 = m, v_0 \]

\[ \vec{v}_1 = M + m v_0 \]

\[ \vec{v}_C \]

After collision, in laboratory system

\[ \vec{v}_1 = M + m v_0 \]

\[ \vec{v}_C \]

\[ \Theta : \text{scattering angle in the CM system} \]

\[ \theta \]

\[ \text{Center-of-mass} \]

\[ \text{Recoil nucleus} \]

The speed of the scattered neutron is

\[ \vec{v}' = \vec{v}_1 + \vec{v}_C. \]

Therefore

\[ v'^2 = v_1^2 + v_C^2 - 2 v_1 v_C \cos(\pi - \theta) = v_1^2 + v_C^2 + 2 v_1 v_C \cos \theta. \]

The kinetic energy of the scattered neutron, \( E' \), is

\[ E' = \frac{1}{2} m v'^2, \]

and

\[ \frac{E'}{E_0} = \frac{M^2 + m v_0^2}{M + m} + \frac{m}{M + m} v_0^2 \cos \theta. \]

Therefore,

\[ \frac{E'}{E_0} = \frac{\frac{M^2 + m v_0^2 + 2 M m v_0^2 \cos \theta}{(M + m)^2}}. \]

Energy Spectrum of the Scattered Neutrons

Since the scattering in the CM system is isotropic, the probability of the scattered neutron falling into an angular interval \([\theta, \theta + d\theta]\) is

\[ p(\theta) \cdot d\theta = \frac{(2 \pi) \sin \theta \cdot d\theta}{4 \pi} = \frac{1}{2} \sin \theta \cdot d\theta. \]

The probability of the outgoing neutron carrying a kinetic energy falling into a given window \([E', E' + dE']\) is given by

\[ p(E') \cdot dE' = -p(\theta) \cdot d\theta \]

\[ = \frac{1}{2} \sin \theta \cdot d\theta \]

\[ = \frac{1}{2} \sin \theta \left( \frac{M + m}{4 M m} \right)^{\frac{1}{2}} \cdot dE' \]

\[ = \frac{1}{2} \sin \theta \cdot \left( \frac{(M + m)^2}{4 M m} \right)^{\frac{1}{2}} \cdot \frac{1}{E_0} \cdot dE' \]

\[ = \frac{1}{2} \sin \theta \cdot \left( \frac{(M + m)^2}{4 M m} \right)^{\frac{1}{2}} \cdot \frac{1}{E_0} \cdot dE' \]

\[ = \frac{(M + m)^{\frac{1}{2}}}{4 M m} \cdot \frac{1}{E_0} \cdot dE' \]

\[ = \frac{(M + m)^{\frac{1}{2}}}{4 M m} \cdot \frac{1}{E_0} \cdot dE' \]
The fraction of energy carried by the scattered neutron is

\[
P(E') = \frac{1}{1 - \alpha E_0}, \\
E' \in \{a E_0, E_0\}.
\]

The distribution of the energy of the scattered neutrons is given by

\[
p(E' = \frac{1}{1 - \alpha E_0}, E' \in \{a E_0, E_0\}).
\]

Average energy carried by the scattered neutron:

\[
E_{\text{avg, energy}} = E_0 - E_{\text{avg}} = \frac{2Mm}{(M + m)^2} \cdot E_0
\]

Average energy transferred to the recoil nucleus:

\[
E_{\text{avg, energy}} = E_0 - E_{\text{avg}} = \frac{2Mm}{(M + m)^2} \cdot E_0
\]

Average logarithmic energy decrement of a neutron in multiple scattering.

\[
\xi = \Delta \ln E = \ln E_0 - \ln E = \ln \frac{E_0}{E} = -\ln \frac{E_0}{E}
\]

and

\[
\xi = 1 + \frac{\alpha \ln \alpha}{1 - \alpha}
\]

for the scattered neutron

where

\[
\alpha = \frac{(M - m)/(M + m)}
\]

The average logarithmic energy decrement is independent of the neutron energy and is a function only of the mass of the scattering nuclei.
Average Logarithmic Energy Decrement of Scattered Neutrons

The average logarithmic energy decrement per collision is defined as

$$\bar{\xi} = \frac{1}{E_i} \int_0^\infty \frac{\sigma_i}{E} \frac{dE}{E} = -\ln \frac{E_f}{E_i}$$

$$\bar{\xi} = \frac{1}{E_i} \int_0^\infty \frac{\sigma_i}{E} \frac{dE}{E}$$

$$\bar{\xi} = -\ln \left( \frac{E_f}{E_i} \right) = -\ln \left( \frac{E_0}{E_i} \right)$$

$$\bar{\xi} = -\ln \left( \frac{E_0}{E_i} \right)$$

$$\bar{\xi} = -\ln \left( \frac{E_0}{E_i} \right)$$

where

$$\sigma_i = \left( \frac{E}{E_i} \right)^k$$

Since

$$\ln \frac{E}{E_0} = -\xi$$

$$\frac{E}{E_0} = e^{-\xi}$$

The median fraction of the incident neutron's energy that is transferred to the nucleus during a collision is

$$f = 1 - \frac{E}{E_0} = 1 - e^{-\xi}$$

Fast- and Thermal-Diffusion Lengths

Fast-diffusion length: the average straight-line distance covered by fast neutrons traveling in a given medium.

Thermal-diffusion length: the average distance covered by thermalized neutrons before it is absorbed. It is measured by the thickness of a slowing down medium that attenuates the beam of thermal neutrons by a factor of e. Thus, the attenuation of a beam of thermal neutrons by a substance of thickness t (cm), whose thermal diffusion length is L (cm) is given by

$$R = t \exp(-t/L)$$

<table>
<thead>
<tr>
<th>Substance</th>
<th>Fast Diffusion Length, cm</th>
<th>Thermal Diffusion Length, cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>He</td>
<td>5.75</td>
<td>7.04</td>
</tr>
<tr>
<td>D_2O</td>
<td>11</td>
<td>171</td>
</tr>
<tr>
<td>Bi</td>
<td>9.9</td>
<td>24</td>
</tr>
<tr>
<td>C (graphite)</td>
<td>17.3</td>
<td>50</td>
</tr>
</tbody>
</table>
Interaction of Slow Neutrons (E<0.5eV)

The most important interactions between slow neutrons and absorbing materials are neutron-induced reactions, such as (n,γ), (n,α), (n,p) and (n, fission) etc. These interactions lead to more prominent signatures for neutron detection.

neutron + target nucleus ⇒ recoil nucleus
protons
alpha particles
fission fragments

Neutron Induced Reactions

\[ _1^1n + _{\text{He}}^3\text{He} \rightarrow _1^2\text{H} + _1^1\text{p} \]

- Cross section for thermal neutron is 5330 barns.
- \( Q = 765 \text{keV} \)
- Commonly used in proportional counters for fast neutron detection.

Neutron Induced Reactions

\[ _1^3\text{Li} + _1^1\text{H} \rightarrow _1^2\text{He} + _1^1\text{H} \]

- Cross section for thermal neutron is 940 barns.
- \( Q = 4.78 \text{MeV} \)
- Widely used for thermal neutron detection.

Neutron sensitive Li scintillator can be made or Li can be added to other scintillator to register neutrons.

\(^6\text{Li} \) is 7.42% abundant and Li enriched in the isotope \(^6\text{Li} \) is available.
Neutron Induced Reactions

\[ { }_0^1n + { }_{10}^{27}B \rightarrow { }_2^4Li + { }_2^4He \]

- Cross section for thermal neutron is 3840 barns.
- Q=2.31 MeV when the daughter nucleus is in an excited state (93%) and 2.79 MeV when the Li nucleus is in ground state (7%).
- Widely used for thermal neutron detection.
- BF₃ is a gas that can be used directly in a neutron counter.
- Boron is also used as a liner inside the tubes of proportional counters for neutron detection.

Energetics of Threshold Reactions

Consider the following reaction

\[ { }_0^1n + { }_{16}^{32}S \rightarrow { }_{15}^{32}P + { }_1^1p \]

- The neutrons must have an energy of above a certain threshold to enable this reaction.
- These reactions are called endothermic reactions, in which energy is converted into mass and therefore Q<0.

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Energetics of Threshold Reactions

Example:

Calculate the threshold energy for the reaction $^9\text{Be}(p,p')^9\text{Be}$.

Solution:

The reaction

\[ ^9\text{Be} + p \rightarrow ^9\text{Be} + p' \]

The atomic mass difference is given by

\[ \Delta = A_{^9\text{Be}} + A_p - A_{^9\text{Be}} - A_{p'} = 0.937 \text{ MeV} \]

The threshold energy is

\[ E_{\text{th}} = \gamma (\Delta M_{^9\text{Be}}) \text{ m.p.v.} \left( \frac{1}{A_{^9\text{Be}} + A_p} \right) \]

\[ = 0.937 \text{ MeV} \]

---

<table>
<thead>
<tr>
<th>Nuclide</th>
<th>Natural Abundance (%)</th>
<th>Type of Decay</th>
<th>Half-Life</th>
<th>Major Radiation, Energy (MeV), and Frequency per Disintegration (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{9}$Be</td>
<td>100</td>
<td>$^p$</td>
<td>2.06 y</td>
<td>$^p$ (0.546 MeV, avg 0.546 MeV) y $^p$, 0.937 MeV, y, and y'</td>
</tr>
<tr>
<td>$^{10}$Be</td>
<td>96.75</td>
<td>$^p$</td>
<td>1.56 h</td>
<td>$^p$ (0.546 MeV, avg 0.546 MeV) y $^p$, 0.937 MeV, y, and y'</td>
</tr>
<tr>
<td>$^{8}$Be</td>
<td>13.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{9}$Be</td>
<td>12.1</td>
<td>$^p$</td>
<td>7.16 x 10$^5$ y</td>
<td>$^p$ (0.546 MeV, avg 0.546 MeV) y $^p$, 0.937 MeV, y, and y'</td>
</tr>
<tr>
<td>$^{8}$Be</td>
<td>8.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{10}$Be</td>
<td>0.01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{9}$Be</td>
<td>2.5</td>
<td>$^p$</td>
<td>14.29 d</td>
<td></td>
</tr>
<tr>
<td>$^{8}$Be</td>
<td>0.006</td>
<td>$^p$</td>
<td>3.56 x 10$^6$ y</td>
<td>$^p$ (0.546 MeV, avg 0.546 MeV) y $^p$, 0.937 MeV, y, and y'</td>
</tr>
<tr>
<td>$^{9}$Be</td>
<td>0.001</td>
<td>$^p$</td>
<td>2.87 h</td>
<td>$^p$ (0.546 MeV, avg 0.546 MeV) y $^p$, 0.937 MeV, y, and y'</td>
</tr>
<tr>
<td>$^{10}$Be</td>
<td>0.0004</td>
<td>$^p$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Daughter radiation from $^{9}$Be.
Energetics of Threshold Reactions

Energy release: \[ Q = M_1 + M_2 - (M_3 + M_4) \] (1)

Conservation of energy: \[ E_1 = E_3 + E_4 + Q \Rightarrow E_1 = E_3 + E_4 \] (2)

Conservation of momentum: \[ p_1 = p_3 + p_4 \Rightarrow (2M_3E_3^{1/2}) = (2M_3E_3^{1/2} + 2M_4E_4^{1/2}) \] (3)

Substitute (2) into (3), we have
\[ (2M_3E_3^{1/2}) = (2M_3E_3^{1/2} + 2M_4E_4^{1/2}) \]

After some algebraic manipulations,
\[ E_1 \geq -Q \left(1 + \frac{M_1}{M_3 + M_4 - M_1}\right) \] (7)

Neutron Activation

For endothermic reactions, the minimum energy carried by the neutron [the threshold energy] can be derived based on the conservation of energy and momentum:
\[ E_1 = E_3 + E_4 - Q \]

The threshold energy is slightly greater than the \( Q \) value (the mass difference before and after the reaction).
\[ E_{th} = -Q \left(1 + \frac{M_1}{M_3 + M_4 - M_1}\right) \]

Neutron Activation

Neutron activation is the production of a radioactive isotope by the absorption of a neutron, such as in the \((n,p)\) reaction.

Neutron activation is important to health physicists for several reasons.
(a) Materials irradiated by neutrons may become radioactive. A radiation hazard may therefore persist after the irradiation by neutron is terminated.
(b) Neutron activation provides a convenient way to measure neutron flux.
(c) By spectroscopic examination of the induced radiation, quantitative analysis of the unknown samples is also possible.
Neutron Induced Reactions

\[ ^{1}n + ^{1}H \rightarrow ^{2}H + ^{0}H \]

- Neutron absorption followed by the immediate emission of a gamma ray photon.
- Since the thermal neutron has negligible energy by comparison, the gamma photon has the energy \( Q = 2.22 \text{MeV} \) released by the reaction, which represents the binding energy of the deuteron.
- The capture cross section per atom is 0.33 barn.
- When tissue is exposed to thermal neutrons, this reaction provides a source of gamma rays that delivers dose to the tissue.

Cross section for thermal neutron is 0.534 barns.
Q=0.626 MeV.
\( ^{23}\text{Na} \) undergo radioactive decay with the emission of two gamma rays, having energies of 2.75 MeV and 1.37 MeV per disintegration.
Since \( ^{23}\text{Na} \) is a normal constituent of blood, activation of blood sodium can be used as a dosimetric tool when persons are exposed to relatively high doses of neutrons, for example, in a criticality accident.

Neutron Induced Reactions

\[ ^{1}n + ^{14}\text{N} \rightarrow ^{14}\text{C} + ^{1}p \]

Cross section for thermal neutron is 1.70 barns.
Q=0.626 MeV.
Since the range of the proton and the \( ^{14}\text{C} \) nucleus are relatively small, their energy is deposited locally at the site where the neutron was captured.
Capture by hydrogen and nitrogen are the only two processes through which neutron deliver a significant dose to soft tissue.
Neutron Activation

The radioactivity induced by neutron activation (the number of disintegration of the activated daughter atoms per second) is given by

$$\lambda N = \phi \sigma n (1 - e^{-\lambda t})$$

where
- $\phi$ = flux, neutrons per cm$^2$ per s,
- $\sigma$ = activation cross section, cm$^2$,
- $\lambda$ = transformation constant of the induced activity,
- $N$ = number of radioactive atoms,
- $n$ = number of target atoms.

The saturation activity is given by $\phi \sigma$. For an infinitely long irradiation time, it represents the maximum obtainable activity with any given neutron flux.

The analysis leading to these results is identical to that used for analyzing the secular equilibrium for radioactive decay chains, in which the daughter has a much shorter decay time than that of the parent.

Statistical nature of radiation and radiation interaction:

- How much energy will a 1 MeV photon lose in its next collision with an atomic electron?
- Will a 400 keV photon penetrate a 2 mm lead shielding without interaction?
- When we use measured count-rate to estimate the activity of a source, and how certain are we on the estimation?
Radioactive Disintegration – Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- It consists of N trials.
- Each trial has a binary outcome: success or failure (decay or not).
- The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- The trials are independent.

In statistics, these four conditions characterize a Bernoulli process.

Binomial Distribution

Given, \( p, N \) and \( t \), what is the probability of observing \( n \) disintegrations within a time \( t \)?

The number of ways to choose \( n \) atoms from a total of \( N \) atoms in the sample is

\[
\binom{N}{n} = \frac{N!}{n!(N-n)!}
\]

So the probability of the \( n \) atoms chosen decayed during the time span \( t \) is

\[
P_n = \binom{N}{n} p^n q^{N-n}
\]

The above equation describes the so-called Binomial distribution.

What are the mean and standard deviation of a Binomial distribution?

Mean

The mean value \( \mu \) of the binomial distribution is defined by Eq. (11.15):

\[
\mu = \sum_{x=0}^{N} x \cdot p^x q^{N-x} = \sum_{x=0}^{N} x \cdot \binom{N}{x} p^x q^{N-x}.
\]  

(E.1)

To evaluate this sum, we first use the binomial expansion to write, for an arbitrary (continuous) variable \( x \),

\[
(px + q)^N = \sum_{x=0}^{N} \binom{N}{x} p^x q^{N-x} = \sum_{x=0}^{N} x^N P_x.
\]  

(E.2)

Differentiation with respect to \( x \) gives

\[
Np(px + q)^{N-1} = \sum_{x=0}^{N} x^{N-1} P_x = \sum_{x=0}^{N} x^{N-1} nP_n
\]  

(E.3)

Letting \( x = 1 \) and remembering that \( p + q = 1 \) gives

\[
Np = \sum_{x=0}^{N} nP_n = \mu.
\]  

(E.4)
For a binomial distribution, the mean or the expectation of the number of disintegration in time \( t \) is given by

\[
\mu = \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np
\]

and the fluctuation on the number of disintegrations is given by the variance or the standard deviation of the

\[
\sigma^2 = \sum_{n=0}^{N} (n - \mu)^2 \cdot P_n = Npq
\]

and

\[
\sigma = \sqrt{\sum_{n=0}^{N} (n - \mu)^2 \cdot P_n} = \sqrt{Npq}
\]

An Example Binomial Distribution

Example

More realistically, consider a \(^{40}\)K source with an activity of \(37 \text{ Bq} (= 1 \text{ nCi})\). The source is placed in a counter, having an efficiency of 100%, and the numbers of counts in one-second intervals are registered.

(a) What is the mean disintegration rate?
(b) Calculate the standard deviation of the disintegration rate.
(c) What is the probability that exactly 40 counts will be observed in any second?

The decay constant for \(^{40}\)K is \(\lambda = 0.0559 \text{ h}^{-1} = 1.55 \times 10^{-4} \text{ s}^{-1}\).

The number of atoms present is

\[
N = \frac{\frac{Q}{4.536 \times 10^{18} \text{ atoms}}}{1.55 \times 10^{-3} \text{ s}^{-1}} = 2.39 \times 10^9.
\]

From Eq. (11.18), we obtain for the standard deviation of the disintegration rate

\[
\sigma_n = \frac{\sqrt{Npq}}{t} = \frac{\sqrt{2.39 \times 10^9 \times 0.0000155 \times 0.9999845}}{1 \text{ s}} = 0.09 \text{ s}^{-1},
\]

which is about 16% of the mean disintegration rate.

Example

(a) The mean disintegration rate is the given activity, \(r_n = 37 \text{ s}^{-1}\).
(b) The standard deviation of the disintegration rate is given by Eq. (11.18). We work with the time interval, \(t = 1 \text{ s}\). Since the decay constant is \(\lambda = 0.0559 \text{ h}^{-1} = 1.55 \times 10^{-4} \text{ s}^{-1}\), we have

\[
q = e^{-\lambda t} = e^{-1.55 \times 10^{-4} \times 1} = 0.9999845
\]

and

\[
p = 1 - q = 0.0000155.
\]

The number of atoms present is

\[
N = \frac{Q}{4.536 \times 10^{18} \text{ atoms}} = \frac{37 \text{ s}^{-1}}{1.55 \times 10^{-3} \text{ s}^{-1}} = 2.39 \times 10^9.
\]

From Eq. (11.18), we obtain for the standard deviation of the disintegration rate

\[
\sigma_n = \frac{\sqrt{Npq}}{t} = \frac{\sqrt{2.39 \times 10^9 \times 0.0000155 \times 0.9999845}}{1 \text{ s}} = 0.09 \text{ s}^{-1},
\]

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Radioactive Disintegration – Bernoulli Process (Revisited)

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- It consists of \( N \) trials.
- Each trial has a binary outcome: success or failure (decay or not).
- The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- The trials are independent.

In statistics, these four conditions characterize a Bernoulli process.

Binomial Distribution (Revisited)

The probability of exactly \( n \) decays out of a total of \( N \) atoms in the source is

\[
P_n = \binom{N}{n} p^n q^{N-n}
\]

For a binomial distribution, the mean or the expectation of the number of disintegration within the measurement period is given by

\[
\mu = \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np
\]

and the fluctuation on the number of disintegrations is quantified by the variance

\[
\sigma^2 = \sum_{n=0}^{N} (n - \mu)^2 \cdot P_n = Npq
\]
Chapter 3: Counting Statistics

Poisson Distribution

Remember the conditions for Binomial distribution to be approximated by Poisson Distribution:
1. The number of trials, N, is very large, e.g. N>>1.
2. Each trial is independent.
3. The probability that each single trial is successful is a constant and approaching zero, p<<1. So the number of successful trials is fluctuating around a finite number.

The probability of having n successful trials can be approximated with the Poisson distribution.

\[ P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu} \]

and the mean and the variance of number of successful trial are given by

\[ \text{Mean}(n) = \mu = N \cdot p \]

\[ \text{Std}(n) = \sigma = \sqrt{\mu = \sqrt{Np}} \]

An example: Consider the following particle counting experiment.

- The detector covers 10% solid angle.
- Detection efficiency: \( \lambda = 55\% \).
- The measurement takes \( T = 1 \) s.
- \( N \) particles reached the detector.
- \( k \) detected particles.

What do we learn from this experiment?

\[ P(N \mid m) = \frac{m^N}{N!} e^{-m} \]

Suppose there are, in average, \( m \) particles reaching the detector during the given time period \( T \), the probability \( P(N \mid m) \) of having \( N \) particles reaching the detector during a given experiment would follow...

Poisson distribution:

\[ P(N \mid m) = \frac{m^N}{N!} e^{-m}. \]

Once the \( N \) particles reached the detector, the number of particles detected would follow...

the Binomial distribution, so that the probability of detecting \( k \) particles is

\[ P(k \mid N) = \binom{N}{k} \lambda^k (1 - \lambda)^{N-k}. \]
Chapter 3: Counting Statistics

Poisson Distribution – An example

Therefore, given the mean (average) number of particles reaching the detection is \( m \), the total probability of detecting \( k \) counts is

\[
P(k = m) = \sum_{k=m}^{\infty} P(k|N)P(N|m) = \sum_{k=m}^{\infty} \frac{N^k}{(N-k)!} e^{-(N-k)} e^{-m} \]

which is

\[
= \frac{(m)^k}{k!} e^{-m} = e^{-m \lambda} \]

If we would like to ensure 90% chance of detecting at least 1 particle, then we could set

\[
1 - P(k = 0) = e^{-m \lambda} = 0.9,
\]

then the mean number of particles reaching the detector during the measurement should be

\[
m = 4.2.
\]

So finally, we can conclude that:

Because we did not record any count, we have 90% confidence to claim that the source strength (average number of particles emitted per second) should not exceed

\[
A \leq 4.2 \times 10\% \cdot 1 \text{ s} = 42 \text{ particles per sec} = 42 \text{ Bq}
\]

The detector covers 10% solid angle with respect to the point source.

- detection efficiency: \( \lambda = 55\% \).
- The measurement takes \( T = 1 \text{ s} \).
- \( N \) particles reached the detector.
- detected \( k \)-\( 0 \) count.

The probability of having \( n \) successful trials can be approximated with the Poisson distribution.

\[
P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}
\]

and the mean and the variance of number of successful trial are given by

\[
Mean(n) = \mu = N \cdot p
\]

\[
Std(n) = \sigma = \sqrt{\mu} = \sqrt{Np}
\]
The Gaussian (Normal) Distribution

As \( p \) (the prob. of an atom decay within \( t \)) is getting even smaller and \( N \) is getting larger, both Binomial and Poisson distributions are approaching an extremely useful form of distribution – the Gaussian distribution.

Gaussian distribution is defined for a continuous variable \( x \)

\[
p(x | \mu, \sigma) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

but it is very useful for describing the counting fluctuation on discrete numbers.

Binomial and Poisson distributions practically match the normal distribution when \( \mu > 30 \).

Binomial distribution

The probability of observing \( n \) successful trials out of a total of \( N \) independent trials:

\[
P(n | N, p) = \binom{N}{n} p^n (1-p)^{N-n}
\]

Mean of \( n \):

\[
\mu = np
\]

Standard deviation:

\[
\sigma = \sqrt{np(1-p)}
\]

Poisson distribution when \( N \gg 1, np \ll 1 \)

\[
P(n | \lambda) = \frac{e^{-\lambda} \lambda^n}{n!}
\]

Mean of \( n \):

\[
\mu = \lambda
\]

Standard deviation:

\[
\sigma = \sqrt{\lambda}
\]

If \( N \) is further increased, and \( p \) is further decreased

\[
p(x | \mu, \sigma) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]
Chapter 3: Counting Statistics

The Gaussian (Normal) Distribution

Binomial and Poisson distributions practically match the normal distribution when \( \mu \geq 30 \).

The Gaussian (Normal) Distribution

For a variable, \( x \), following the normal distribution, the probability that it takes a value between \( x_1 \) and \( x_2 \) is equal to the area under the curve \( p(x) \) between these two values:

\[
P(x_1 \leq x \leq x_2) = \frac{1}{\sqrt{2\pi} \sigma} \int_{x_1}^{x_2} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} \, dx.
\]

Many common manipulations when carried out on counting data that were originally Gaussian distributed will produce derived values that also follow Gaussian shape:

- Multiplying or dividing the data by a constant,
- Combining two Gaussian-distributed variables through addition, subtraction, or multiplication or,
- Calculating the average of a series of independent measurements.

Central Limit Theorem

The sum or average of a large number of independent random variables tends to follow Gaussian (Normal) distribution.

The distribution of an average tends to be NORMAL, even when the distribution of the underlying variables from which the average is computed is decidedly non-Normal!

Central Limit Theorem

Consider a series of independent and identically distributed (i.i.d.) random variables, \( x_1, x_2, \ldots, x_n \), whose probability density function are given by

\[
p_x(x) = \begin{cases} 
1, & 0 \leq x \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]
Error and Error Propagation

Two ways to express the error associated with a given measurement:

Probable error:

- The symmetric range about the mean, within which there is 50% chance that a measurement will fall.
- The width of the range depends on the distribution of the variable. For example, for Gaussian distributed error, the probable error is $\pm 0.675 \sigma$.

Fractional standard deviation:

- The ratio of the standard deviation and the mean of the distribution of the random variable.
- For Poisson distributed random variable, the fractional standard deviation is simply $\frac{\sigma}{\mu} = \frac{1}{\sqrt{\mu}}$.

Error Propagation

Case 1: Sums or differences of counts – $u$ is the sum or difference of two random numbers representing counts measured in two independent experiments.

$$u = x + y \quad \text{or} \quad u = x - y$$

$$\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}$$

or

$$\sigma_u \approx \sqrt{\sum \left( \frac{\partial Q}{\partial x_i} \right)^2 \sigma_i^2}.$$

Example: estimating the net counts from a sample.

net counts = total counts – background counts

$$u = x - y$$

Error Propagation

Case 2: Multiplication or division by a constant

$$u = Ax$$

$$\sigma_u = A \sigma_x$$

Example: estimating the count rate, counting rate $= r = \frac{x}{t}$

Assuming that the error in the measuring time is negligible, we get

$$\sigma_r = \frac{\sigma_x}{t}$$
Error Propagation in Net Count Rate Measurement

As an application of the error-propagation formula, Eq. (11.46), we find the standard deviation of the net count rate of a sample, obtained experimentally as the difference between gross and background count rates, \( r_g \) and \( r_b \). As with gross counting, one also measures the number \( n_b \) of background counts in a time \( t_b \).

The net count rate ascribed to the sample is then the difference

\[
r_n = r_g - r_b = \frac{n_g}{t_g} - \frac{n_b}{t_b}.
\]

To find the standard deviation of \( r_n \), we apply Eq. (11.46) with \( Q = r_n \), \( x = n_g \), and \( y = n_b \). From Eq. (11.49) we have \( \frac{\partial r_n}{\partial n_g} = \frac{1}{t_g} \) and \( \frac{\partial r_n}{\partial n_b} = -\frac{1}{t_b} \). Thus, the standard deviation of the net count rate is given by

\[
\sigma_n = \sqrt{\sigma_{n_g}^2 + \sigma_{n_b}^2} = \sqrt{\frac{\sigma_{n_g}^2}{t_g^2} + \frac{\sigma_{n_b}^2}{t_b^2}}.
\]

(11.50)

\[\sigma_{n_g}^2 = \sigma_g^2 + \sigma_b^2, \quad \sigma_{n_b}^2 = \sigma_b^2, \quad \sigma_{n_g} = \sigma_g, \quad \sigma_{n_b} = \sigma_b,\]

Assuming no error on \( t \)

where \( \sigma_g^2 = \sigma^2(n_g) \), \( \sigma_b^2 = \sigma^2(n_b) \), \( \sigma_{g}^2 = \sigma^2(n_g) \), and \( \sigma_{b}^2 = \sigma^2(n_b) \). The net count rate ascribed to the sample is then the difference

\[
r_n = r_g - r_b = \frac{n_g}{t_g} - \frac{n_b}{t_b}.
\]

(11.49)

To find the standard deviation of \( n_b \), we apply Eq. (11.46) with \( Q = r_n \), \( x = n_g \), and \( y = n_b \). From Eq. (11.49) we have \( \frac{\partial r_n}{\partial n_g} = \frac{1}{t_g} \) and \( \frac{\partial r_n}{\partial n_b} = -\frac{1}{t_b} \). Thus, the standard deviation of the net count rate is given by

\[
\sigma_n = \sqrt{\sigma_n^2 + \sigma_{n_b}^2} = \sqrt{\frac{\sigma_{n_g}^2}{t_g^2} + \frac{\sigma_{n_b}^2}{t_b^2}}.
\]

(11.50)

\[\sigma_{n_g}^2 = \sigma_g^2 + \sigma_b^2, \quad \sigma_{n_b}^2 = \sigma_b^2, \quad \sigma_{n_g} = \sigma_g, \quad \sigma_{n_b} = \sigma_b,\]

Assuming no error on \( t \)

Turner, pp. 324.
Chapter 3: Counting Statistics

Error Propagation in Net Count Rate Measurement

Example
A long-lived radioactive sample is placed in a counter for 10 min, and 1426 counts are registered. The sample is then removed, and 2561 background counts are observed in 90 min. (a) What is the net count rate of the sample and its standard deviation? (b) If the counter efficiency with the sample present is 28%, what is the activity of the sample and its standard deviation in Bq? (c) Without repeating the background measurement, how long would the sample have to be counted in order to obtain the net count rate to within ±5% of its true value with 95% confidence? (d) Would the time in (c) also be sufficient to ensure that the activity is known to within ±5% with 95% confidence?

Solution:
(b) Since the counter efficiency is $\epsilon = 0.28$, the inferred activity of the sample is $A = \frac{t_s \epsilon}{t_b} = \frac{(114 \text{ min}^{-1})(0.28)}{0.28} = 407 \text{ dpm} = 6.78 \text{ Bq}$. The standard deviation of the activity is $\sigma_{A} = \frac{A}{\epsilon^2} = \frac{6.78 \text{ dpm}}{0.28^2} = 3.82 \text{ dpm} = 0.227 \text{ Bq}.

$$u = Ax$$

$$\sigma_u = A \sigma_x$$

Turner, pp. 324.
Chapter 3: Counting Statistics

Error Propagation in Net Count Rate Measurement

The chance of the measured net count rate to fall within ±5% of its true value is therefore 95%.

If we assume that the measured net count rate of 114 cpm is close enough to the true net count rate, then to ensure there is 95% chance that the measured net count rate would fall within ±5% of its true value, we need

114 (cpm) × 5% = 1.96 × σ_{true}.

Remember that

\[
\sigma_{true} = \sqrt{\frac{\sigma_1^2}{A_1^2} + \frac{\sigma_2^2}{A_2^2}}
\]

then

5.71 (cpm) = 1.96 \times \sqrt{\frac{\sigma_1^2}{A_1^2} + \frac{\sigma_2^2}{A_2^2}}
\Rightarrow \sigma_{true} = 17.5 \text{ (min)}

Chapter 3: Counting Statistics

Error Propagation

Let each individual measurement \( x_i \) be given a weighting factor \( a_i \) and the best value \( \langle x \rangle \) computed from the linear combination

\[
\langle x \rangle = \sum_{i=1}^{N} a_i x_i
\]

We now seek a criterion by which the weighting factors \( a_i \) should be chosen in order to minimize the expected error in \( \langle x \rangle \).

For brevity, we write

\[
\alpha = \sum_{i=1}^{N} a_i
\]

so that

\[
\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\alpha} \quad \text{and} \quad \sigma_{\langle x \rangle}^2 = \sigma^2_{\langle x \rangle} = \sum_{i=1}^{N} \left( \frac{\partial x_i}{\partial x} \right)^2 \sigma_i^2.
\]

Now apply the error propagation formula [Eq. (3.37)] to this case:

\[
\sigma^2_{\langle x \rangle} = \sum_{i=1}^{N} \left( \frac{\partial x_i}{\partial x} \right)^2 \sigma_i^2
\]

Knoll, p. 91.

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Error Propagation

Case 5: Combination of independent measurements with unequal errors

If \( N \) independent measurements of the same quantity have been carried out and not all the measurements have the same precision, what is the best way to estimate the best estimate of the mean value of the quantity to be measured?

The best estimate of the quantity, \( \langle x \rangle \), can be achieved by the weighted average

\[
\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}
\]

How to assign the weighting factors \( a_i \)’s?

Chapter 3: Counting Statistics

Error Propagation

Now apply the error propagation formula [Eq. (3.37)] to this case:

\[
\sigma^2_{\langle x \rangle} = \sum_{i=1}^{N} \left( \frac{\partial x_i}{\partial x} \right)^2 \sigma_i^2
\]

Knoll, p. 91.

where

\[
\alpha = \sum_{i=1}^{N} a_i \quad \beta = \sum_{i=1}^{N} a_i^2 \quad \sigma_{\langle x \rangle}^2 = \frac{\beta}{\alpha^2} \sum_{i=1}^{N} a_i^2 \sigma_i^2
\]

In order to minimize \( \sigma_{\langle x \rangle}^2 \), we must minimize \( a_i^2 \) from Eq. (3.46) with respect to a typical weighting factor \( a_j \):

\[
0 = \frac{\partial \sigma_{\langle x \rangle}^2}{\partial a_j} = \frac{a_j}{\alpha^2} - 2\beta a_j \frac{\partial a_j}{\partial a_j} \quad \text{or} \quad \sigma_{\langle x \rangle}^2 = \frac{1}{\alpha^2} \sum_{i=1}^{N} a_i^2 \sigma_i^2
\]
Error Propagation

\[ 0 = \frac{\partial^2 \xi}{\partial \xi_i^2} \frac{\partial \xi}{\partial \xi_i} - \frac{\partial \xi}{\partial \xi_i} \frac{\partial^2 \xi}{\partial \xi_i^2} \]

Putting these results into Eq. (3.47), we obtain

\[ a = \sum_{i=1}^{N} a_i \]

and solving for \( a \), we find

\[ a_i = \frac{1}{\sigma_i^2} \]

If we choose to normalize the weighting coefficients,

\[ \sum_{i=1}^{N} a_i = 1 \]

\[ a_i = \frac{1}{\sigma_i^2} \]

Therefore, the proper choice for the normalized weighting coefficient for \( x_i \), is

\[ a_i = \frac{1}{\sigma_i^2} \left( \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \right)^{-1} \]

We therefore see that each data point should be weighted inversely as the square of its own error.
Chapter 3: Counting Statistics

Error Propagation

Case 5: Combination of independent measurements with unequal errors (continued)

The proper choice for the normalized weighting factors for \( x_i \) is

\[
(x) = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}
\]

And the error (variance) on the weighted average is

\[
\sigma_{\bar{x}}^2 = \left( \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1}
\]

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Error Propagation in Net Count Rate Measurement

To find the standard deviation of \( r_n \), we apply Eq. (11.46) with \( Q = r_n \), \( x = n_2 \), and \( y = n_0 \). From Eq. (11.49) we have \( \partial r_n/\partial n_2 = 1/T_b \) and \( \partial r_n/\partial n_0 = -1/T_b \). Thus, the standard deviation of the net count rate is given by

\[
\sigma_{n_2} = \sqrt{\sigma_{n_2}^2 + \sigma_{n_0}^2} = \sqrt{\sigma_2^2 + \sigma_0^2}.
\]

(11.50)

Here \( \sigma_2 \) and \( \sigma_0 \) are the standard deviations of the numbers of gross and background counts, and \( \sigma_2 \) and \( \sigma_0 \) are the standard deviations of the gross and background count rates. Equation (11.50) expresses the well-known result for the standard deviation of the sum or difference of two Poisson or normally distributed random variables. Using \( n_2 \) and \( n_0 \) as the best estimates of the means of the gross and background distributions and assuming that the numbers of counts obey Poisson statistics, we have \( \sigma_2^2 = n_2 \) and \( \sigma_0^2 = n_0 \). Therefore, the last equation can be written

\[
\sigma_{n_2} = \sqrt{\frac{n_2}{T_s} + \frac{n_0}{T_b}} = \sqrt{\frac{T_s}{n_2} + \frac{T_b}{n_0}}.
\]

(11.51)

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Optimization of Counting Experiments

Case 6: Measuring the net count rate from a long-lived radioisotope.

\[
S = \text{counting rate due to the source alone without background}
\]

\[
B = \text{counting rate due to background}
\]

The measurement of \( S \) is normally carried out by counting the source plus background (at an average rate of \( S + B \)) for a time \( T_{S+B} \) and then counting background alone for a time \( T_B \). The net rate due to the source alone is then

\[
S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B}
\]

(2)

where \( N_1 \) and \( N_2 \) are the total counts in each measurement.

If the total measurement \( T = T_{S+B} + T_B \) is fixed, how to minimize the statistical error on the measured net count rate?

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Applying the results of error propagation analysis to Eq. (3.52), we obtain

\[
\sigma_S = \sqrt{\left( \frac{S}{T_{S+B}} \right)^2 + \left( \frac{B}{T_B} \right)^2}
\]

\[
\sigma_S = \sqrt{\left( \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B} \right)^2}
\]

\[
\sigma_S = \sqrt{\left( \frac{S + B}{T_{S+B}} \right)^2 + \left( \frac{B}{T_B} \right)^2}
\]

\[
\sigma_S = \sqrt{\left( \frac{N_1 - N_2}{T_{S+B}} \right)^2 + \left( \frac{B}{T_B} \right)^2}
\]

If we now assume that a fixed total time \( T = T_{S+B} + T_B \) is available to carry out both measurements, the above uncertainty can be minimized by optimally choosing the fraction of \( T \) allocated to \( T_{S+B} \) (or \( T_B \)). We square Eq. (3.53) and differentiate

\[
2\sigma_S dS = -\frac{S}{T_{S+B}} dT_{S+B} - \frac{B}{T_B} dT_B
\]

and set \( d\sigma_S = 0 \) to find the optimum condition. Also, because \( T \) is a constant, \( dT_{S+B} + dT_B = 0 \). The optimum division of time is then obtained by meeting the condition

\[
\frac{T_{S+B}}{T_B} = \sqrt{\frac{S + B}{B}}
\]

(3.54)
Limits of Detectability

For a counting system, it is useful to set a detection limit. That is, the amount of activity can be detected reliably.

The basic procedure could be

1. Setting a certain confidence level – the probability that a decision (on whether or not a source is present) is correct.
2. Define a quantity based on which the decision can be made. In the source counting case, it is the net count per unit time

\[ n_s = n_g - n_b \]

where

- \( n_s \): net counts
- \( n_g \): gross counts
- \( n_b \): background counts

3. Finding a critical level, \( L_c \). If \( n_s \) exceeds \( L_c \) we assume source activity is present, otherwise we assume that the source does not contain activity.

False Positive and False Negative Errors

Due to the statistical fluctuation on the counts measured within a given time, there will be

1. many instances in which a positive \( n_s \) is above the critical level even for samples with no activity, which leads to the false positive.
2. and similarly, measured counts is lower than the critical level even when the source contains non-zero activity, which leads to the false negative.

False Positive Rate and Minimum Significant Net Count Rate – An Example

Example

A sample, counted for 10 min, registers 530 gross counts. A 30-min background reading gives 1500 counts. (a) Does the sample have activity? (b) Without changing the counting times, what minimum number of gross counts can be used as a decision level such that the risk of making a type-I error is no greater than 0.050?

Example

A sample, counted for 10 min, registers 530 gross counts. A 30-min background reading gives 1500 counts. (a) Does the sample have activity? (b) Without changing the counting times, what minimum number of gross counts can be used as a decision level such that the risk of making a type-I error is no greater than 0.050?

Solution

(a) The numbers of gross and background counts are \( n_g = 530 \) and \( n_b = 1500 \); the respective counting times are \( t_g = 10 \) min and \( t_b = 30 \) min. The gross and background counts are \( n_g = n_g t_g = 530 \) cpm and \( n_b = n_b t_b = 50 \) cpm. The question of whether activity is present cannot be answered in an absolute sense from these measurements. The observed net rate could occur randomly with or without activity in the sample. We can, however, compute the probability that the result would occur randomly when we assume that the sample has no activity. To do this, we compute the net count rate with its estimated standard deviation \( \sigma_n \), given by Eq. (11.51):

\[ \sigma_n = \sqrt{\frac{t_b}{t_g} \sigma_b^2 + \frac{t_g}{t_b} \sigma_g^2} \]

where \( \sigma_g = \frac{33}{\sqrt{50}} = 3.66 \) and \( \sigma_b = \frac{50}{\sqrt{30}} = 2.64 \) cpm.

The observed net rate differs from 0 by 3.66: 1.14 standard deviations. As found in Table 11.1, the area under the standard normal curve to the right of this value is 0.125. Assuming that the activity is zero, as shown in Fig. 11.4, we conclude that an observed net count rate greater than the observed \( 1.14 \sigma_n = 3 \) cpm would occur randomly with a probability of 0.127. This single set of measurements, gross and background, is thus consistent with the conclusion that the sample likely contains little or no activity. However, one does not know where the bell-shaped curve in Fig. 11.4 should be centered. Based on this single measurement, the most likely place is \( \sigma_n = 3 \) cpm, with the sample activity corresponding to that value of the net count rate.
Possible conclusion #1: If there is no activity in the source, there will be 87% of chance of observing less than or equal to 3 cpm, the fact that we measured 3 cpm seems consistent with the assumption that there is no activity – so we can conclude that there is NO ACTIVITY in the source.

Possible conclusion #2: We don’t know where this bell-shaped distribution is. Based on the single measurement and the fact that we see 3 cpm, we may conclude that the source HAS ACTIVITY ...
To derive the minimum significant measured net count rate \( r_1 \), we write

\[
r_1 = k_\alpha \sqrt{\sigma^2_{br} + \sigma^2_{gr}} = k_\alpha \sqrt{\frac{r_1 + r_b}{t_g} + \frac{r_b}{t_b}}.
\]

Solving for \( r_1 \), we get the minimum significant measured net count rate \( r_1 \) as

\[
r_1 = \frac{k_\alpha^2}{2t_g} + \frac{k_\alpha}{2 \sqrt{\frac{t_g}{t_b}}} + 4r_b \left( \frac{t_g}{t_b} \right).
\]

When the gross and background only counting times are equal \( t \), we can derive the minimum significant count difference, \( \Delta_1 \), as

\[
\Delta_1 = r_1 t = \frac{1}{2} k_\alpha^2 + \frac{1}{2} k_\alpha \sqrt{k_\alpha^2 + 8n_b} + \frac{k_\alpha}{\sqrt{8n_b}} \left( 1 + \frac{k_\alpha^2}{8n_b} \right).
\]

In many instances, we have \( k_\alpha / \sqrt{n_b} \ll 1 \). Then

\[
\Delta_1 \approx k_\alpha \sqrt{2n_b}.
\]
False Positive Rate and Minimum Significant Measured Count Difference

Often, the background can be measured accurately. The expected number of background counts B in time t is known. In such case, if there is no source activity, the standard deviation of the net count is equal to \( \sqrt{B} \). It follows that the minimum significant net count difference is

\[ \Delta_1 = k_e \sqrt{B} \quad \text{(Background accurately known)}. \]

\[ \Delta_1 = k_e \sqrt{2n_b} \quad k_e / \sqrt{n_b} \ll 1. \]

The minimum significant net count difference is lowered by a factor of 1.414 when the background is well known.

Minimum Significant Measured Activity

Consider that the measurements were done with a detector of efficiency \( \epsilon \). Then the minimum significant measured activity is

\[ A_I = \frac{\Delta_1}{\epsilon t}. \]

If the measured net activity \( A > A_0 \), we state that the source contains activity, with the probability of false positive is \( < \alpha \).

\[ \Delta_1 = r_I t = \frac{1}{2} k_n^2 + \frac{1}{2} k_n \sqrt{k_n^2 + 8n_b} + \frac{k_n \sqrt{2n_b}}{\sqrt{8n_b}} \left( \frac{k_n}{\sqrt{8n_b}} + \sqrt{1 + \frac{k_n^2}{8n_b}} \right). \]

Example

A 10-min background measurement with a certain counter yields 410 counts. A sample is to be measured for activity by taking a gross count for 10 min. The maximum acceptable risk for making a type-1 error is 0.05. The counter efficiency is such that 3.5 disintegrations in a sample result, on average, in one net count.

(a) Calculate the minimum significant net count difference and the minimum significant measured activity in Bq.

(b) How much error is made in (a) by using the approximate formula (11.72) in place of (11.69)?

(c) What is the decision level for type-1 errors in terms of the number of gross counts in 10 min?

Solution

(a) With equal counting times, \( t_b = t_s = t = 10 \text{ min} \), one can use Eq. (11.69) in place of the general expression (11.68). For \( \alpha = 0.05 \) and \( k_n = 1.65 \). With \( n_b = 410 \), we obtain

\[ \Delta_1 = \frac{1}{2}(1.65)^2 + \frac{1}{2}(1.65) \sqrt{(1.65)^2 + 8(410)} = 48.6 \quad 49 \quad (11.74) \]

for the minimum significant count difference in 10 min (rounded upward to the nearest integer). The counter efficiency is \( \epsilon = 1/3.5 = 0.286 \text{ dpm/cpm} \). It follows from Eq. (11.71) that the minimum significant measured activity is \( A_I = 48.6/(0.286 \times 10 \text{ min}) = 17.0 \text{ dpm} = 0.283 \text{ Bq} \).

\[ \frac{\Delta_1}{\epsilon t} \]
Minimum Significant Measured Activity

(b) How much error is made in (a) by using the approximate formula (11.72) in place of (11.69)?

(c) What is the decision level for type-I errors in terms of the number of gross counts in 10 min?

Solution

\[ \Delta n = r_{1} = \frac{1}{2} k^{2} + \frac{1}{2} k \sqrt{k^{2} + 8n_{o}}. \]

Minimum Significant Measured Activity

The value \( n_{1} = 459 \) in the last example can serve as a decision level for screening samples for the presence of activity by gross counting for 10 min. A sample showing \( n_{b} < 459 \) counts can be reported as having less than the “minimum significant measured activity,” \( A_{1} = 0.283 \) Bq. A sample showing \( n_{b} \geq 459 \) counts can be reported as having an activity \( (n_{b} - n_{o})/\epsilon = (n_{b} - 410)/2.86 \text{ dpm} \). "having no reportable activity"

False Negative and Minimum True Activity

Type-II Errors (False Negative) – wrongly conclude that no activity is present when there is actually activity in the source.
False Negative and Minimum True Activity

Type-II Errors (False Negative) – wrongly conclude that “no active source is present” when there is an active source.

If we set a threshold level, \( r_1 \), what would be the minimum true source activity \( (A_{II}) \), so that the decision rule based on the threshold value \( r_1 \) can correctly detect the presence of the source with a probability of \( 1-\beta \)? or equivalently with the probability of making Type II error being \( \beta \)?

\[ A > A_{II} : \text{probability of a false negative (type-II error) is less than } \beta. \]

\( A_{II} \) is called the minimum detectable true activity.

False Negative and Minimum True Count Rate, \( r_2 \)

To determine \( r_2 \), we would write

\[ r_1 - r_2 = -k\beta \sqrt{\frac{t_1}{t_0} + \frac{t_0}{t_1}}. \quad (1) \]

We would further assume that \( r_2 \equiv r_1 + r_0 \) and substitute into the above equation

\[ r_1 - r_2 = -k\beta \sqrt{\frac{t_1 + t_0}{t_0} + \frac{t_0}{t_1}}. \quad (2) \]

or

\[ r_2 = r_1 + k\beta \sqrt{\frac{t_1 + t_0}{t_0} + \frac{t_0}{t_1}}. \]

Substituting for \( r_1 \) from Eq. (11.68),

\[ r_1 = \frac{k_2}{2} + \frac{k_0}{2} \sqrt{\frac{t_1}{t_2} + \frac{t_2}{t_1}} \left( t_0 + \frac{t_1 + t_0}{t_0} \right). \]

To determine \( r_2 \), we would write

\[ r_1 - r_2 = -k\beta \sqrt{\frac{t_1}{t_0} + \frac{t_0}{t_1}}. \quad (1) \]

We would further assume that \( r_2 \equiv r_1 + r_0 \) and substitute into the above equation

\[ r_1 - r_2 = -k\beta \sqrt{\frac{t_1 + t_0}{t_0} + \frac{t_0}{t_1}}. \quad (2) \]

or

\[ r_2 = r_1 + k\beta \sqrt{\frac{t_1 + t_0}{t_0} + \frac{t_0}{t_1}}. \]

Substituting for \( r_1 \) from Eq. (11.68),

\[ r_1 = \frac{k_2}{2} + \frac{k_0}{2} \sqrt{\frac{t_1}{t_2} + \frac{t_2}{t_1}} \left( t_0 + \frac{t_1 + t_0}{t_0} \right). \]
False Negative and Minimum True Activity, $A_{II}$

From $r_2$, we can derive the minimum true activity, $A_{II} = \frac{r_2}{\sqrt{2}}$.

Decision threshold $r_n$: measured net count rate, with given $\alpha, \beta$.

$\Delta_2$: Minimum detectable true count difference

Special case 1:

When the latter are equal ($t_0 = t_1$), Eq. (11.78) gives for the number of net counts with the minimum detectable true activity

$$\Delta_2 = r_2 = \sqrt{2n_0} \left[ k_\alpha \frac{k_\alpha}{\sqrt{8n_0}} + \sqrt{1 + \frac{k_\alpha^2}{8n_0}} \right]$$

With the help of Eq. (11.70), we can also write

$$\Delta_2 = \Delta_1 + k_\beta \sqrt{2n_0} \left[ 1 + \frac{k_\beta^2}{4n_0} + \frac{k_\beta}{\sqrt{2n_0}} \left( 1 + \frac{k_\beta^2}{8n_0} \right)^{1/2} \right]$$

The minimum detectable true activity is given by

$$A_{II} = \frac{\Delta_2}{\sqrt{2}}$$

False Negative and Minimum Detectable True Count Difference

Special case 2:

When $k_\alpha / \sqrt{n_0} << 1$, the number of net counts that is corresponding to the minimum detectable true activity is given as

$$\Delta_2 \equiv (k_\alpha + k_\beta) \sqrt{2n_0}$$

$$\Delta_2 = r_2 = \sqrt{2n_0} \left[ k_\alpha \frac{k_\alpha}{\sqrt{8n_0}} + \sqrt{1 + \frac{k_\alpha^2}{8n_0}} \right]$$

$$+ k_\beta \left[ 1 + \frac{k_\beta^2}{4n_0} + \frac{k_\beta}{\sqrt{2n_0}} \left( 1 + \frac{k_\beta^2}{8n_0} \right)^{1/2} \right]$$

(11.79)

Special case 3:

When the background count $B$ is accurately known, we have seen by Eq. (11.73) that the minimum significant count difference is $\Delta_1 = k_\beta \sqrt{B}$. If a sample has exactly the minimum detectable true activity, then the expected number of net counts $\Delta_2$ is just $k_\beta$ standard deviations greater than $\Delta_1$. The standard deviation of the net count rate is $\sqrt{B + \Delta_2}$. Thus,

$$\Delta_2 = k_\beta \sqrt{B + \Delta_2}$$

Solving for $\Delta_2$, we find

$$\Delta_2 = \sqrt{B} \left( k_\beta + \sqrt{\frac{k_\beta^2}{2B} + k_\beta \frac{k_\beta}{\sqrt{B}} \left( 1 + \frac{k_\beta^2}{4B} \right) \frac{2B}{k_\beta^2} \right)$$

(11.84)

(Background accurately known.)