

Chapter 1: Mathematical Preliminaries

Contents

- Signals and Systems.
- Fourier Transform Basics
- Analytical Image Reconstructing Techniques
- Iterative Reconstruction Methods
- Image Quality Assessment and System Optimization.

Reference book: Chapter 2 in <<Medical Imaging Signals and Systems>>, Prince and Links, Prentice Hall, 2006.

Part I: Signals and Systems

Reading Material:

Chapter 2 in

Medical Imaging Signal and Systems, 2nd Ed.

J. L. Prince et. al,

Prentice Hall, 2012.

Signals

Learning Objectives:

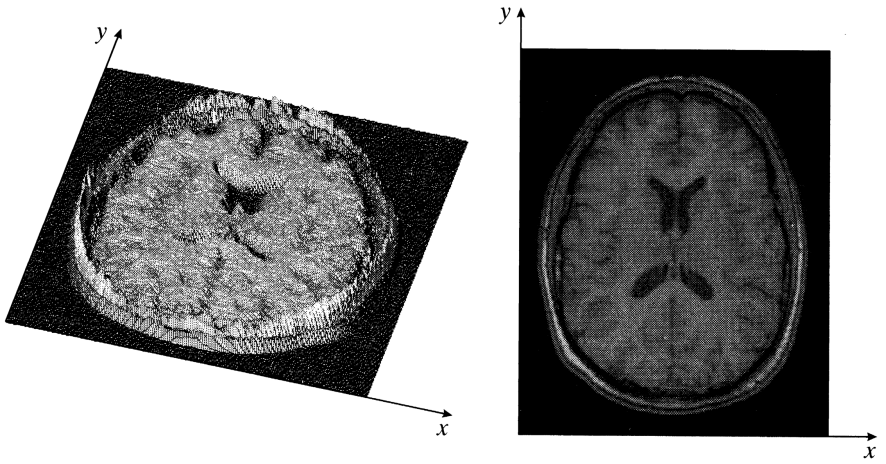
1. Mathematically, **signals** are typically described as functions of independent variables, typically time or space, with the signal itself being the dependent variable (amplitude).
2. The idea of decomposing an arbitrary signal into a linear combination of a series of ***basis*** functions.
    a. Why do we want to do this?
    b. Examples of the basis functions and their properties, e.g., the Fourier series, delta function, and sampling function …
3. Other useful signal functions, e.g., the line-impulse function, square function, triangle functions, Gaussian function...
4. Periodic signals and separable signal functions.

Introduction to Signals

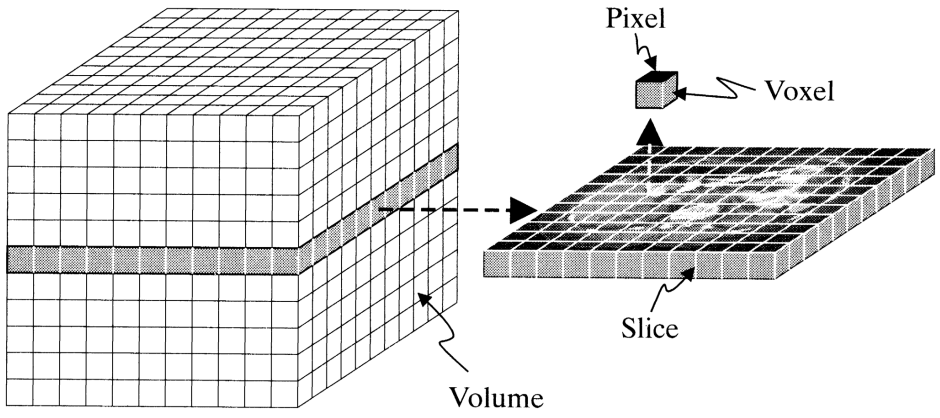
- Continuous signal:

  A continuous 2-D signal

$$f(x, y), \quad -\infty \le x, y \le \infty$$



- Discrete signal: Pixel and voxel representations of a continuous signal



Continuous Fourier Transform

- For any square-integrable function f(x,y), a continuous Fourier transform is defined as

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$where\ j = \sqrt{-1}$$

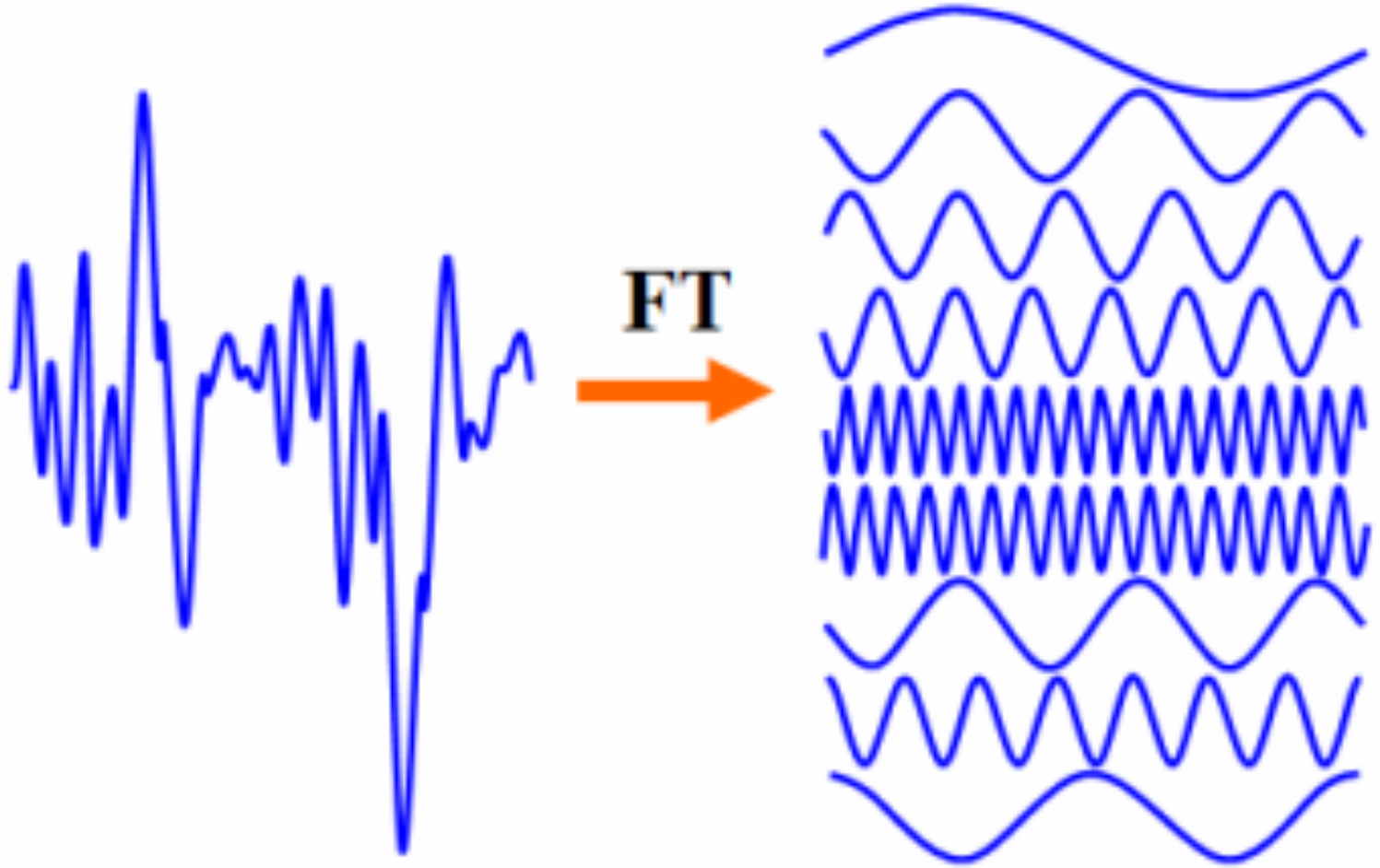
- We can also define an inverse Fourier transform as

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

- Both f(x,y) and F(u,v) have infinite support.
- Both f(x,y) and F(u,v) are defined on a continuum of values.
- f(x,y) and F(u,v) must contain the same information?

$$e^{-j\cdot 2\pi(ux+vy)} = \boldsymbol{cos}[2\pi(ux+vy)] - j\cdot \boldsymbol{sin}[2\pi(ux+vy)]$$

Continuous Fourier Transform



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Discrete Fourier Transform in 1-D

The discrete Fourier transform (DFT) is defined as

$$F_n = \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}, \mathrm{n} = 0,1,2,...\mathrm{N}-1$$

$\mathrm{n} = 0$ corresponding to the DC component (spatial frenquency is zero)

$\mathrm{n} = 1,..., \mathrm{N}/2 - 1$ are corresponding to the positive frenquencies $0 < \mathrm{u} < \mathrm{u_c}$

$\mathrm{n} = \mathrm{N}/2, ..., \mathrm{N} - 1$ are corresponding to the negative frenquencies $-\mathrm{u_c} < \mathrm{u} < 0$

The inverse DFT is defined as

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{-\frac{j2\pi nk}{N}}, \mathrm{k} = 0,1,2,...\mathrm{N}-1$$

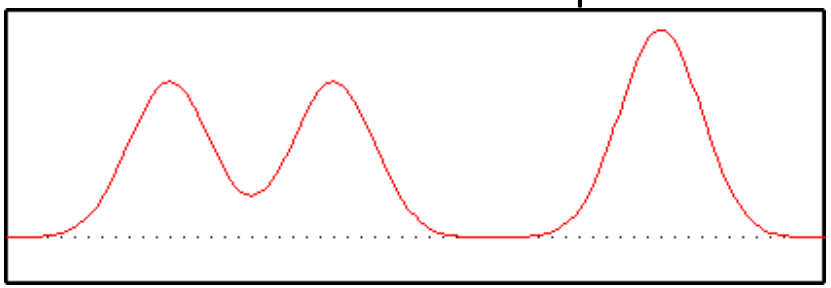
$$e^{-j\frac{2\pi nk}{N}} = cos\left[\frac{2\pi nk}{N}\right] - j \cdot sin[\frac{2\pi nk}{N}]$$

Continuous Fourier Transform

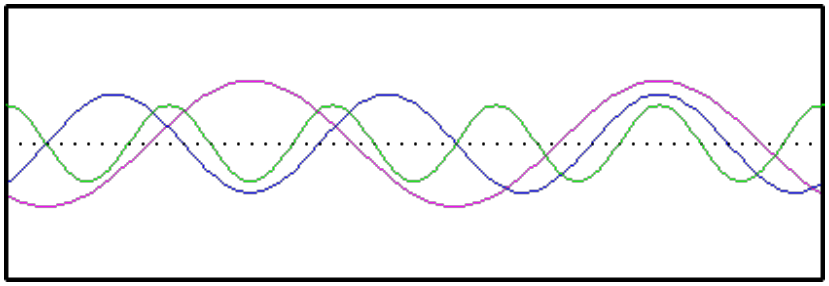
A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

So what does this mean exactly?

Let's start with an example…in 1-D

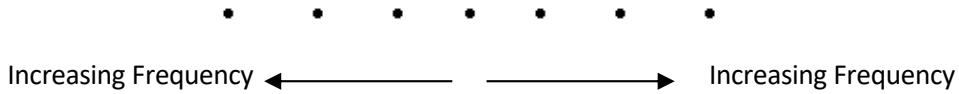


Can be represented by:



When you let these three waves interfere with each other you get your original wave function!

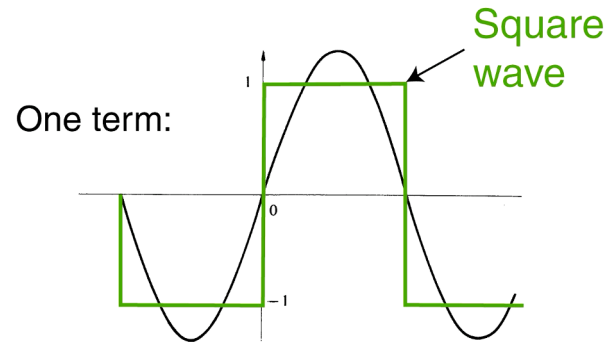
Since this object can be made up of 3 fundamental frequencies an ideal Fourier Transform would look something like this:

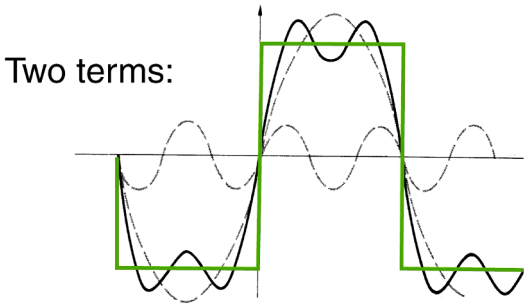


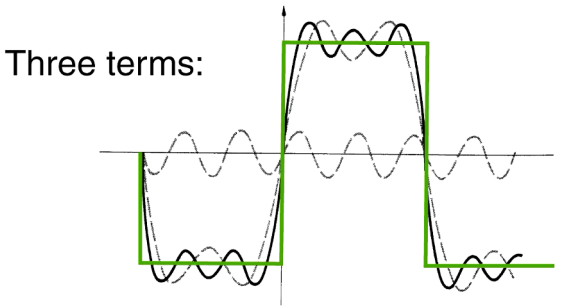
Notice that it is symmetric around the central point and that the number of points radiating outward corresponds to the distinct frequencies used in creating the image.

Fourier Transform and Spatial Frequency

Fourier transform provides information on the sinusoidal composition of a signal at different spatial frequencies.







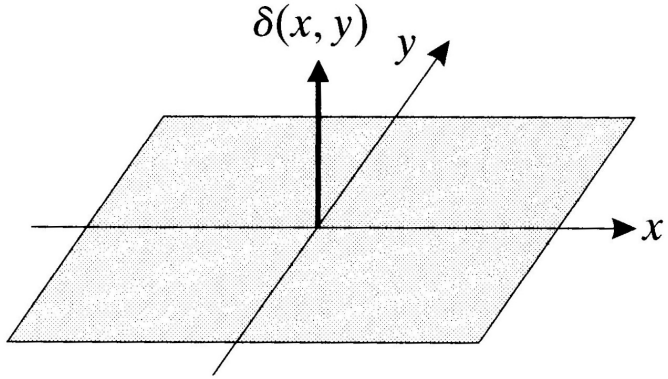
Point Impulse Signal

- A point source is mathematically represented by the delta function or Dirac function.

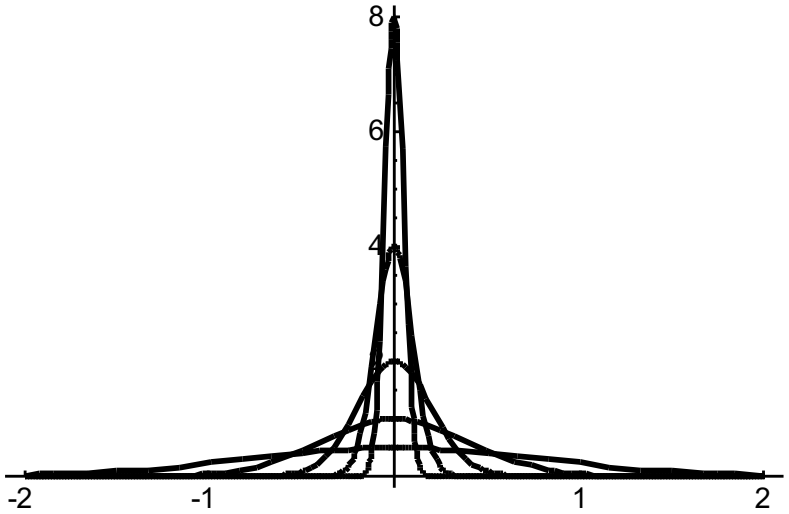
$$\delta(x,y)\begin{cases}\neq 0, & x=0 \text{ and } y=0\\ =0, & otherwise\end{cases}$$

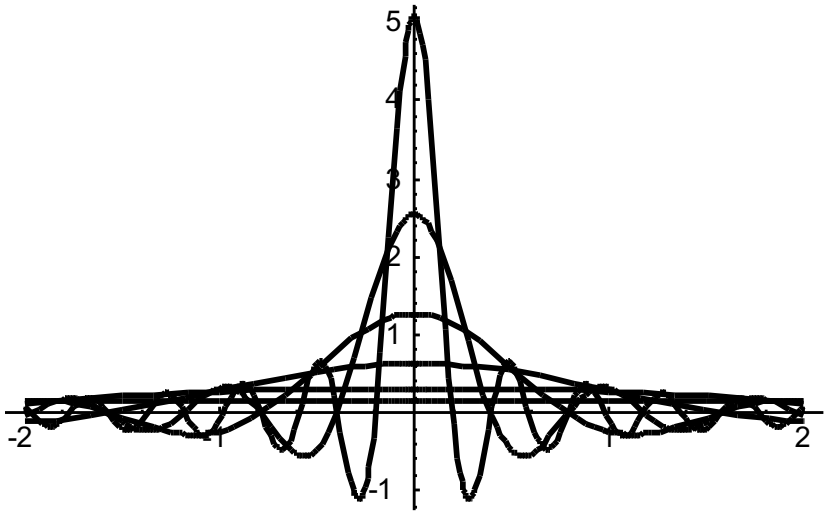
and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy=1$$



Point Impulse Signal





$$\delta(x) = \lim_{\alpha \to \infty} \alpha \; e^{-\pi \alpha^2 x^2}$$

$$\delta(x) = \lim_{\alpha \to \infty} \frac{\alpha}{\pi} \mathrm{sinc}(\alpha \mathrm{x}); \; \mathrm{sinc}(\mathrm{x}) = \frac{\sin(x)}{x}$$

Point Impulse Signal

- The sampling property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-\xi,y-\eta)dxdy = f(\xi,\eta)$$

- The scaling property

$$\delta(ax,by) = \frac{1}{|ab|}\delta(x,y)$$

$$\delta(x,y)\begin{cases}\neq 0, & x=0 \text{ and } y=0\\ =0, & otherwise\end{cases}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy = 1$$

Comb and Sampling Function

- The 2-D comb function

$$comb(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n)$$

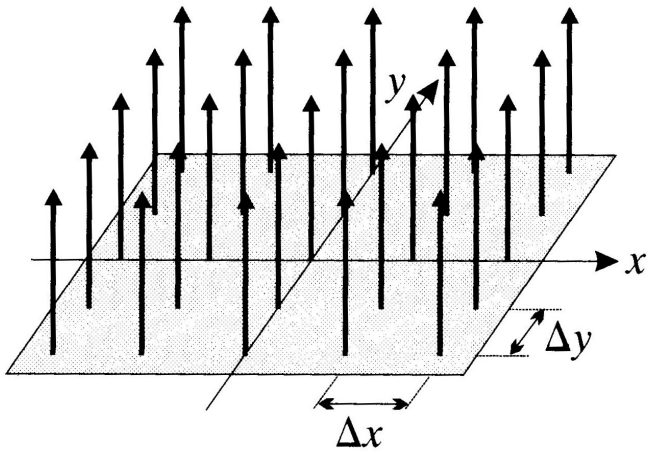
Comb and Sampling Function

- The 2-D sampling function

$$\delta_s(x,y,\Delta x,\Delta y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x,y-n\Delta y)$$

where Δx and Δy are the sampling intervals

$$\delta_s(x,y,\Delta x,\Delta y)=\frac{1}{\Delta x\Delta y}comb\left(\frac{x}{\Delta x},\frac{y}{\Delta y}\right)$$



$$comb(x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n)$$

$$\delta(ax,by)=\frac{1}{|ab|}\delta(x,y)$$

Comb and Sampling Function

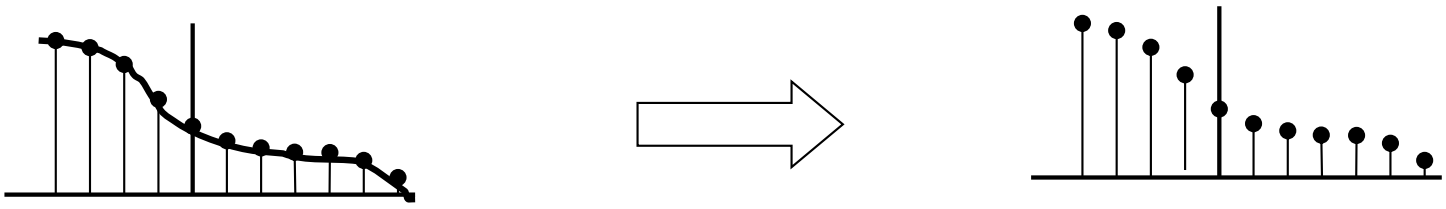
- The sampling function is critical for the discretization of continuous signals.
- The sampled signal function is then

$$\delta_s(x,y,\Delta x,\Delta y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x, y-n\Delta y)$$

where Δx and Δy are the sampling intervals

$$f_s(x,y) = f(x,y)\cdot\delta_s(x,y)$$

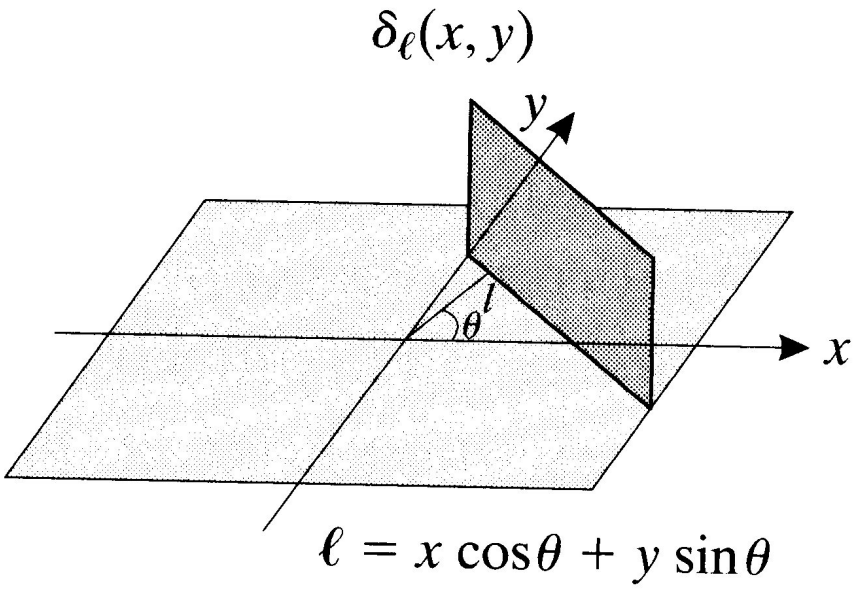
$$= f(x,y)\cdot\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x, y-n\Delta y)$$



Line Impulse Signal

$$\delta_L(x,y) = \delta(x\cos\theta + y\sin\theta - l)$$

$$where\ \delta(x) = \begin{cases} > 0, & x\cos\theta + y\sin\theta = l \\ 0, & otherwise \end{cases}$$



Line Impulse Signal

$$\delta_L(x, y) = \delta(x\cos\theta + y\sin\theta - l)$$

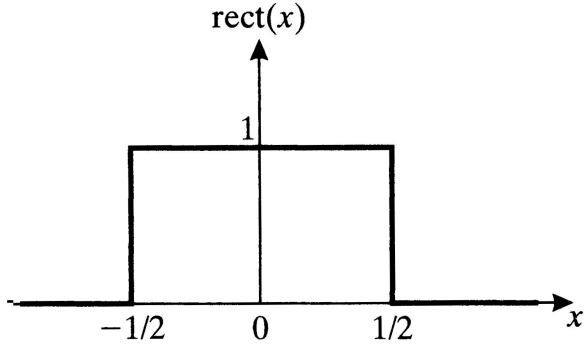
- It can be used to measure the spatial resolution of a given imaging system.

- It is used to calculate the line-integral projection data for a given 2-D object.

Rect Function

- Rect function:

$$rect(x,y)=\begin{cases}1, \text{for } |x|<\frac{1}{2} \text{ and } |y|<\frac{1}{2}\\ 0, \text{otherwise}\end{cases}$$



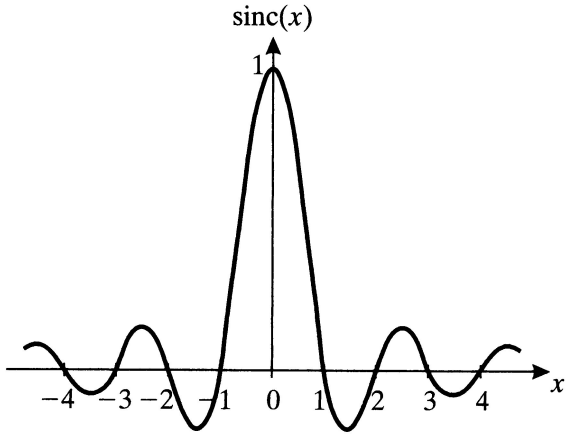
- It is normally used to pick up a particulate section of a given function:

$$f(x,y)\cdot rect(\frac{x-\xi}{w_X},\frac{y-\eta}{w_Y})$$

Sinc Function

- The sinc function is defined as

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



- The sinc function is normalized.

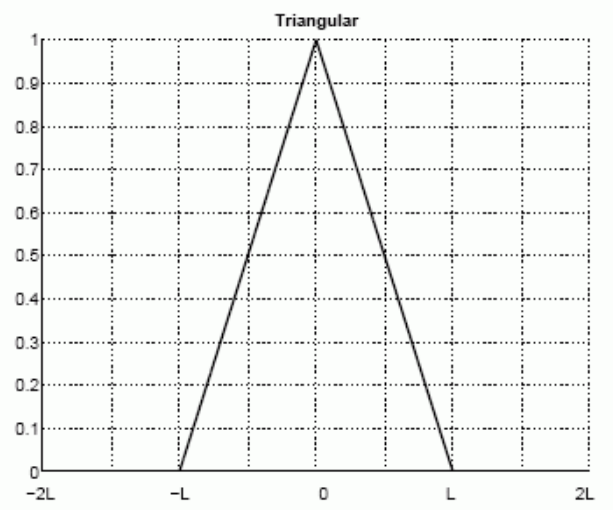
$$\int_{-\infty}^{\infty} \mathrm{sinc}(x)\,dx = 1$$

Any arbitrary band-limited signal can be written as a weighted sum of multiple sinc functions … (the Nyquist Sampling Theorem)

Triangular Signals and Gaussian Signals

- Triangular function:

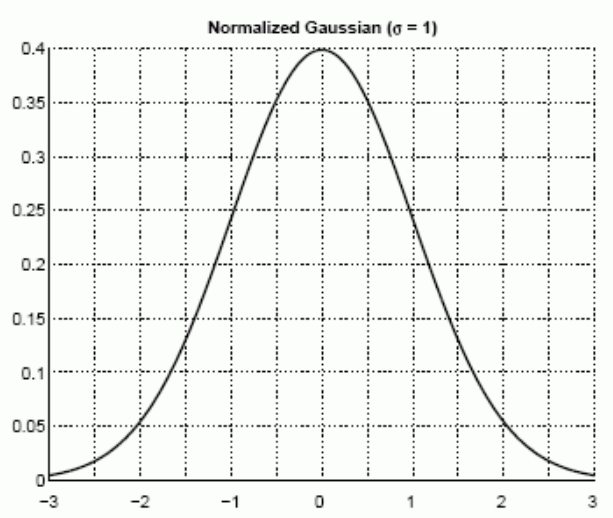
$$Tri(\frac{x}{2L}) = 1 - \frac{|x|}{L} \quad \text{for} \quad |x| < L$$
$$= 0 \quad \text{for} \quad |x| > L$$



- Normalized Gaussian function:

$$G_{1D}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$G_{2D}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



Separable Signals and Periodic Signals

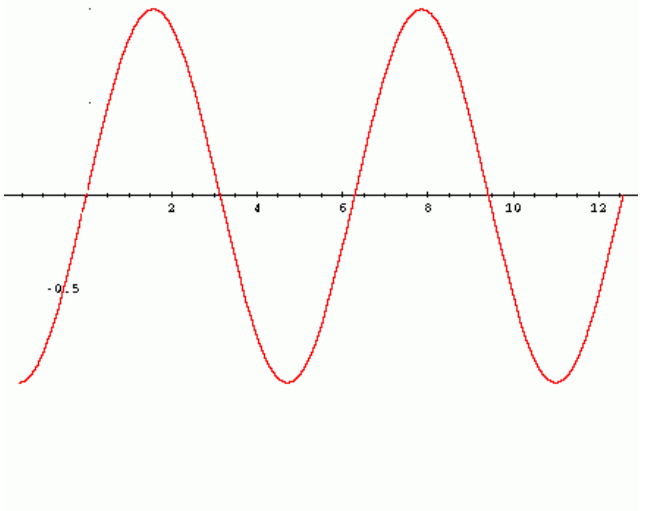
- The separable signals is a class of continuous signals that satisfy

$$f(x,y) = f_1(x) \cdot f_2(y)$$

- A 2-D signal is periodic if

$$f(x,y) = f(x+X,y) = f(x,y+Y)$$

where X and Y are the signal periods



System

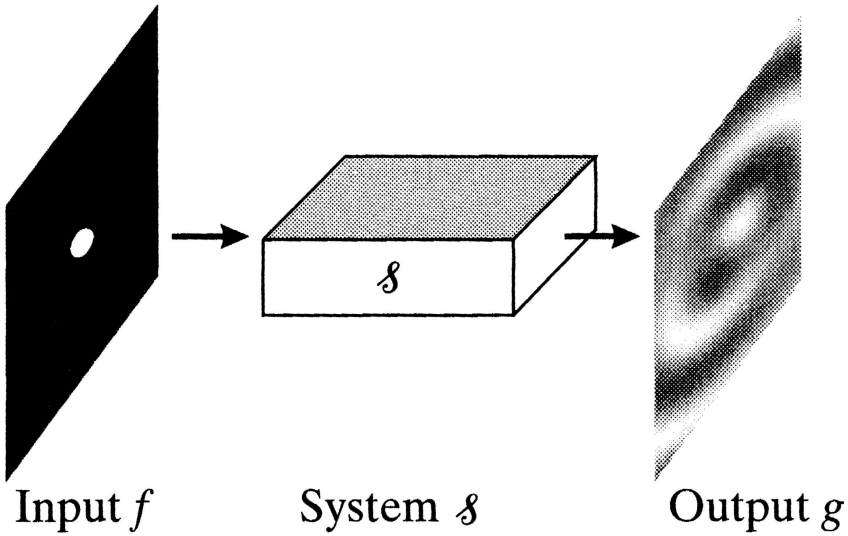
Learning Objectives:

1. Mathematical modeling of an imaging system: Mapping from the input to the output
2. **Linear** systems
    a. The superposition principle
    b. Example of linear systems
    c. Why does it matter?
3. **Shift-invariant** systems
4. The response of a linear and shift-invariant (LSI) system.

General Concept of a System

- A continuous-to-continuous system is defined as

$$g(x, y) = \mathcal{S}[f(x, y)]$$



- A system is a mapping process from an input signal to the output signal



Linear Systems

Linear Systems

A system is linear if it satisfies the **superposition principle**

$$S\left[f(x,y) = \sum_{i=1}^{I} w_i \cdot f_i(x,y)\right] = \sum_{i=1}^{I} w_i \cdot S[f_i(x,y)]$$

where

$f(x,y)$ is the input signal,

$S[\cdot]$ is an operator that represents the system,

$f(x,y)$ is the total input signal, and

w_is are weighting factors.

Linear Systems

A linear imaging system processes an input signal (such as light intensity from an object) to produce an output signal (the resulting image). To be considered "linear," the system must satisfy two core properties:

1. **Additivity:** If you have two different inputs, $f(x)$ and $g(x)$, the output of the system operating on their sum is equal to the sum of the outputs if the system had operated on each input individually:

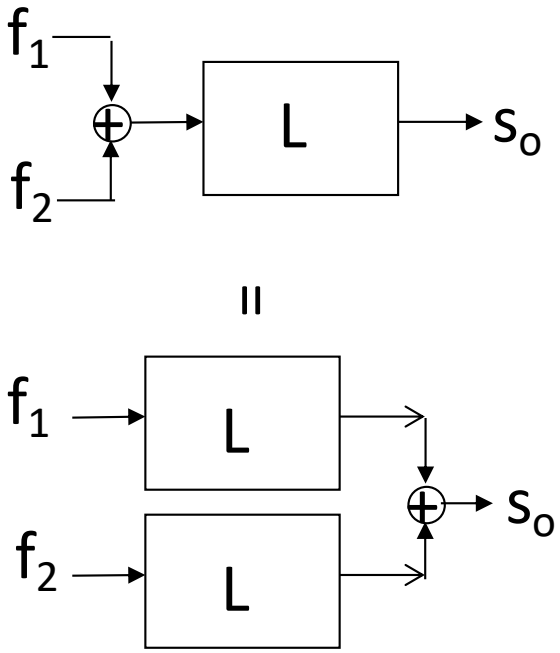
$$S[f(x) + g(x)] = S[f(x)] + S[g(x)].$$

1. **Homogeneity:** A change in the input signal's amplitude results in a corresponding, scaled change in the output signal's amplitude.

$$S[a \cdot f(x)] = a \cdot S[f(x)].$$

Linear Systems – An Example

For example, consider an amplifier with gain A:



$$S[w_1 f_1 + w_2 f_2] = A(w_1 f_1 + w_2 f_2)$$
$$= Aw_1 f_1 + Aw_2 f_2 = w_1 S[f_1] + w_2 S[f_2]$$

It satisfies the Superposition Principle

$$S\left[f(x,y) = \sum_{i=1}^{I} w_i \cdot f_i(x,y) \right] = \sum_{i=1}^{I} w_i \cdot S[f_i(x,y)]$$

Linear Systems – Why Important?

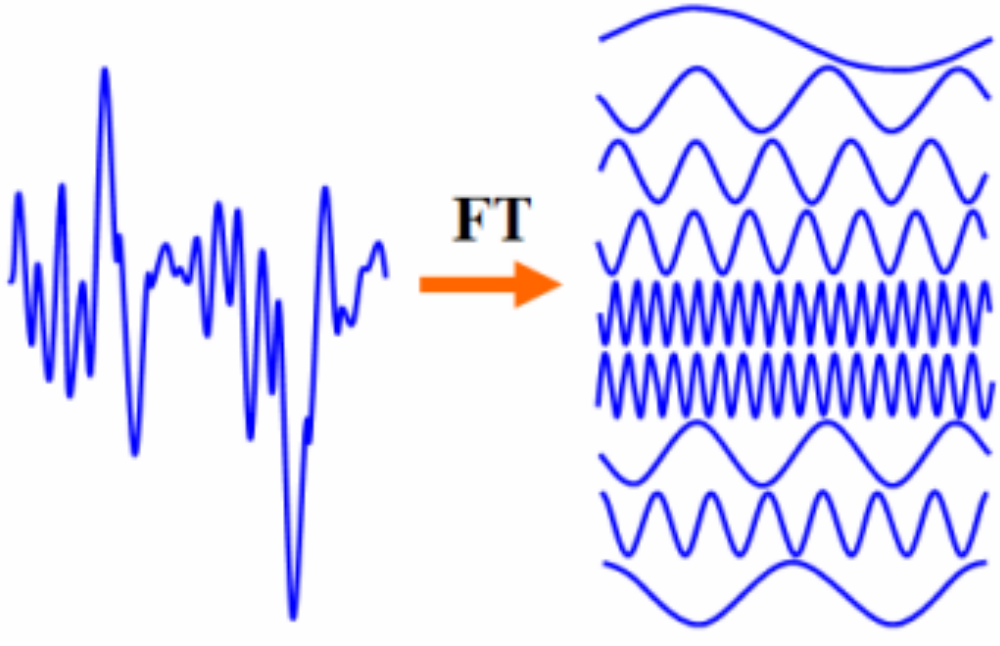
- Linear systems are mathematically easier to model.

$$g(x,y) = S\left[f(x,y)\right] = S\left[\sum_{i=1}^{I} w_i \cdot f_i(x,y)\right] = \sum_{i=1}^{I} w_i \cdot S\left[f_i(x,y)\right]$$

- Many imaging systems used in medical and other applications as designed to be linear systems.

Linear Systems – Why Important?

Consider that we can break up an arbitrary signal through Fourier transform



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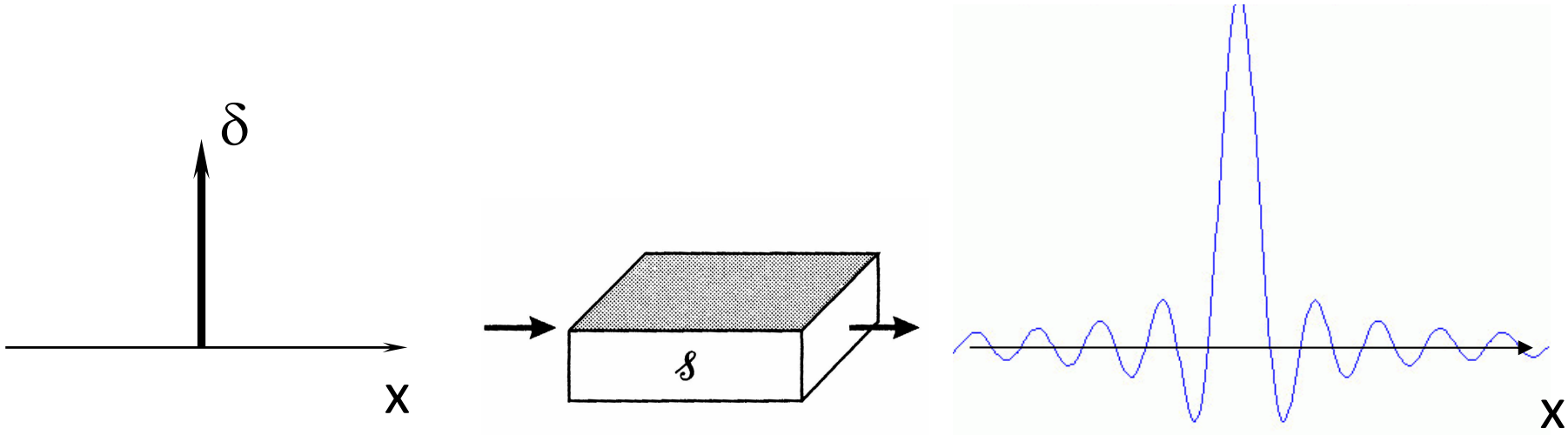
If we send the signal into a linear system, then the output is given as

$$g(x,y) = S\left[f(x,y)\right] = S\left[\sum_{i=1}^{I} w_i \cdot f_i(x,y)\right] = \sum_{i=1}^{I} w_i \cdot S\left[f_i(x,y)\right]$$

Impulse Response Function

Impulse Response Function

One of the most common shape for impulse responses used in imaging application



The **impulse response function** (in the 2-D case) is defined as

$$h(x, y, \xi, \eta) \equiv \mathrm{S}[\delta(x - \xi, y - \eta)].$$

Impulse Response Function (IRF)

For a linear system, knowing the IRF enables one to compute the output from any arbitrary input function.

$$
\begin{aligned}
g(x,y) &= \mathrm{S}[f(x,y)] \\
&= \mathrm{S}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,\delta(x-\xi,y-\eta)d\xi d\eta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{S}[f(\xi,\eta)\delta(x-\xi,y-\eta)]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,\mathrm{S}[\delta(x-\xi,y-\eta)]d\xi d\eta.
\end{aligned}
$$

Remember that the **impulse response function** is defined as

$$h(x,y,\xi,\eta) \equiv \mathrm{S}[\delta(x-\xi,y-\eta)].$$

So the response of the above linear system to an arbitrarily defined input signal, $f(x,y)$, becomes

$$g(x,y) = \mathrm{S}[f(x,y)] \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,h(x,y,\xi,\eta)d\xi d\eta.$$



Shift-Invariant Systems

Impulse Response Function

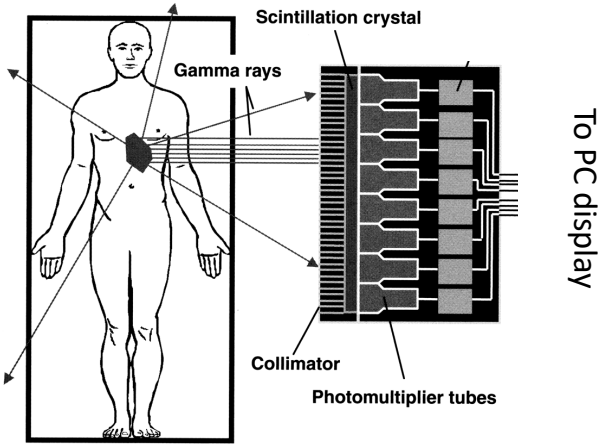
- For a 2-D problem, the impulse response is a 4-D function.

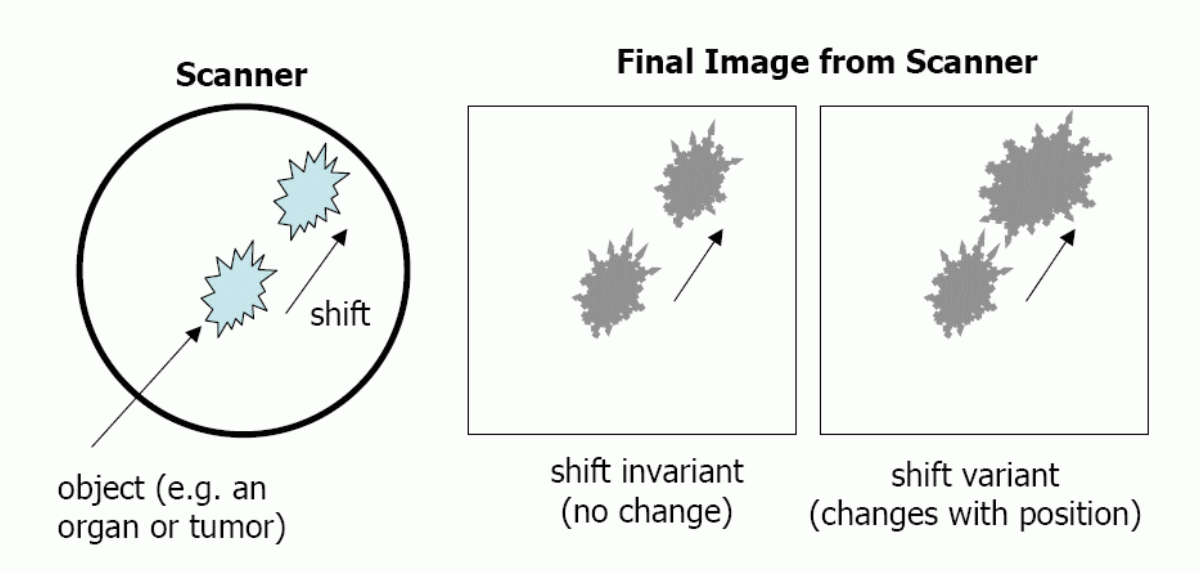
$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(x,y,\xi,\eta) d\xi d\eta$$

- The computation can be greatly reduced with further simplifications …

What exactly is this function again? What does it tell us about the system?

Shift Invariant Systems





Shift Invariant Systems

Remember that a system is called shift-invariant if

$$g(x-\Delta x, y-\Delta y) = S\left[f(x-\Delta x, y-\Delta y)\right]$$

Note that shift-invariance does not require or imply linearity.

The impulse response function of a shift invariant system is

$$h(x, y, \xi, \eta) = S[\delta_{\xi,\eta}(x, y)] = h(x-\xi, y-\eta)$$

4-D → 2-D



Linear and Shift-Invariant (LSI) Systems

Linear and Shift-Invariant (LSI) Systems

- The response of a **linear and shift-invariant (LSI)** system to an arbitrarily given signal, $f(x, y)$, is

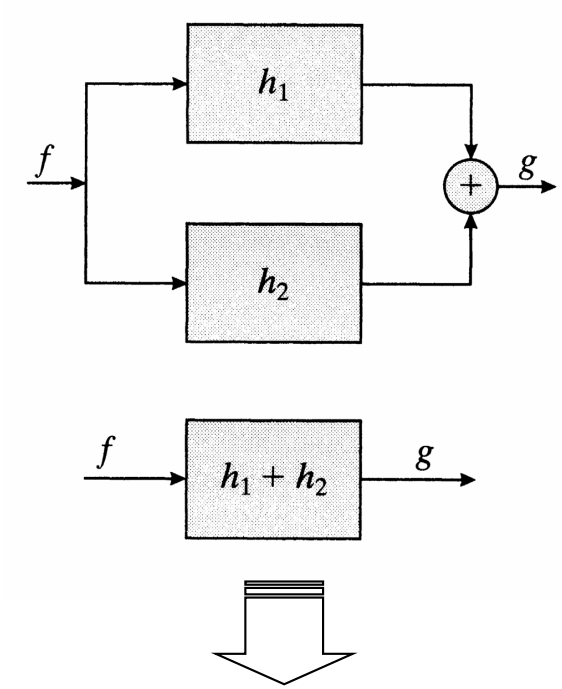
$$\begin{aligned} g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(x,y,\xi,\eta)\, d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(x-\xi,y-\eta)\, d\xi d\eta \\ &= f(x,y) * h(x,y), \end{aligned}$$

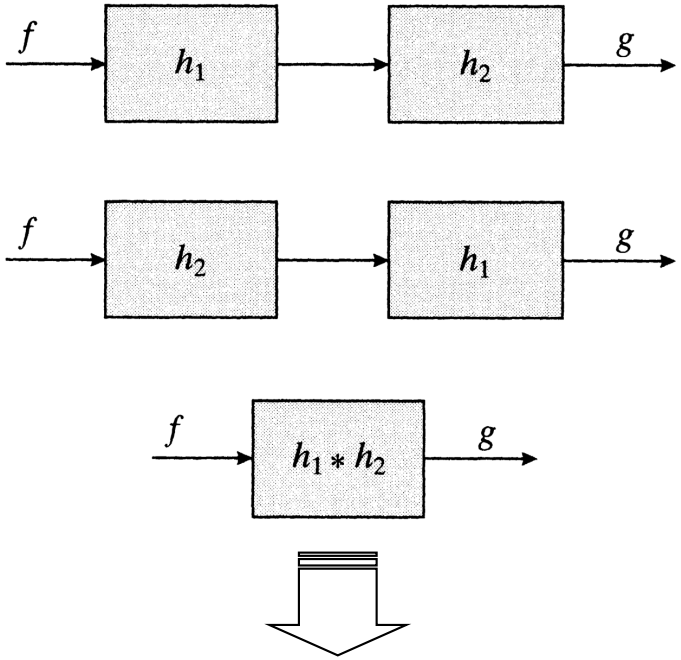
where * denotes the convolution operation, and

$$h(x,y) \equiv S[\delta(x,y)].$$

- The output of a linear and shift-invariant (LSI) system is the **input signal convolved with the impulse response function**.
- The impulse response function is also referred to as the **point-spread function** (PSF).

Connection of LSI Systems





$$g(x,y) = \left[h_1(x,y) + h_2(x,y)\right] * f(x,y)$$
$$= \left[h_2(x,y) + h_1(x,y)\right] * f(x,y)$$

$$g(x,y) = h_2(x,y) * \left[h_1(x,y) * f(x,y)\right]$$
$$= h_1(x,y) * \left[h_2(x,y) * f(x,y)\right]$$

LSI systems may be decomposed into the combination of multiple sub-systems. This may lead to a simplified mathematical representation of the complete system…

Separable Systems

A system is called separable if

$$h(x, y) = h_1(x)h_2(y)$$

in which case, the convolution between the input and the impulse response function is

Separable Systems

For a separable system, the 2-D convolution operation can be re-write as two 1-D convolution operations.

$$g(x,y) = h(x,y) * f(x,y) = h_1(x) * \left[h_2(y) * f(x,y) \right]$$

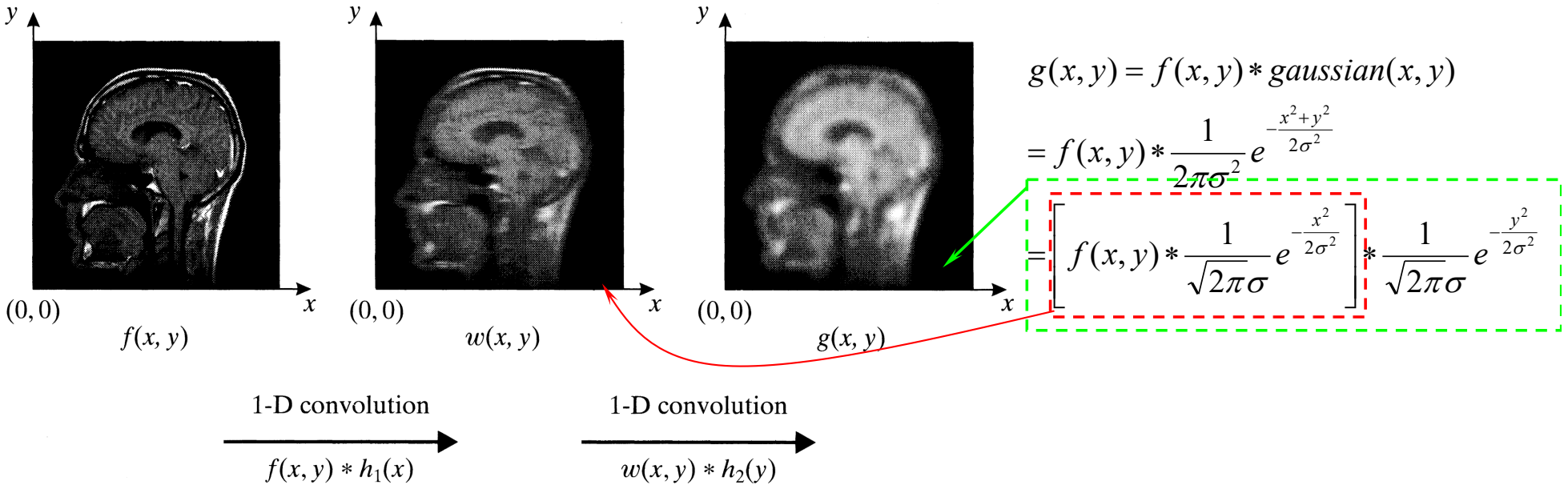
An example

$$gaussian(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \right) \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \right)$$

Separable Systems – An Example

An input image pass through a separable system having a impulse response function described by a 2-D Gaussian function

$$gaussian(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$$



$$g(x,y) = f(x,y) * gaussian(x,y)$$

$$= f(x,y) * \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$= \left[f(x,y) * \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \right] * \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$$

Fourier Transform and Sampling

Reading Material:

Chapter 2, Medical Imaging Signals and Systems, 2'nd Edition,

by Prince and Links, Prentice Hall, 2006.



Fourier Transform Basics

Fourier Transform Basics

Learning Objectives:

Fourier transform

a. What is the Fourier transform?

b. What is the so-called **spatial frequency**?

   i. What are the high and low spatial frequency components of a signal?

   ii. In the context of an image signal, what kind of information is carried by the higher spatial frequency components and the lower spatial frequency components?

c. The properties of the Fourier transform

   i. Typical Fourier transform pairs

   ii. The shifting property, scaling property, and convolution theorem …

Fourier Transform – What and Why?

What is Fourier Transform?

- A function can be described by a summation of waves with different frequency, amplitudes and phases.

The importance of Fourier Transform in Imaging?

- Signal representations in the frequency domain provide **unique information.**
- Certain **computations** can be performed more efficiently in frequency domain.
- Certain **hardware** naturally measures signals in the frequency domain.

Continuous Fourier Transform

- For any square-integrable function f(x,y), a continuous Fourier transform is defined as

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$where\ j = \sqrt{-1}$$

- We can also define an inverse Fourier transform as

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

- Both f(x,y) and F(u,v) have infinite support.
- Both f(x,y) and F(u,v) are defined on a continuum of values.
- f(x,y) and F(u,v) must contain the same information.

Fourier Transform and Spatial Frequency

- Fourier transform can be viewed as a decomposition of the function f(x,y) into a linear combination of complex exponentials with strength F(u,v).

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

- Fourier transform provides information on the sinusoidal composition of a signal at different spatial frequencies.

$$e^{j2\pi(ux+vy)} = \cos\left[2\pi(ux+vy)\right] + j\sin\left[2\pi(ux+vy)\right]$$

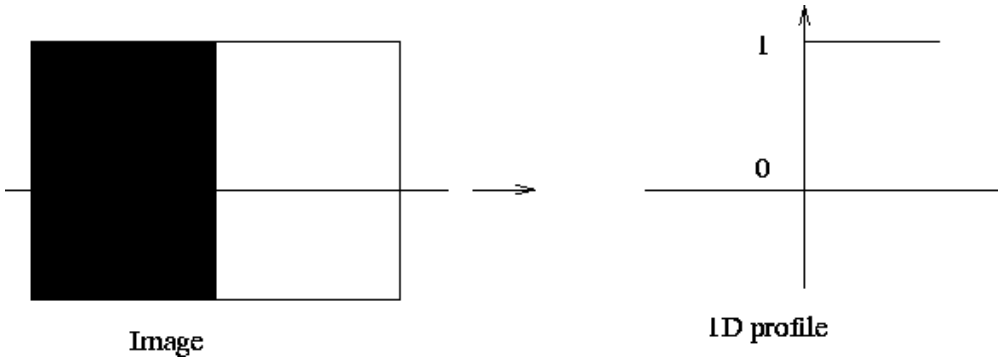
- F(u,v) is normally referred to as the spectrum of the function f(x,y).

What is Spatial Frequency?

and

What are the so-called Higher or Lower Spatial Frequency Components?

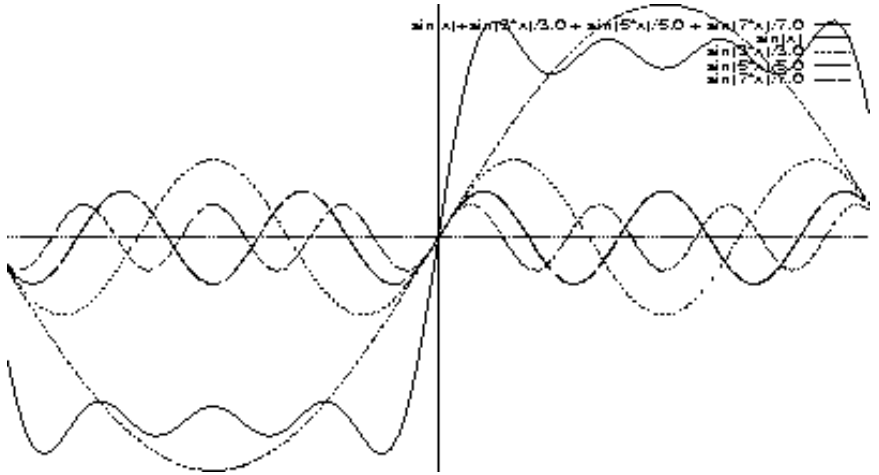
What is Spatial Frequency?



$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{-j2\pi(ux+vy)}dudv$$

$$e^{-j2\pi(ux+vy)}$$

$$= \cos[2\pi(ux+vy)] + j\sin[2\pi(ux+vy)]$$



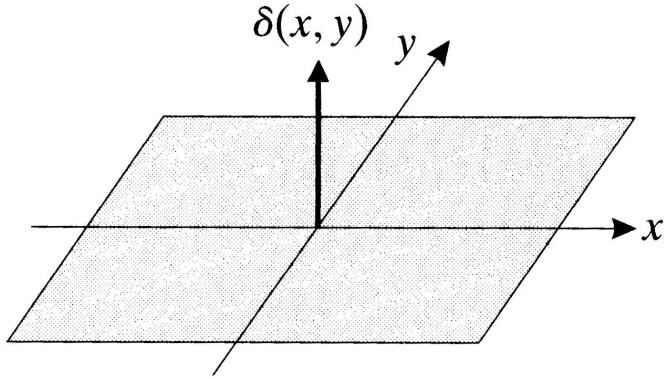
Point Impulse Signal

- A point source is mathematically represented by the delta function or Dirac function.

$$\delta(x,y)\begin{cases}\neq 0, & x=0 \; and \; y=0\\ =0, & otherwise\end{cases}$$

and

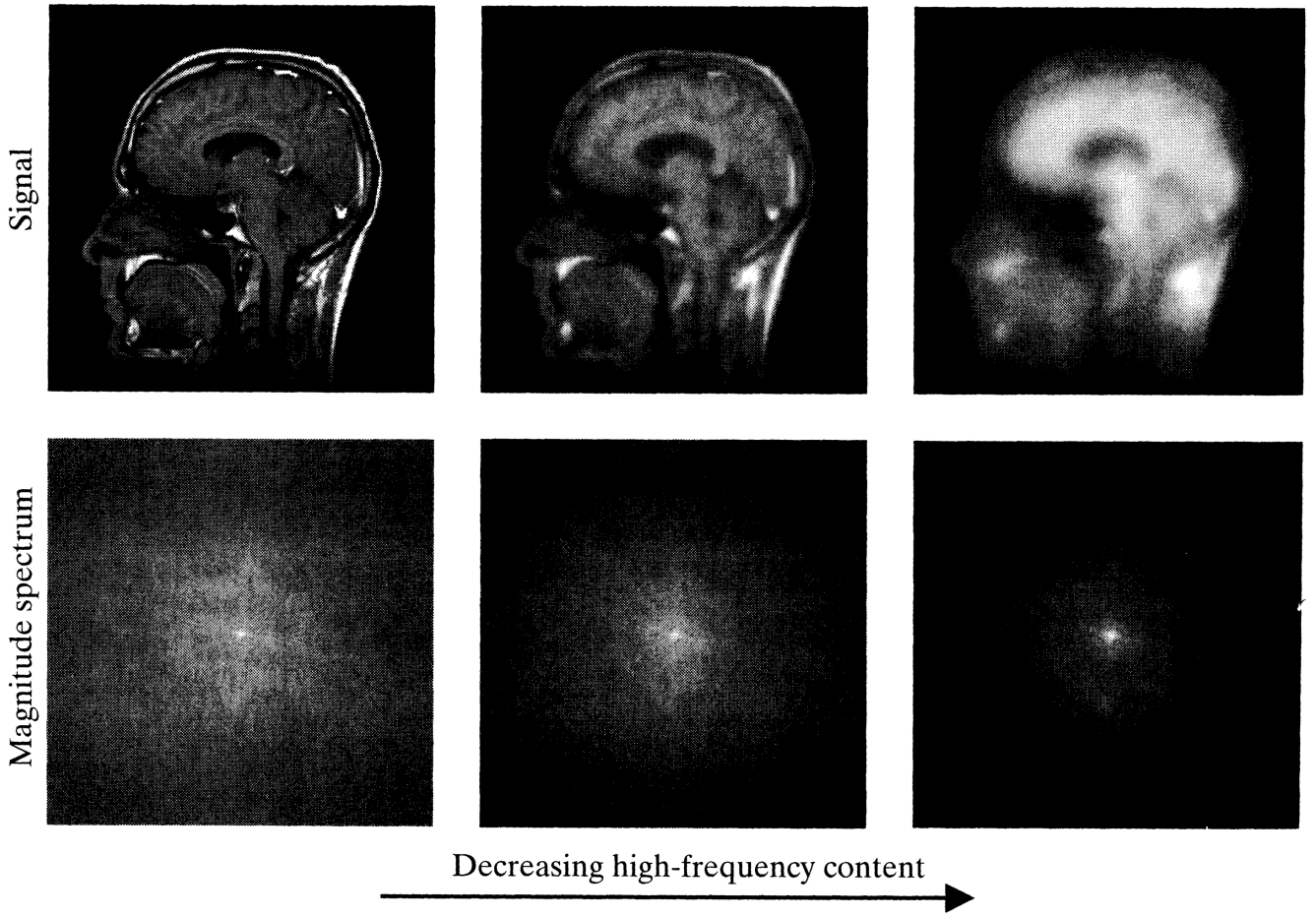
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy=1$$



- The sampling property

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\cdot\delta(x-\xi,y-\eta)d\xi d\eta$$

An Example


Signal
Magnitude spectrum
Decreasing high-frequency content

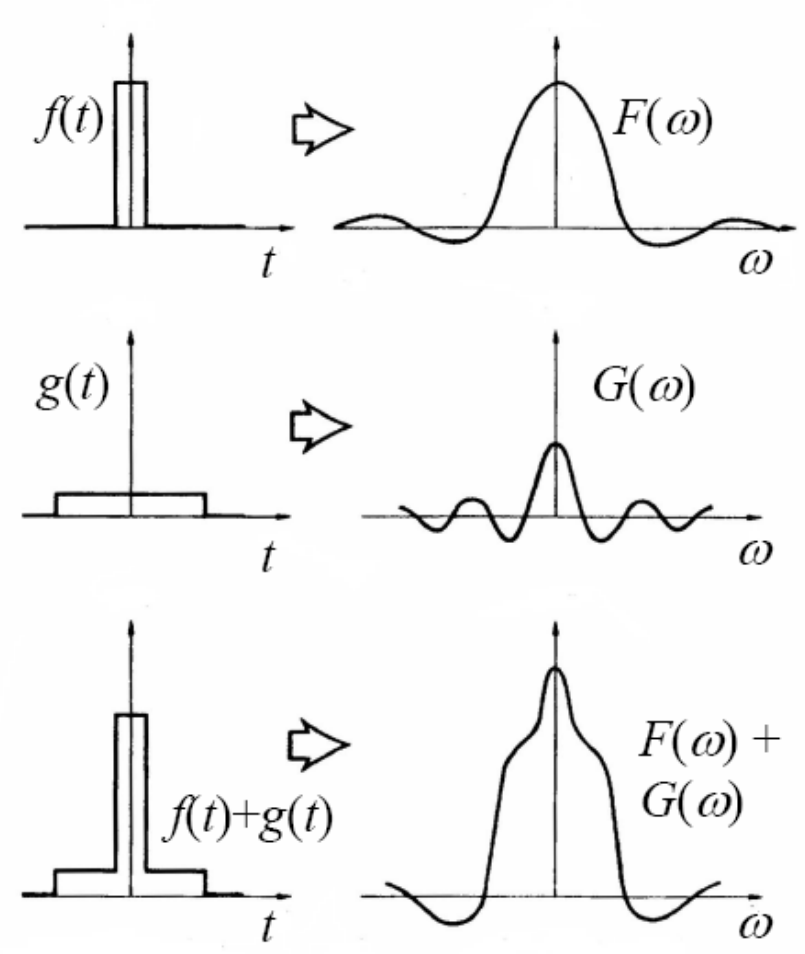
Basic Fourier Transform Pairs

Basic Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0+vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\text{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0x+v_0y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0x + v_0y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0x + v_0y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\text{rect}(x, y)$	$\text{sinc}(u, v)$
$\text{sinc}(x, y)$	$\text{rect}(u, v)$
$\text{comb}(x, y)$	$\text{comb}(u, v)$
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$

Properties of Fourier Transform (1)

- Linearity $$F\left[a_1 f(x,y) + a_2 g(x,y)\right] = a_1 F\left[f(x,y)\right] + a_2 F\left[g(x,y)\right]$$



$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$where\ j = \sqrt{-1}$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

Magnitude and Phase

- In general, Fourier transform is a complex valued signal, even if f(x,y) is real valued.
- It is sometimes useful to consider the magnitude and phase of the Fourier transform separately.

Fourier coefficients are complex: $$F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$$

Magnitude: $$\left|F(u,v)\right| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$$

Phase: $$\angle F(u,v) = \tan^{-1} \frac{F_I(u,v)}{F_R(u,v)}$$

An alternative representation: $$F(u,v) = \left|F(u,v)\right| e^{j\angle F(u,v)}$$

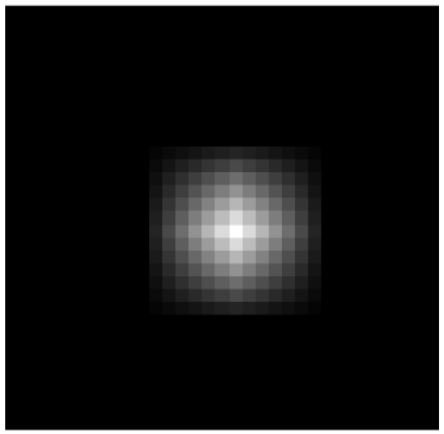
- The square of the magnitude $|F(u,v)|^2$ is referred to as the power spectrum of the original function.

Properties of Fourier Transform (2)

Shifting Property – Shift in spatial domain is equivalent to phase change in spatial frequency domain.

$$\mathcal{F}\left[f(x-x_0, y-y_0)\right] = \mathcal{F}\left[f(u,v)\right]e^{-j2\pi(ux_0+vy_0)}$$

An example



$\Lambda(x/16)\Lambda(y/16)$

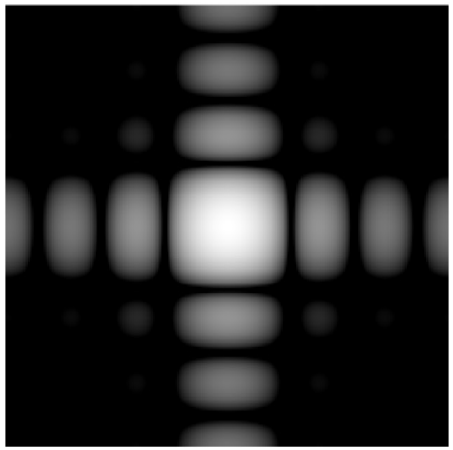
real and even

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$where\ j = \sqrt{-1}$

Properties of Fourier Transform

Shown in Log scale





$\text{Real}\{\text{F(u,v)}\} = 256\ \text{sinc}^2(16\text{u})\text{sinc}^2(16\text{v})$ $\quad Imag\{F(u,v)\} = 0$

$$|F(u,v)| = \sqrt{\{Real[F(u,v)]\}^2 + \{Imag[F(u,v)]\}^2}$$

$$= 256\,\text{sinc}^2(16\text{u})\text{sinc}^2(16\text{v})$$

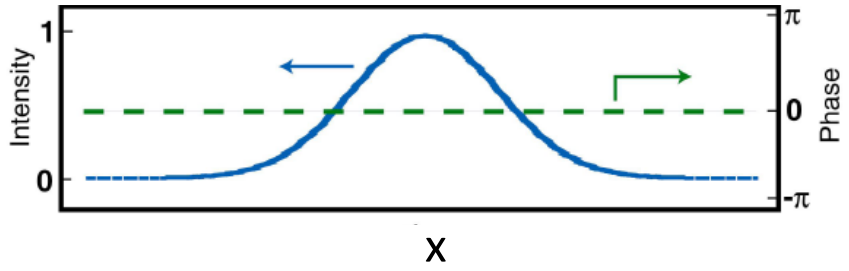
and

$$\angle F(u,v) = \tan^{-1}\frac{Imag[F(u,v)]}{Real[F(u,v)]} = 0$$

Properties of Fourier Transform

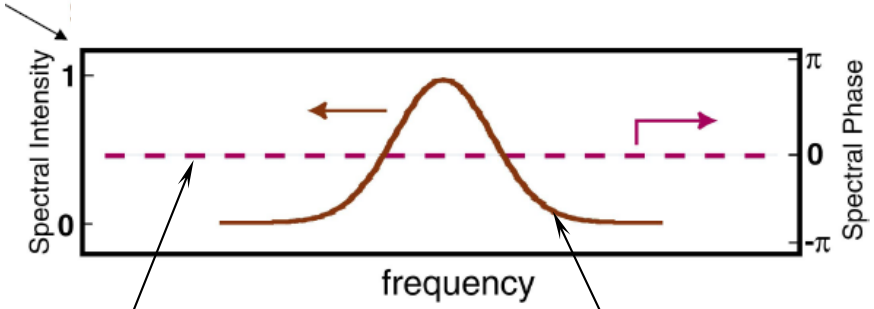
$$1\text{-D Gaussian function}: f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

Spatial Domain



A Gaussian transforms to a Gaussian

Spatial Frequency Domain



$$\mathcal{F}[f(x-x_0, y-y_0)] = \mathcal{F}[f(u,v)]e^{-j2\pi(ux_0+vy_0)}$$

Spectral phase is zero

Magnitude is a Gaussian

Properties of Fourier Transform

$$\boldsymbol{F[f(x-a)] = F[f(x)]exp(-j2\pi ua)}$$

Spatial Domain

$$F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$$

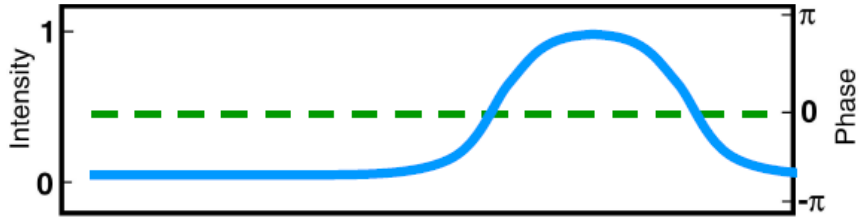
$$|F(u,v)| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$$

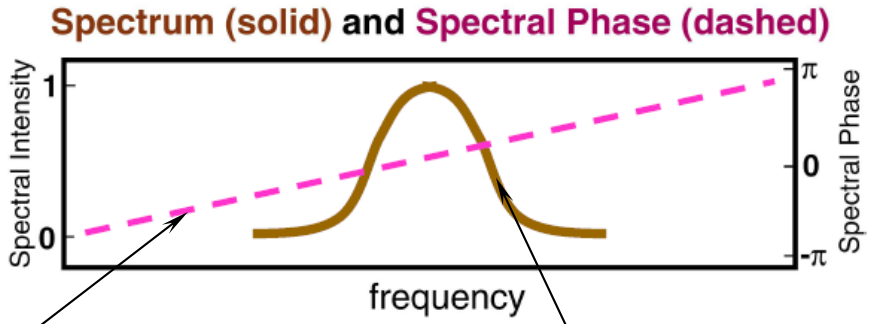
$$\angle F(u,v) = \tan^{-1}\frac{F_I(u,v)}{F_R(u,v)}$$

$$F(u,v) = |F(u,v)|e^{j\angle F(u,v)}$$

Spatial Frequency Domain

Properties





Linear shifting in spatial domain simply adds some linear phase to the pulse

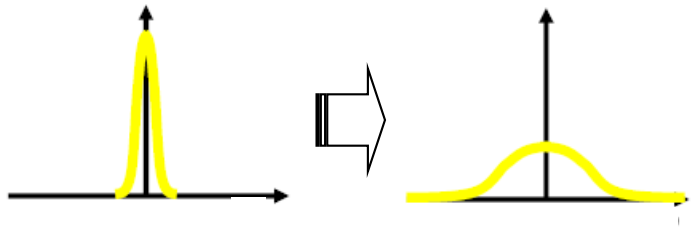
Magnitude is unchanged

Properties of Fourier Transform

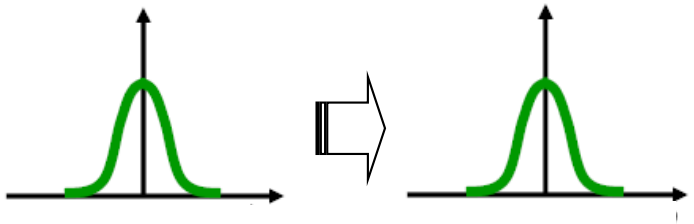
- Scaling

$$F[f(ax,by)] = \frac{1}{ab} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

An 1-D example



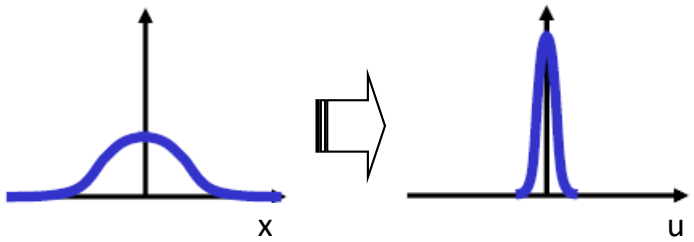
The shorter the pulse, the broader the spectrum !



$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$where\ j = \sqrt{-1}$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$



Spatial domain, f(x)

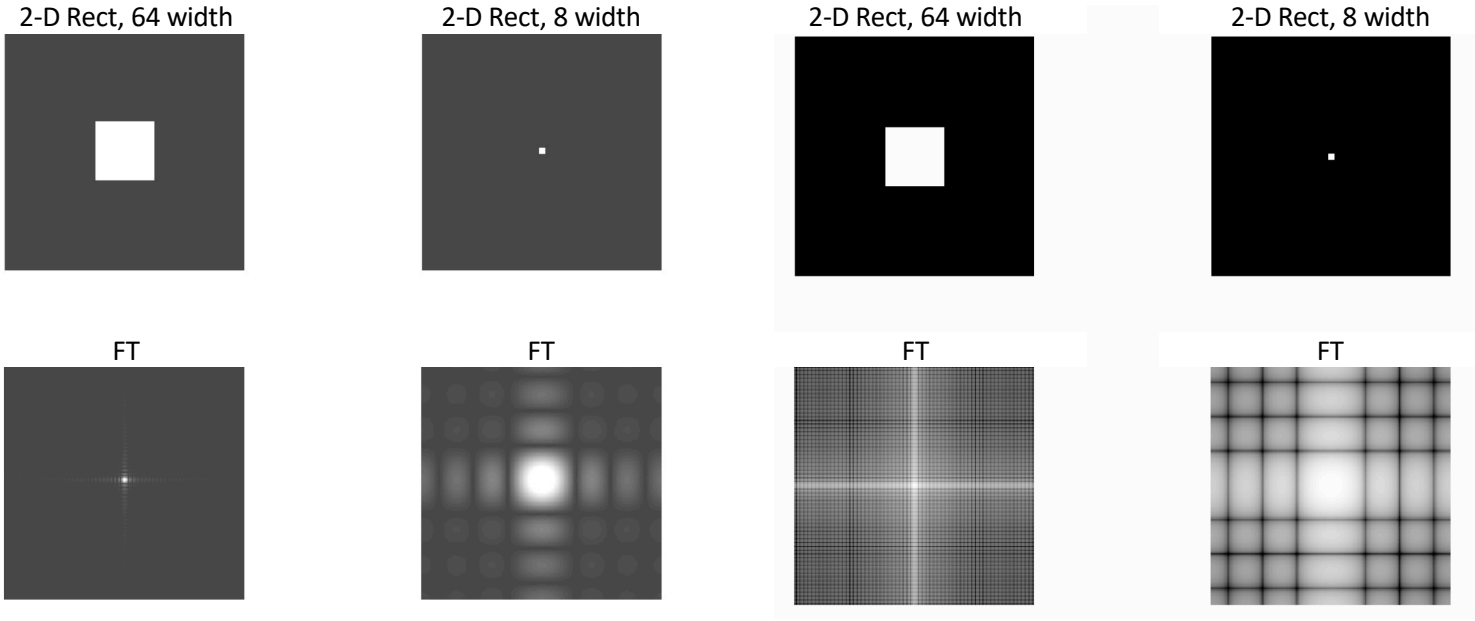
Fourier Transform, F(u)

Properties of Fourier Transform

- Scaling

$$F\left[f(ax,by)\right]=\frac{1}{ab}F\left(\frac{u}{a},\frac{v}{b}\right)$$

A 2-D example



What is the Convolution Theorem of the Fourier Transform?

and

Its Implication to Modeling a System's Response to an Arbitrary Input Signal?

Linear and Shift Invariant Systems Revisited

For a shift-invariant system, the output is the input convolved with the impulse response function.

$$
\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x,y,\xi,\eta)d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta \\
&= f(x,y) * h(x,y)
\end{aligned}
$$

where $h(x,y,\xi,\eta)$ is the response of the system to an delta impulse signal

Convolution Theorem

- The convolution of two 2-D functions is defined as

$$f(x,y) * h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\cdot h(x-\xi, y-\eta)\,d\xi\,d\eta$$

- The Fourier transform of the convolution is equal to the product of the individual Fourier transforms:

$$\mathcal{F}\left[f(x,y) * h(x,y)\right] = \mathcal{F}\left[f(x,y)\right]\cdot\mathcal{F}\left[h(x,y)\right]$$

where $\mathcal{F}[\cdot]$ is the Fourier transform operator.

The output from a **linear and shift-invariant** system is therefore

$$g(x,y) = f(x,y) * h(x,y) = \mathcal{F}^{-1}\left\{\mathcal{F}\left[f(x,y)\right]\cdot\mathcal{F}\left[h(x,y)\right]\right\}$$

Properties of Fourier Transform

- ***Product***

$$\boldsymbol{F}\left[f(x,y)\cdot g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]*\boldsymbol{F}\left[g(x,y)\right]$$

Fourier transform of the product of two functions equals to the convolution of the Fourier transforms of individual function.

- ***The Convolution Theorem***

$$\boldsymbol{F}\left[f(x,y)*g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]\cdot\boldsymbol{F}\left[g(x,y)\right]$$

Properties of Fourier Transform

- Separable product

$$if$$

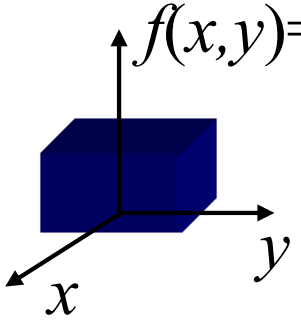
$$f(x,y) = f_1(x) \cdot f_2(y)$$

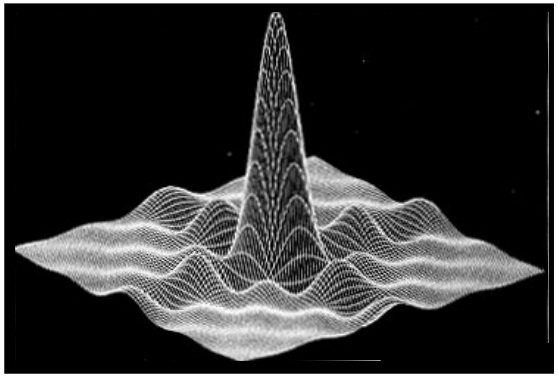
$$then$$

$$\boldsymbol{F}_{u,v}\left[f(x,y)\right] = \boldsymbol{F}_u\left[f_1(x)\right] \cdot \boldsymbol{F}_v\left[f_2(y)\right]$$

$$\mathcal{F}\{f(x,y)\} = \mathrm{sinc}(u)\,\mathrm{sinc}(v)$$

$f(x,y) = \mathrm{Rect}(x)\,\mathrm{Rect}(y)$





Symmetry Properties

Expanding the Fourier transform of a function, f(t):

$$F(\omega) = \int_{-\infty}^{\infty} [\mathrm{Re}\{f(t)\} + i\,\mathrm{Im}\{f(t)\}]\ [\cos(\omega t) - i\sin(\omega t)]\,dt$$

Expanding more, noting that: $\int_{-\infty}^{\infty} O(t)\,dt = 0$ if O(t) is an odd function

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\cos(\omega t)\,dt + \int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\sin(\omega t)\,dt \leftarrow \mathbf{Re}\{F(\omega)\}$$

= 0 if Re{f(t)} is odd; = 0 if Im{f(t)} is even

$$+\ i\int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\cos(\omega t)\,dt - i\int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\sin(\omega t)\,dt \leftarrow \mathbf{Im}\{F(\omega)\}$$

= 0 if Im{f(t)} is odd; = 0 if Re{f(t)} is even

Even functions of w — Odd functions of w

Fourier transform of Circularly Symmetric Functions (1)

A function is circularly symmetric if

$$f_\theta(x,y) = f(x,y), \quad \text{for every } \theta,$$
$$\text{where } f_\theta(x,y) \text{ is a rotated version of } f(x,y).$$

In this case, the function

$$f(x,y) = f(r), \text{ where } r = \sqrt{x^2 + y^2}$$

The Fourier transform is also real and circularly symmetric.

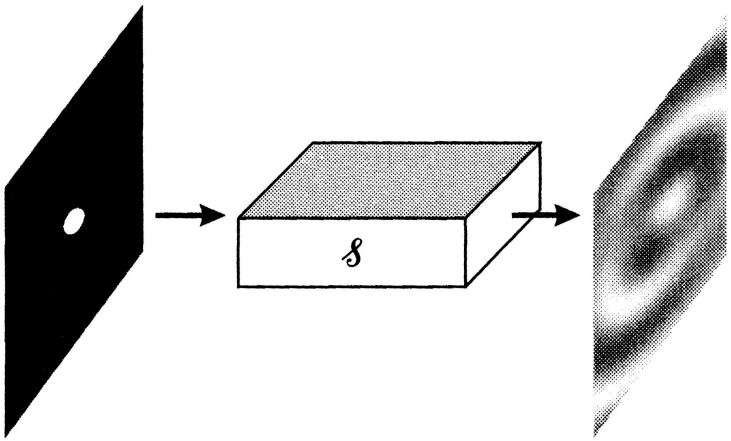
$$\left|F(u,v)\right| = F(u,v) \text{ and } \angle F(u,v) = 0$$
$$and$$
$$F(u,v) = F(q), \text{where } q = \sqrt{u^2 + v^2}$$

Convolution Theorem Revisited

The convolution theorem enables one to perform convolution operation as multiplication process in spatial frequency domain.

$$f(x,y) * h(x,y) = \mathcal{F}^{-1}\left\{\mathcal{F}\left[f(x,y)\right] \cdot \mathcal{F}\left[h(x,y)\right]\right\}$$

By using the Fast Fourier Transform (FFT) algorithms, the convolution operation can be performed very efficiently !! This provide a practical way for modeling linear shift-invariant systems …



$$g(x,y) = f(x,y) * h(x,y)$$

System Transfer Function

The Fourier transform of the impulse response function h is called the system transfer function.

$$H(u,v) = \mathcal{F}\left[h(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)e^{-2\pi j(ux+vy)}\,dx\,dy$$

$$G(u,v) = F(u,v)H(u,v)$$

System Transfer Function

$$G(u,v) = F(u,v)H(u,v)$$

An ideal low-pass filter is defined as

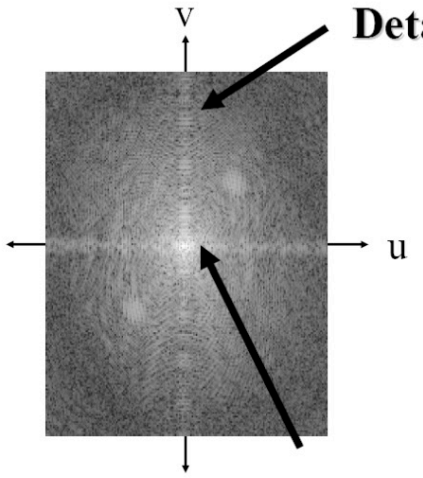
$$H(u,v) = \begin{cases} 1 & \text{for } \sqrt{\mathrm{u}^2 + \mathrm{v}^2} \le c \\ 0 & \text{for } \sqrt{\mathrm{u}^2 + \mathrm{v}^2} > c \end{cases}$$

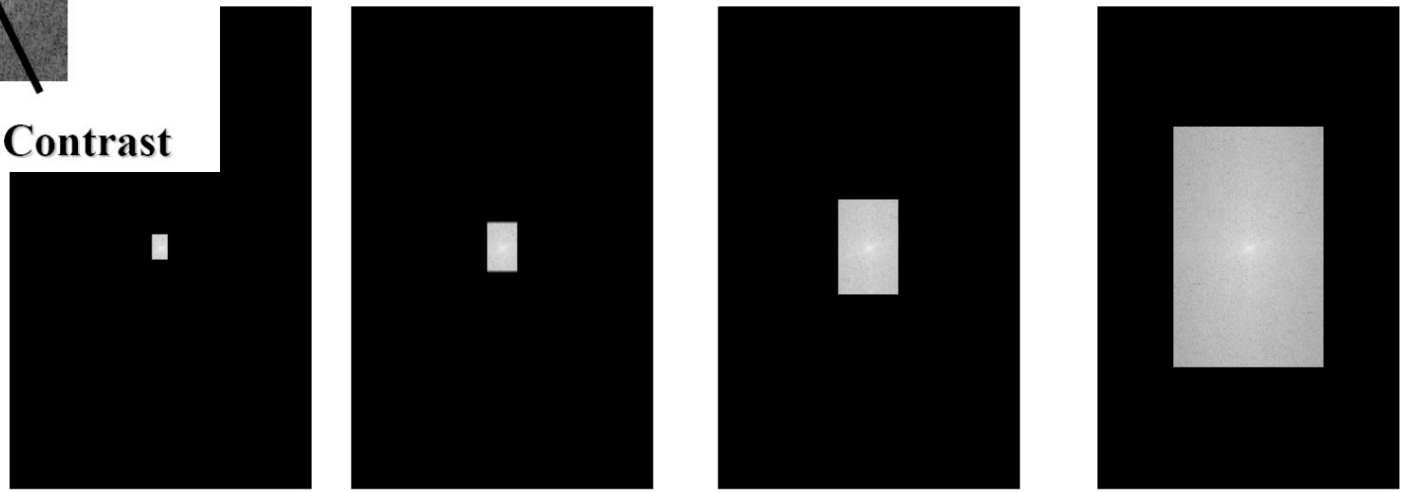
c is called the cut-off frequency.

Image



Fourier Space (log magnitude)



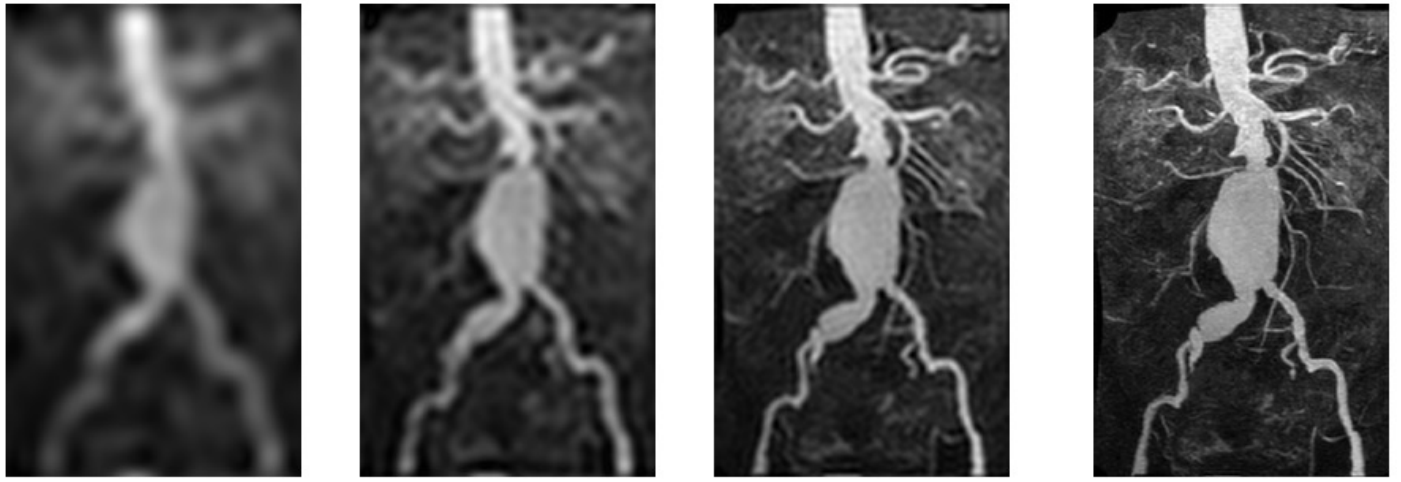


5 %

10 %

20 %

50 %



Sampling

Sampling

Learning Objectives: All about the Nyquist sampling theorem

1. How to sample a continuous signal into a discrete signal?
2. How do we quantify the **information lost** by the continuous-to-discrete sampling process?
    1. What can we learn from the Fourier transform of the sampled signal?
    2. For a band-width-limited signal, what is the condition required for a sampling process to avoid information loss? – the **Nyquist sampling conditions**.
    3. If a band-width-limited signal is sampled by satisfying the Nyquist sampling conditions, how can we restore the original signal from the sampled signal?

What is the Sampling?

and

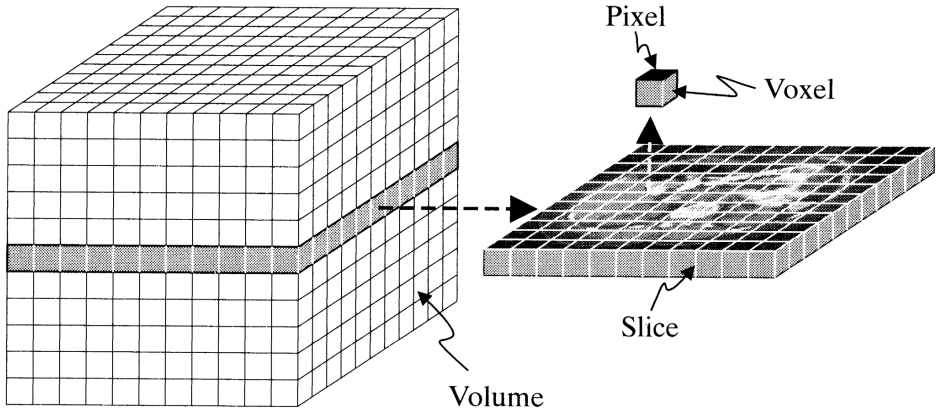
How to Sample a Continuous Signal into Discrete without Losing Information?

What and Why?

Transformation from continuous signals into discrete signals is called ***sampling***. The sampled data can then be processed by digital hardware.

$$f(x,y) \Rightarrow f(m\Delta x, n\Delta y), \text{ for m, n} = 0,1,...$$

where $\Delta x, \Delta y$ are called sampling intervals or sampling periods.



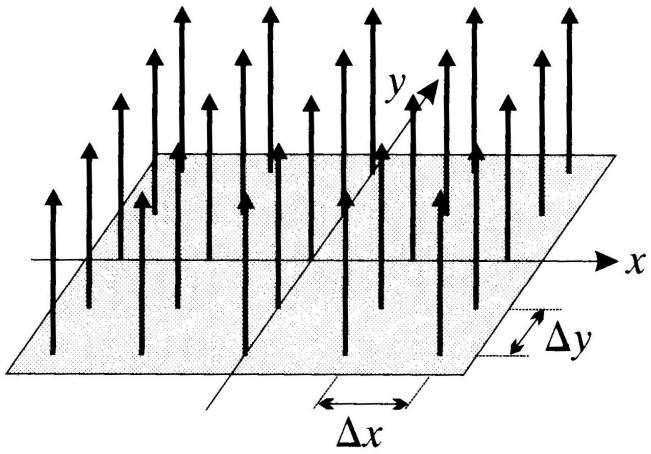
Comb and Sampling Function

- The 2-D sampling function

$$\delta_s(x,y,\Delta x,\Delta y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x, y-n\Delta y)$$

where Δx and Δy are the sampling intervals

$$\delta_s(x,y,\Delta x,\Delta y) = \frac{1}{\Delta x \Delta y}\,comb\left(\frac{x}{\Delta x},\frac{y}{\Delta y}\right)$$

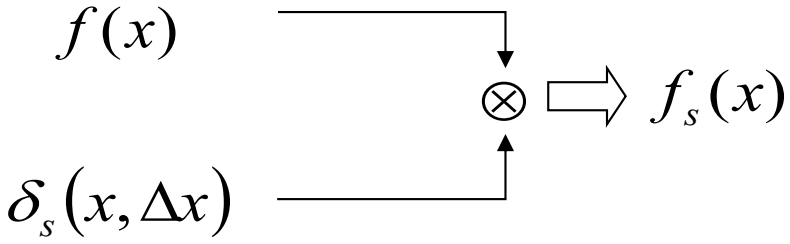


$$comb(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m, y-n)$$

$$\delta(ax,by) = \frac{1}{|ab|}\delta(x,y)$$

Sampling in 1-D

Model

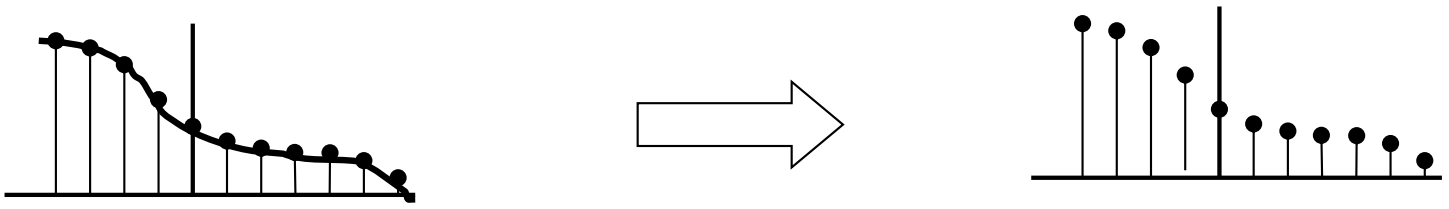


The 1-D sampling function

$$\delta_s(x,\Delta x)=\sum\delta(x-n\cdot\Delta x)$$

The sampled function

$$f_s(\mathrm{x})=f(\mathrm{x})\cdot\delta_s(x,\Delta x)=\sum f(n\Delta x)\delta(x-n\Delta x)$$



Basic Fourier Transform Pairs

Basic Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0+vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\text{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0x+v_0y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0x + v_0y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0x + v_0y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\text{rect}(x, y)$	$\text{sinc}(u, v)$
$\text{sinc}(x, y)$	$\text{rect}(u, v)$
$\text{comb}(x, y)$	$\text{comb}(u, v)$
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$

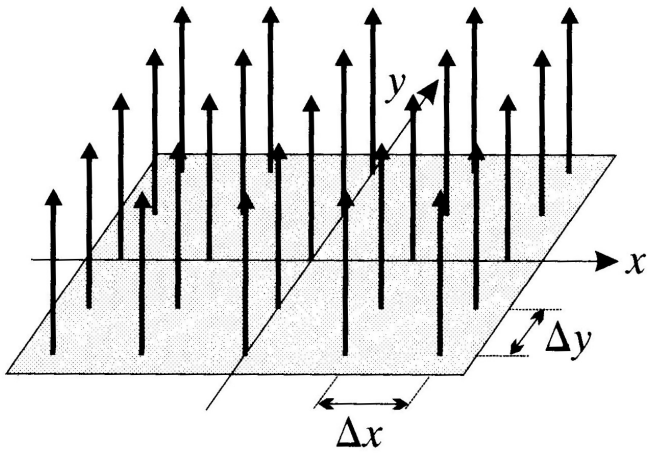
Comb and Sampling Function

- The 2-D sampling function

$$\delta_s(x,y,\Delta x,\Delta y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x,y-n\Delta y)$$

where Δx and Δy are the sampling intervals

$$\delta_s(x,y,\Delta x,\Delta y)=\frac{1}{\Delta x\Delta y}comb\left(\frac{x}{\Delta x},\frac{y}{\Delta y}\right)$$



$$comb(x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n)$$

$$\delta(ax,by)=\frac{1}{|ab|}\delta(x,y)$$

Comb and Sampling Function

- The 2-D comb function

$$comb(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n)$$

Fourier Transform of Sampled Function

$$\mathrm{F}_{f_s}(u) = \mathrm{F}\left[f_s(x)\right] = \mathrm{F}\left[\delta_s(x,\Delta x)\cdot f(x)\right]$$

$$= comb(u\cdot \Delta x) * \mathrm{F}\left[f(x)\right]$$

$$= \left\{\sum_{n=-\infty}^{n=\infty} \delta\left[\Delta x\left(u - \frac{n}{\Delta x}\right)\right]\right\} * \mathrm{F}(u)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{n=\infty}\left\{\delta\left(u - \frac{n}{\Delta x}\right) * \mathrm{F}(u)\right\}$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{n=\infty} \mathrm{F}(u - \frac{n}{\Delta x})$$

$$comb(x) = \sum_{m=-\infty}^{\infty}\delta(x-m)$$

$$\delta_s(x,\Delta x) = \sum_{m=-\infty}^{\infty}\delta(x-m\Delta x)$$

$$\mathcal{F}\left[\delta_s(x,\Delta x)\right] = comb(u\cdot\Delta x)$$

Multiplication in the spatial domain becomes convolution in the spatial frequency domain

Point Impulse Signal

- The sampling property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-\xi,y-\eta)dxdy = f(\xi,\eta)$$

- The scaling property

$$\delta(ax,by) = \frac{1}{|ab|}\delta(x,y)$$

$$\delta(x,y)\begin{cases}\neq 0, & x=0 \text{ and } y=0\\ =0, & otherwise\end{cases}$$

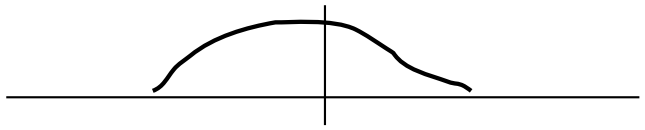
and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy = 1$$

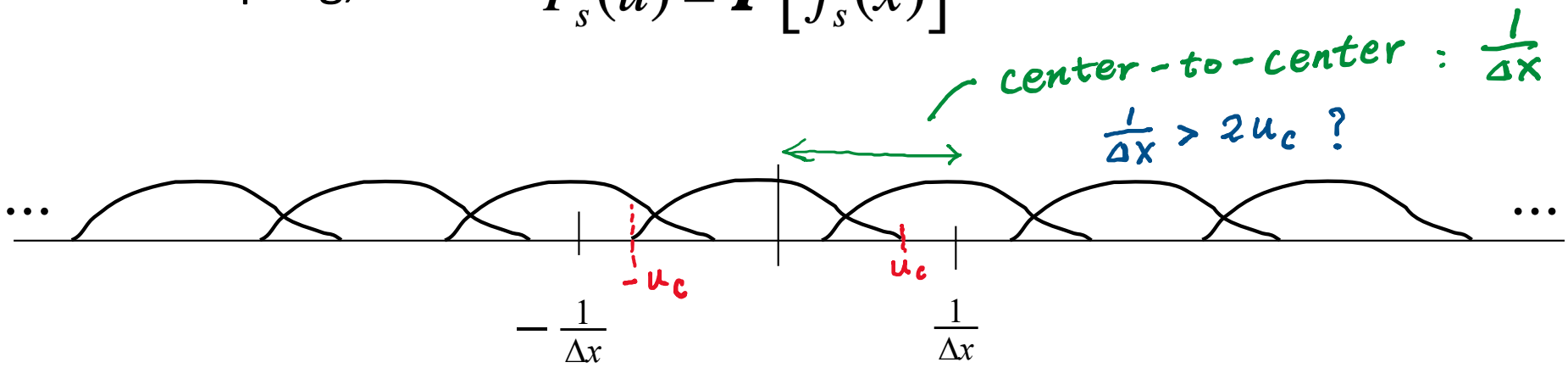
Fourier Transform of Sampled Function

$$\boldsymbol{F}\left[f_s(x)\right] = \frac{1}{\Delta x}\sum_{n=-\infty}^{n=\infty} \mathrm{F}(u - \frac{n}{\Delta x})$$

Prior to sampling, $\mathrm{F}(u) = \boldsymbol{F}\left[f(x)\right]$



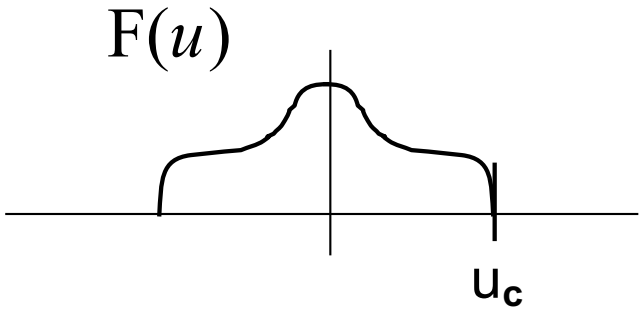
After sampling, $F_s(u) = \boldsymbol{F}\left[f_s(x)\right]$



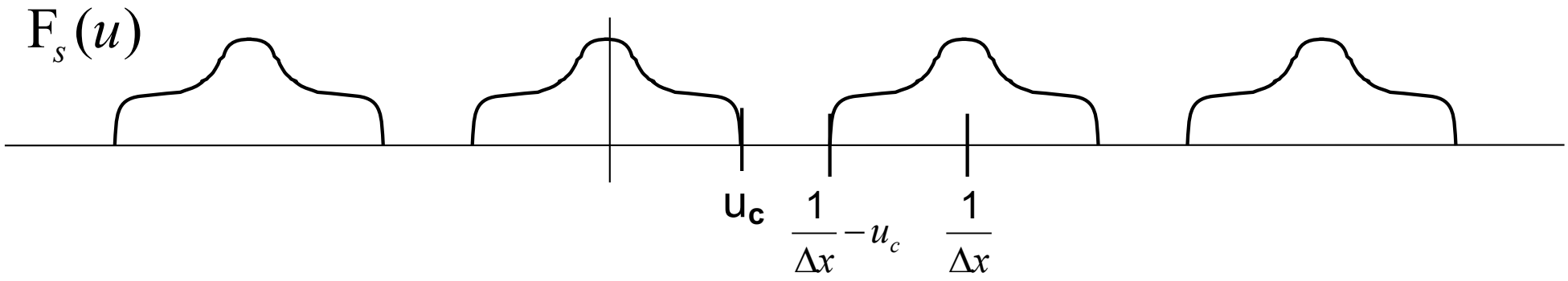
Fourier Transform of Sampled Function

If F(u) is band limited to u_c, (cutoff frequency)

F(u) = 0 for |u| > u_c.



To avoid overlap (aliasing),

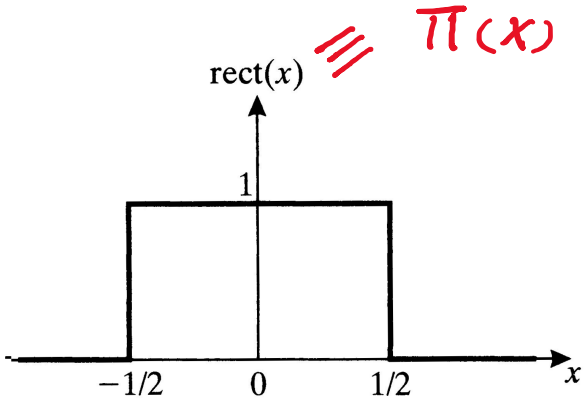


We would need $\frac{1}{\Delta x} - u_c > u_c$ **and therefore** $\frac{1}{\Delta x} > 2u_c$

Rect Function

- Rect function:

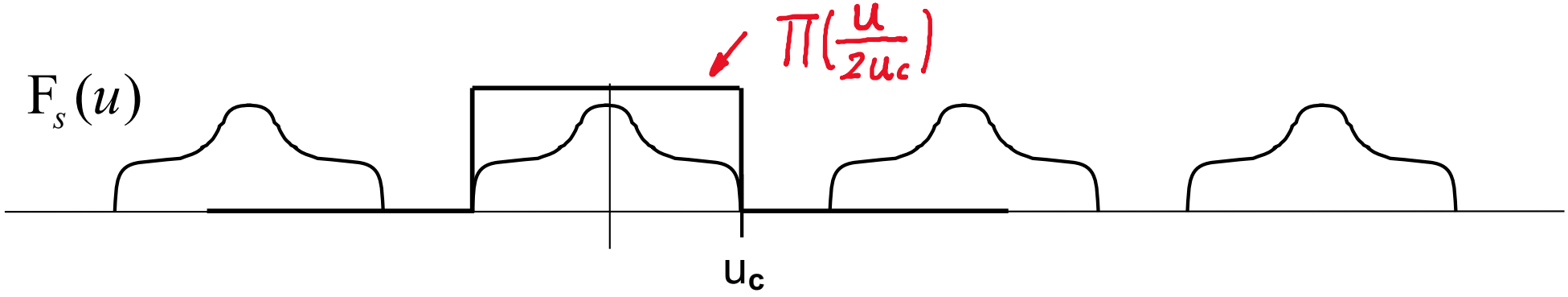
$$rect(x,y)=\begin{cases}1, \text{for } |x|<\frac{1}{2} \text{ and } |y|<\frac{1}{2}\\ 0, \text{otherwise}\end{cases}$$



- It is normally used to pick up a particulate section of a given function:

$$f(x,y)\cdot rect\left(\frac{x-\xi}{w_X},\frac{y-\eta}{w_Y}\right)$$

Restoration of Original Signal



Can we restore g(x) from the sampled frequency-domain signal? Yes, using the Interpolation Filter

$$\mathrm{H}(u) = \Pi\left(\frac{u}{2u_c}\right)$$

$\Updownarrow$ Fourier transform

$$h(x) = 2u_c \cdot \mathrm{sinc}(2u_c x)$$

Restoration of Original Signal

The original function f can be recovered as

$$f(x) = \mathcal{F}^{-1}\left[F_{f_s}(u) \cdot \Pi\left(\frac{u}{2u_c}\right)\right]$$

$$f(x) = f_s(x) * h(x)$$

$$= f_s(x) * \left[2u_c \cdot sinc(2u_c x)\right]$$

$$= \left[\sum_{n=-\infty}^{\infty} f(n\cdot\Delta x)\cdot\delta(x - n\cdot\Delta x)\right] * \left[2u_c \cdot \mathrm{sinc}(2u_c x)\right]$$

$$= \sum_{n=-\infty}^{\infty} 2\cdot u_c \cdot f(n\cdot\Delta x)\cdot sinc(2u_c(x - n\Delta x))$$

$$f_s(x) = \sum f(n\Delta x)\delta(x - n\Delta x)$$

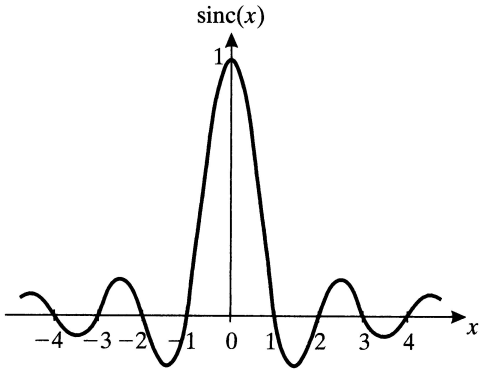
$$\mathrm{H}(u) = \Pi\left(\frac{u}{2u_c}\right)$$

$$\mathcal{F} \Updownarrow$$

$$h(x) = 2u_c \cdot \mathrm{sinc}(2u_c x)$$

$$\int_{-\infty}^{\infty} \delta(u - \xi)\cdot f(u)du = f(\xi)$$

f(x) is restored from a combination of sinc functions, each weighted and shifted according to its corresponding sampling point.



Nyquist Sampling Theorem

Nyquist Theorem:

To restore the original function, the sampling rate must be greater than twice the highest frequency component of the function.



Harry Nyquist

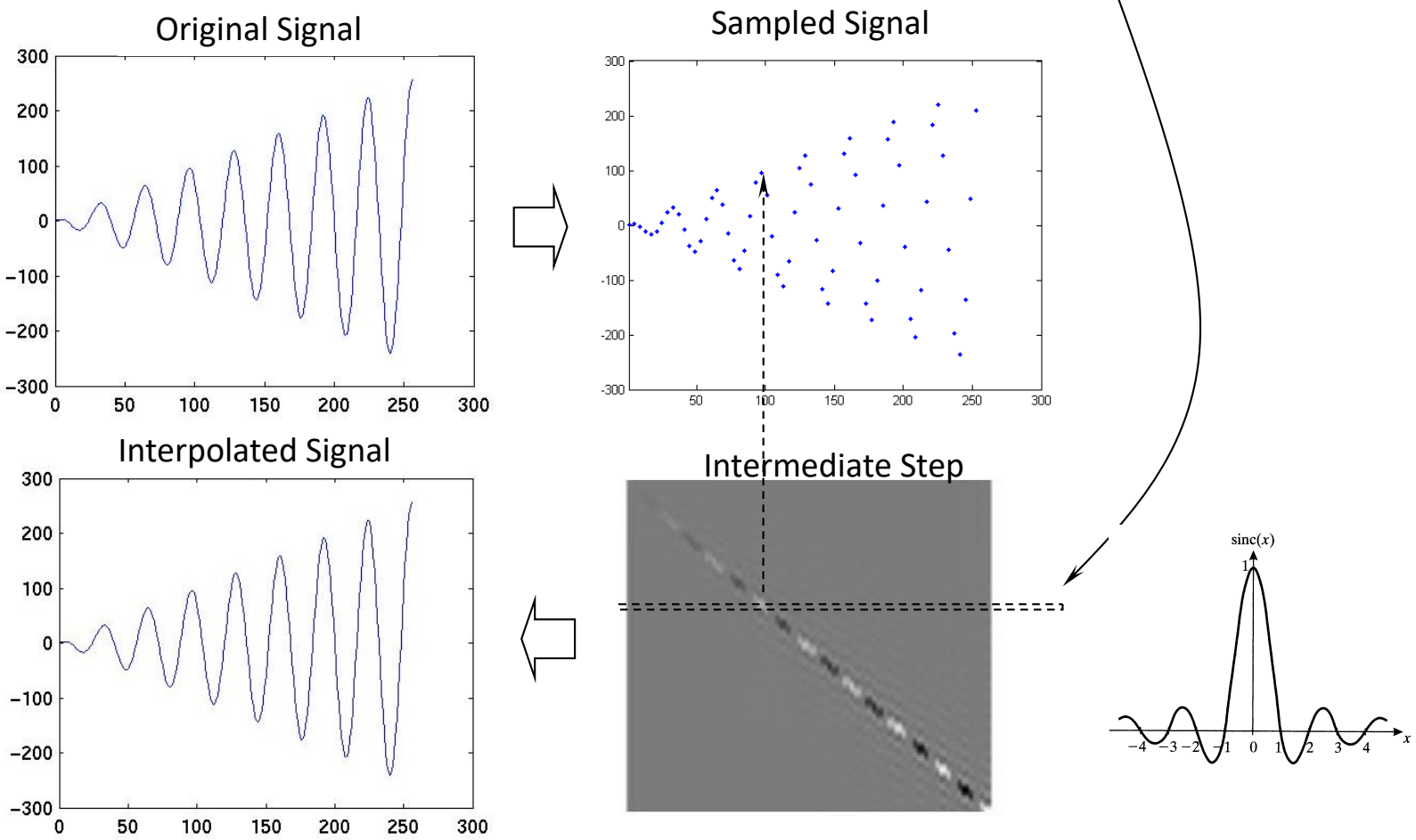
Nyquist Sampling Interval:

The maximum sampling interval allowed without introduce aliasing is

$$\Delta x \leq \frac{1}{2u_c}$$

An Example

$$f(x) = \sum_{n=-\infty}^{\infty} 2 \cdot u_c \cdot f(n \cdot \Delta x) \cdot sinc(2u_c(x - n\Delta x))$$



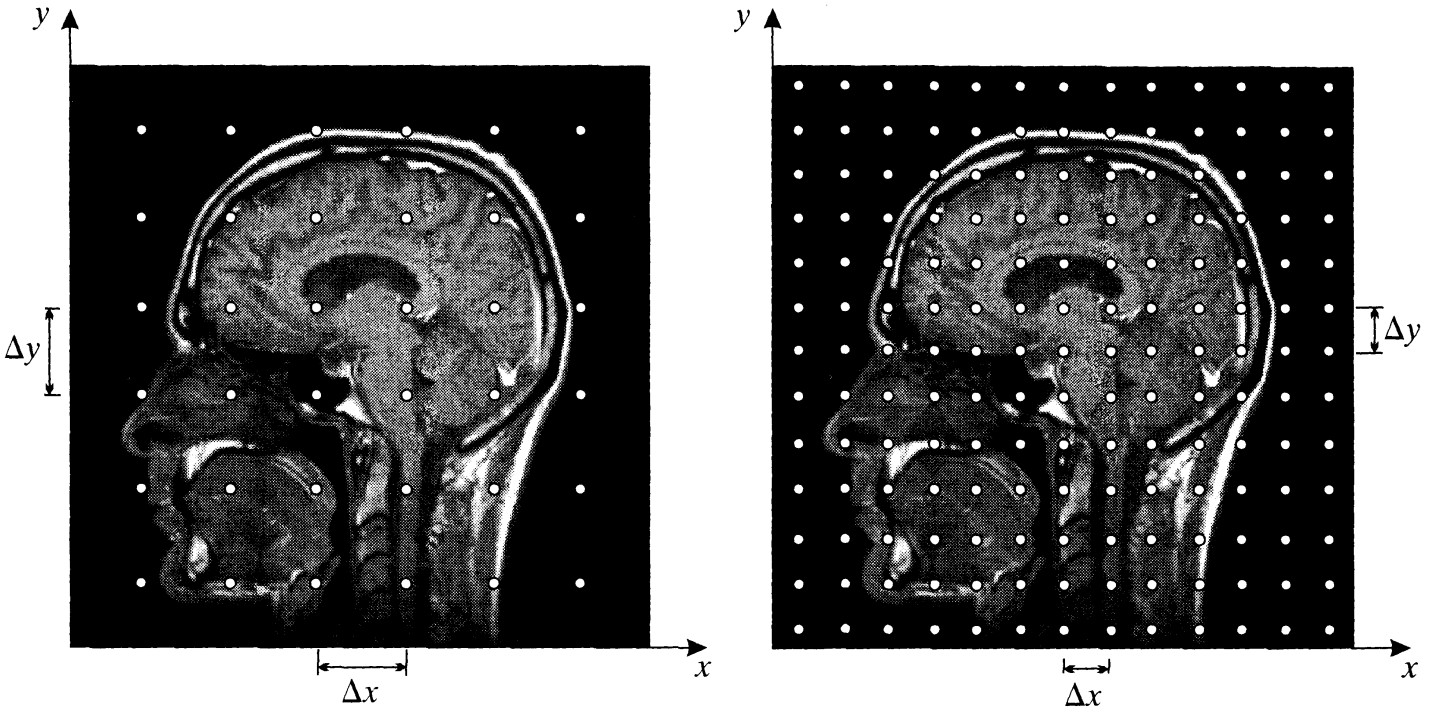
Adding all vertical values gives back the original function

Two-Dimensional Sampling

$$f_s(x,y) = f(x,y)\cdot\delta_s(x,y,\Delta x,\Delta y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(x,y)\cdot\delta(x-n\Delta x, y-m\Delta y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y)\cdot\delta(x-n\Delta x, y-m\Delta y)$$



Fourier Transform of Sampled Image

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(u-\xi,v-\eta)\cdot f(u,v)\,du\,dv = f(\xi,\eta)$$

$$\mathrm{F}_{f_s}(u) = \mathrm{F}\left[f_s(x,y)\right]$$

$$= \mathrm{F}\left[\delta_s(x,y,\Delta x,\Delta y)\cdot f(x,y)\right]$$

$$= comb(u\cdot\Delta x, v\cdot\Delta y) * \mathrm{F}\left[f(x,y)\right]$$

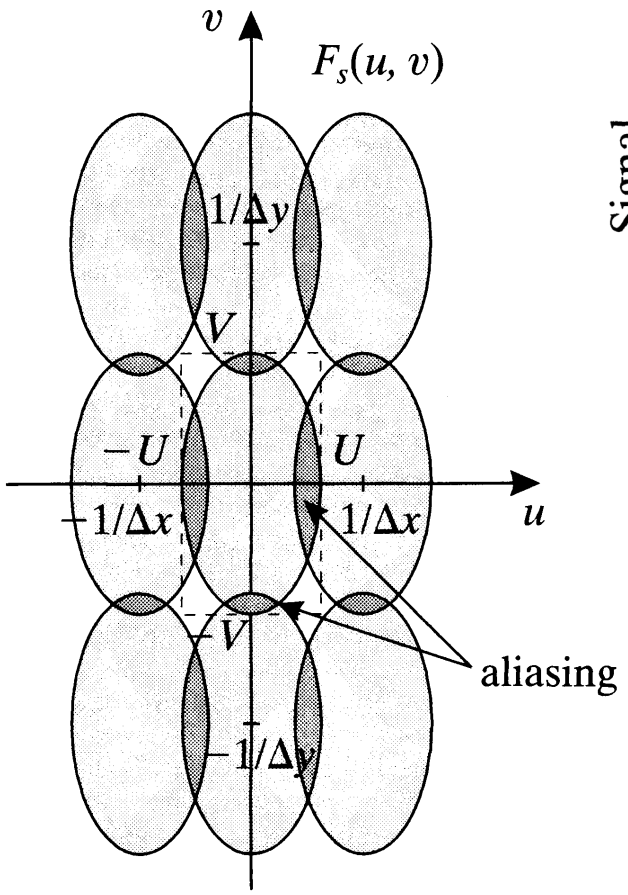
$$= \sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\delta\left[\Delta x\left(u-\frac{n}{\Delta x}\right),\Delta y\left(v-\frac{m}{\Delta y}\right)\right] * \mathrm{F}(u,v)$$

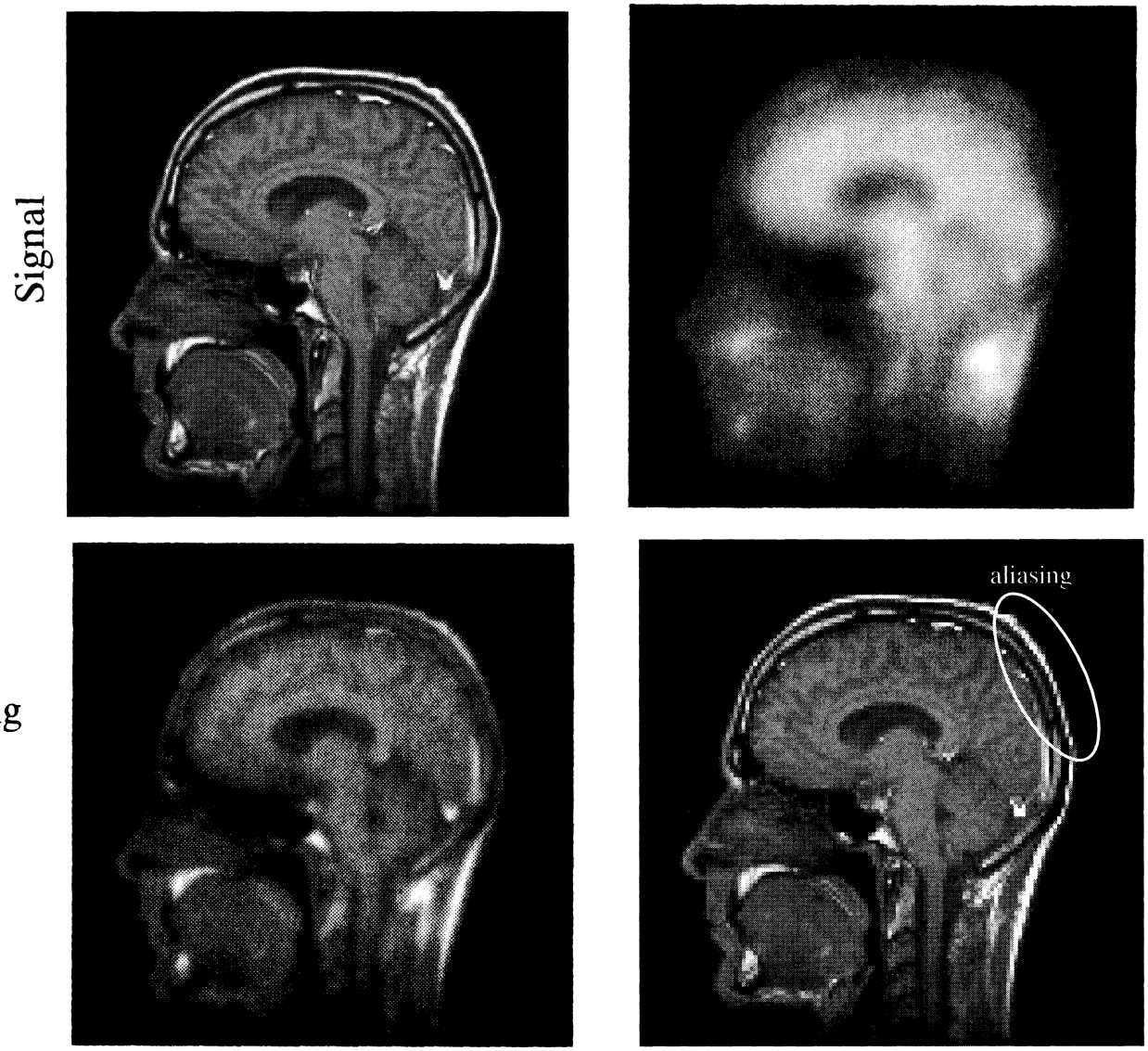
$$= \frac{1}{\Delta x\Delta y}\sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\delta\left(u-\frac{n}{\Delta x},v-\frac{m}{\Delta y}\right) * \mathrm{F}(u,v)$$

$$= \frac{1}{\Delta x\Delta y}\sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\mathrm{F}\left(u-\frac{n}{\Delta x},v-\frac{m}{\Delta y}\right)$$

The result: Replicated F(u,v), or "islands" every 1/ Δx in u, and 1/Δy in v.

Consequence of Under-Sampling





Restoration of the Original 2-D Function

Interpolation Filter in 2-D

$$\mathrm{H}(u,v) = \Pi\left(u\Delta x\right)\Pi\left(v\Delta y\right)$$

$$\Updownarrow$$

$$h(x,y) = \left[\frac{1}{\Delta x}\cdot \mathrm{sinc}\left(\frac{x}{\Delta x}\right)\right]\left[\frac{1}{\Delta y}\cdot \mathrm{sinc}\left(\frac{y}{\Delta y}\right)\right]$$

Restoration of the Original 2-D Function

Given that the Nyquist sampling condition is met, the original function can be recovered exactly.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(u-\xi,v-\eta)\cdot f(u,v)\,du\,dv = f(\xi,\eta)$$

$$f(x,y) = f_s(x,y) * h(x,y)$$

$$= f_s(x,y) * \left[\frac{1}{\Delta x}\cdot \mathrm{sinc}\left(\frac{x}{\Delta x}\right)\right]\left[\frac{1}{\Delta y}\cdot \mathrm{sinc}\left(\frac{y}{\Delta y}\right)\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y)\cdot\delta(x-n\cdot\Delta x, y-m\cdot\Delta y) * \left\{\left[\frac{1}{\Delta x}\cdot \mathrm{sinc}\left(\frac{x}{\Delta x}\right)\right]\left[\frac{1}{\Delta y}\cdot \mathrm{sinc}\left(\frac{y}{\Delta y}\right)\right]\right\}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{\Delta x\Delta y} f(n\Delta x, m\Delta y)\cdot \mathrm{sinc}\left(\frac{x-n\cdot\Delta x}{\Delta x}\right)\cdot \mathrm{sinc}\left(\frac{y-m\cdot\Delta y}{\Delta y}\right)$$



Chapter 2: Mathematical Preliminaries

Mathematical Preliminaries for 2-D Image Reconstructions

Reconstruct 2-D images with Inverse Radon Transform and Filtered Backprojection

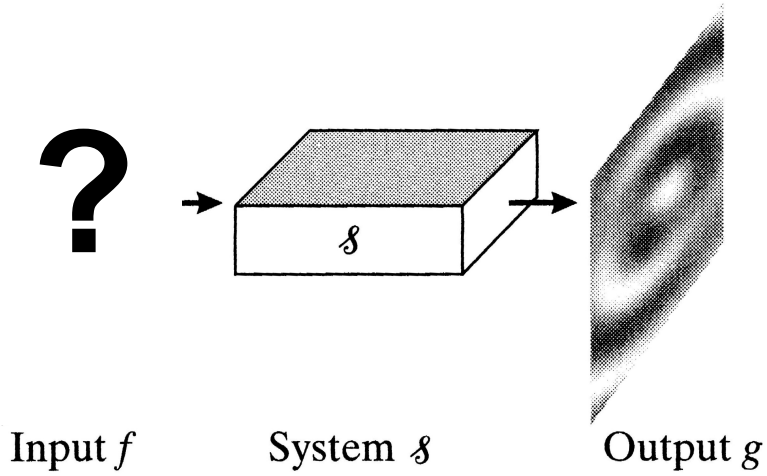
Learning objectives

1. What is the **Radon transform**?
   a. Why do we use the Radon transform to describe the physical data acquisition process in emission and transmission tomography?
   b. What is a projection? What is a sinogram?
2. **Central slice theorem**
   a. What does the projection data tell us about the unknown image function?
3. Based on the above understanding (i.e., the Central Slice theorem), can you design a strategy to restore the unknown image from the projection data?
   a. What is the **inverse Radon transform**? and why?
4. **Filtered-backprojection** (FBP)
   a. What is the difference between the Inverse Radon transform and Filtered-backprojection? What is the rationale for introducing such a difference?



The Inverse Problem

General Inverse Problem and Image Reconstruction



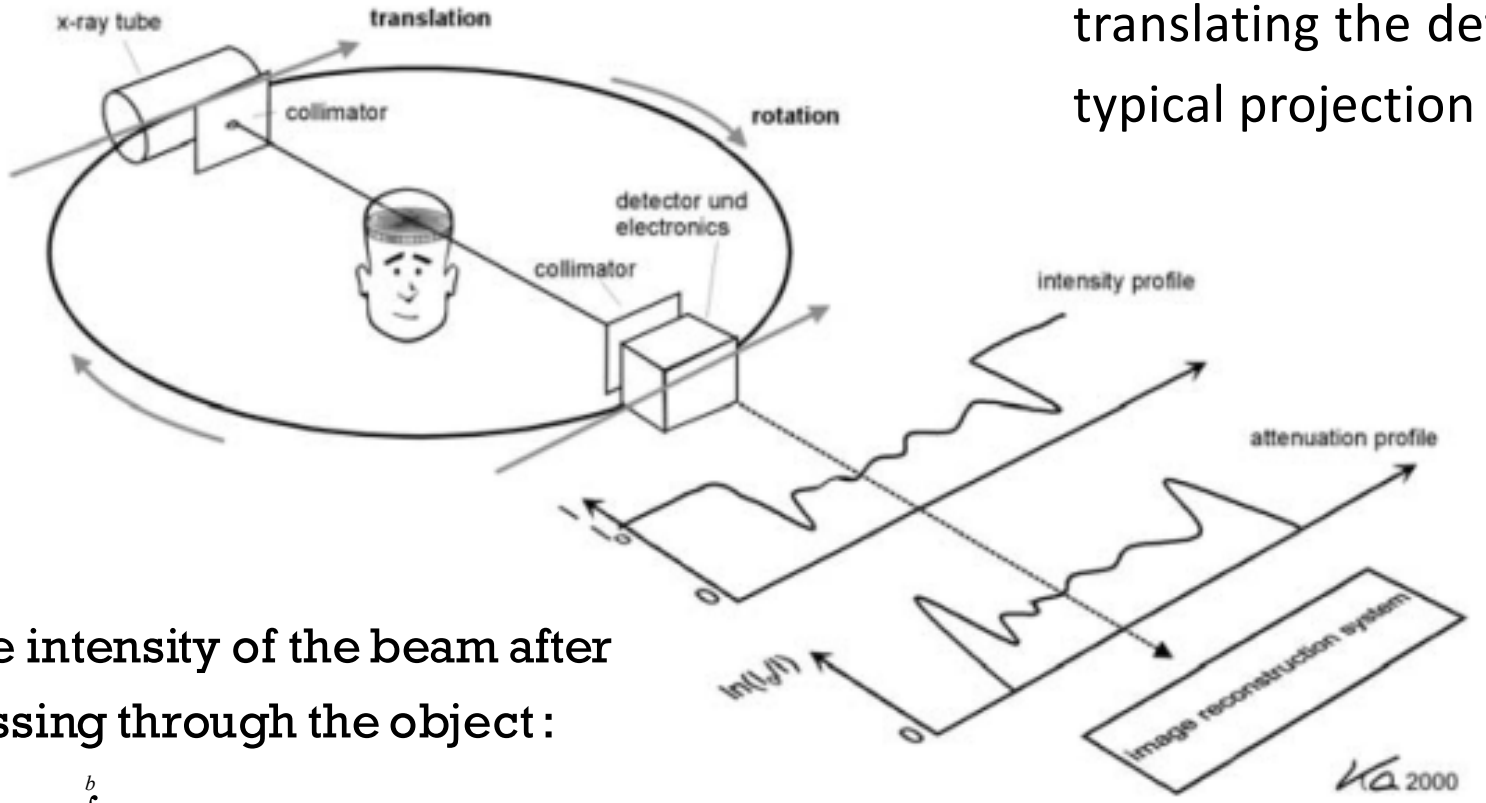
What should be the input signal that gave rise to the output data?



Radon Transform and Projection

Projection Data from Early X-ray CT Systems

Data measured by translating the detector is a typical projection data.



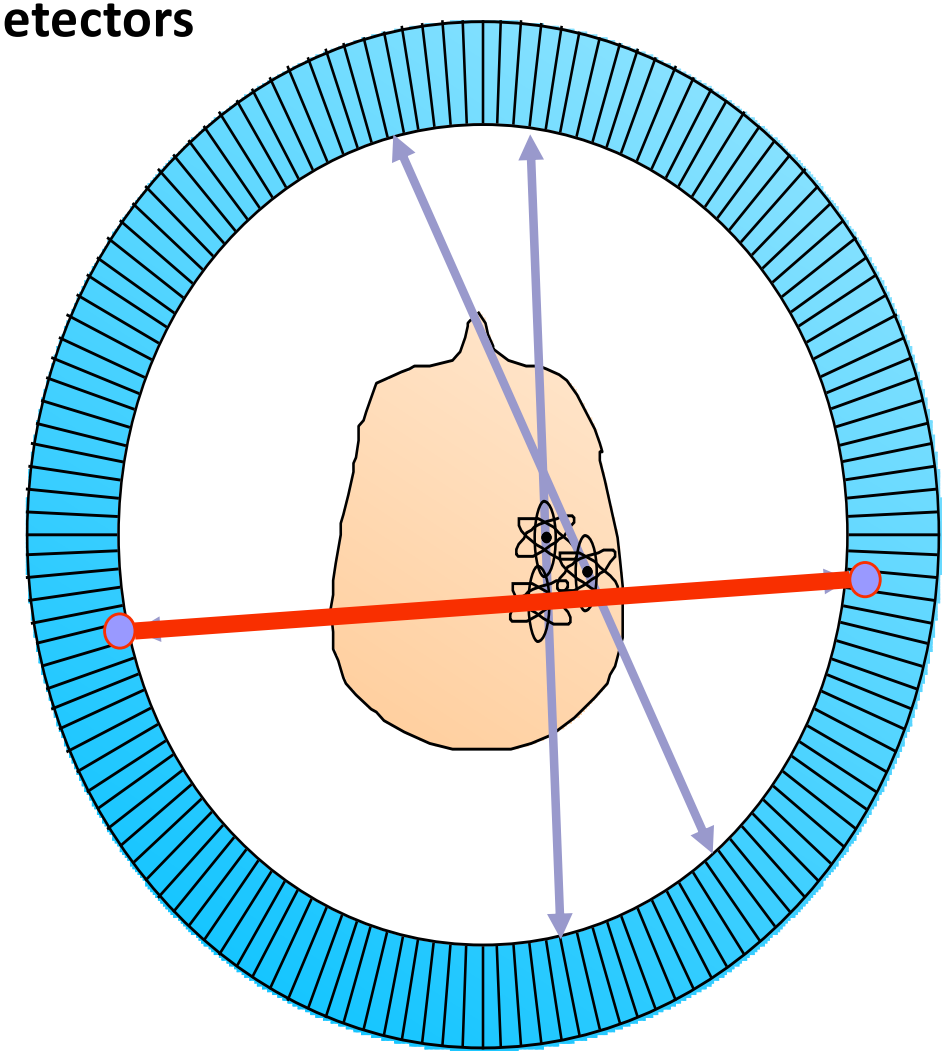
The intensity of the beam after passing through the object :

$$I = I_0 e^{-\int_a^b \mu(t)\cdot dt} \quad \Leftrightarrow \ln(I_0 / I) = \int_a^b \mu(t)\cdot dt$$

From Computed Tomography, Kalender, 2000.

Detect Radioactive Decays

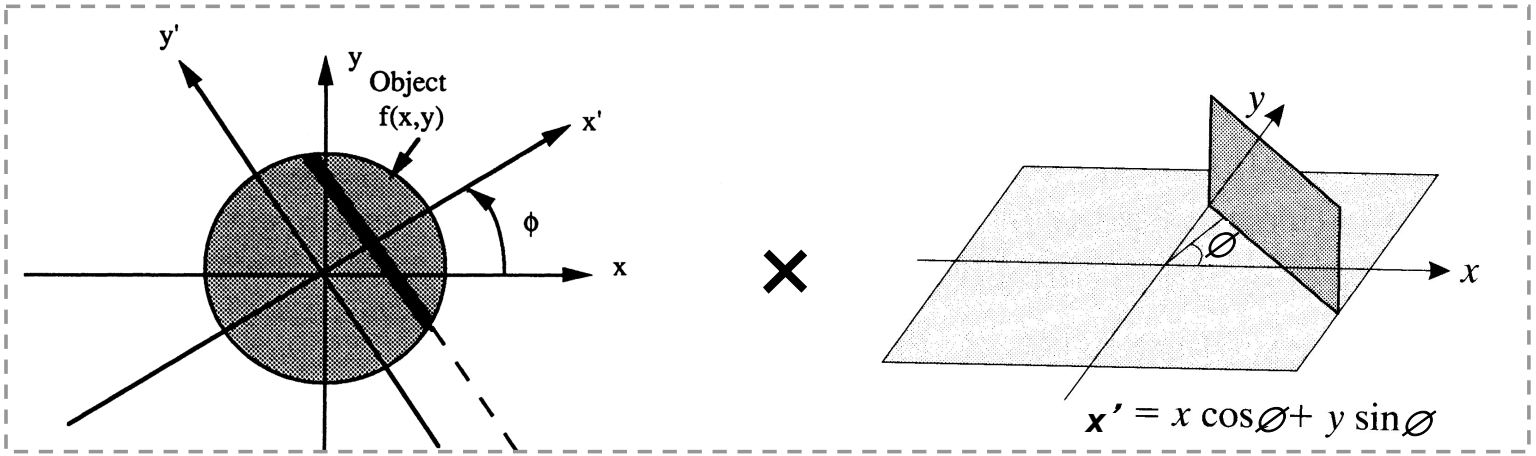
Ring of Photon Detectors



- Radionuclide decays, emitting β^{+}.
- β^{+} annihilates with e^{-} from tissue, forming back-to-back 511 keV photon pair.
- 511 keV photon pairs detected via time coincidence.
- Positron lies on a line defined by a detector pair (known as a chord, a line of response, or a *LOR*).

Detect Pair of Back-to Back 511 keV Photons

Radon Transform



2-D integral

The value of the projection function $p_\phi(x')$ at this point is the integral of the function of f(x,y) along the straight line: $x' = x\cos\phi + y\sin\phi$

The **Radon transform**, which produces the projection data, is defined as

$$p(\phi, x') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\phi + y\sin\phi - x')dxdy$$

Sinogram

Radon Transform and Sinogram

Radon transform maps a 2-D function f(x,y) into a sinogram.

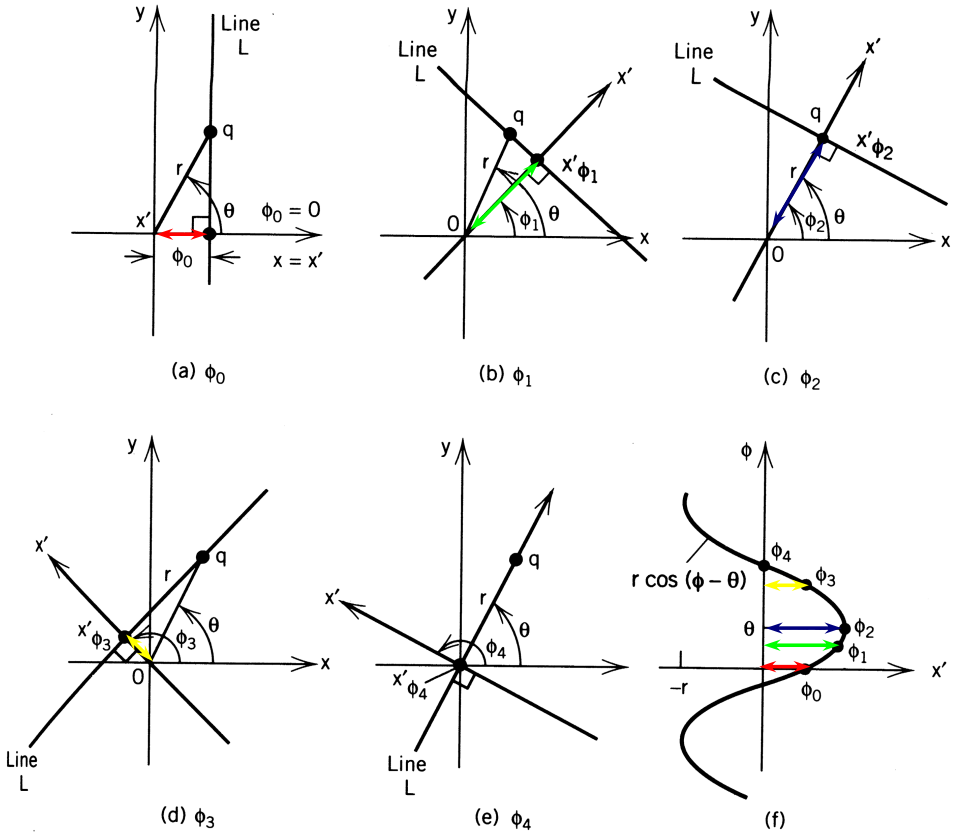
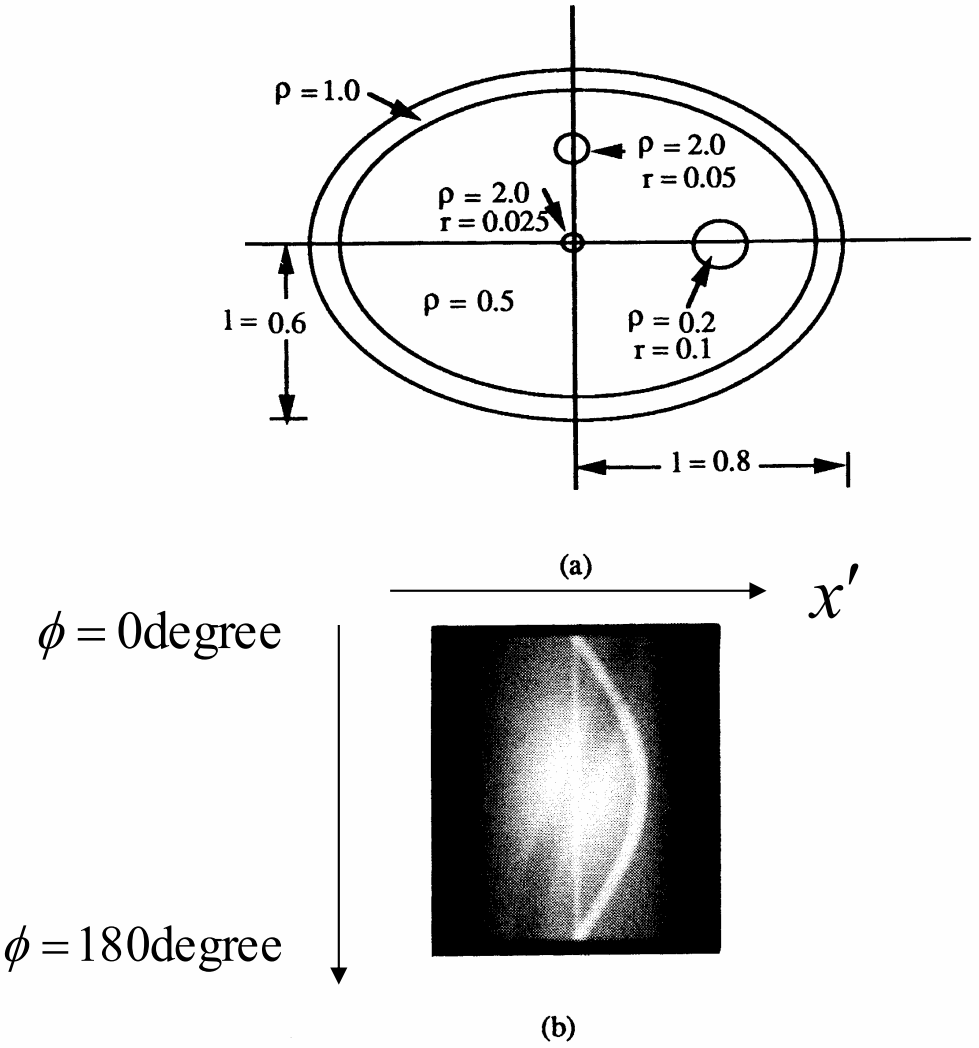


Figure 3-2 Sinogram of a point q at a various angles ϕ_i for a given θ: (a) $\phi_0 = 0$, (b) $\phi_1 = 45°$, (c) $\phi_2 = 60°$, (d) $\phi_3 = 135°$, and (e) $\phi_4 = 150°$. At $i = 0$ or $\phi_i = 0$, the rotated coordinates x' coincide with the reference x-axis and give $x' = r\cos(\phi - \theta) = r\cos\theta$.

$$\mathcal{P}(\phi, x') = \mathcal{R}[f(x,y)]$$

Any point represented by polar coordinates (r,θ) will simply follow the equation $x' = r\cos(\phi - \theta)$ in the sinogram space.

Radon Transform and Sinogram

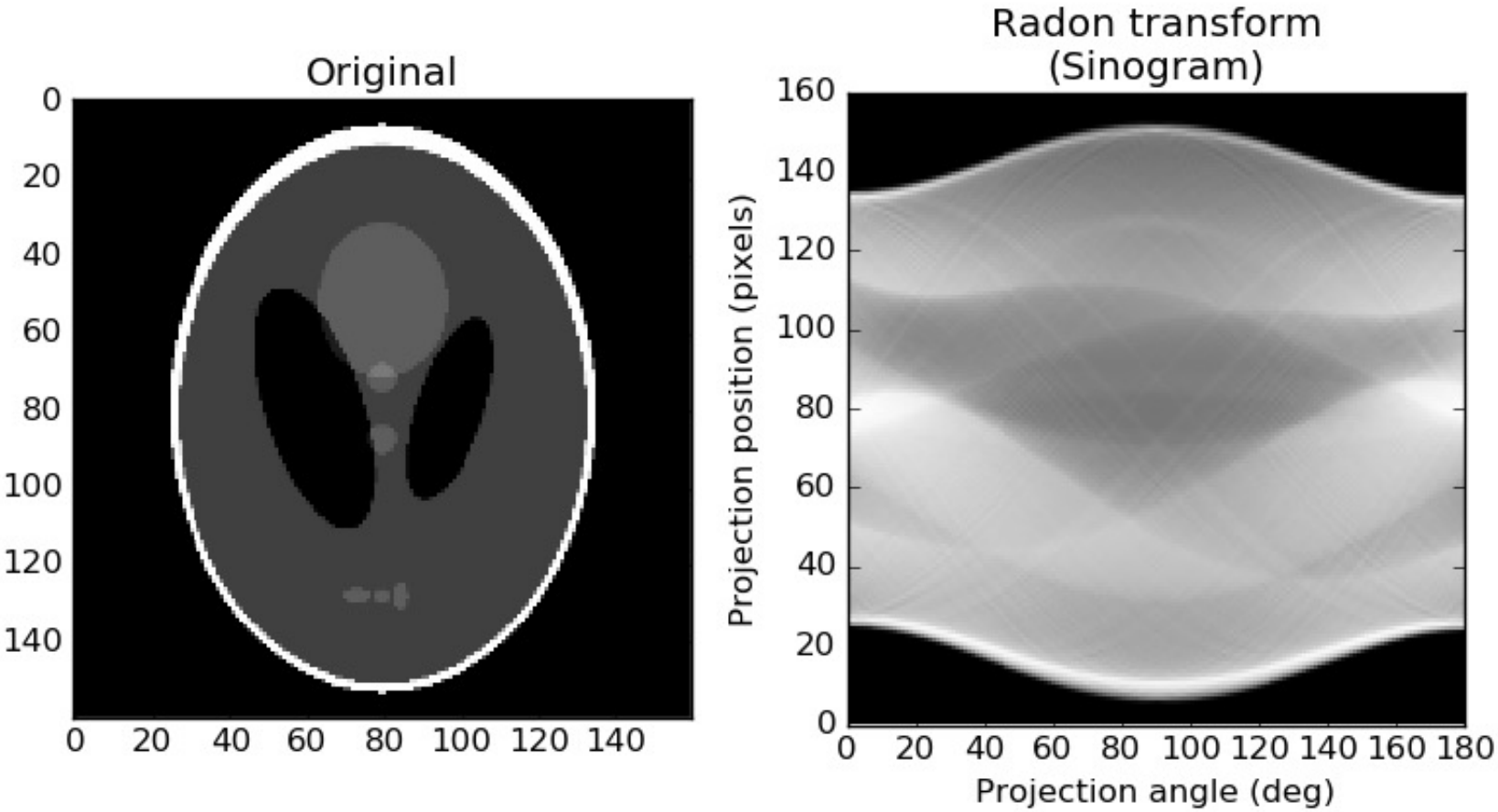


Sinogram is a 2-D function representing the original function f(x,y) into the projection data space.

Sinogram is the basic data format for reconstruction

Figure 3-3 (a) Phantom with several distinct objects at various positions; (b) correspondi sinogram.

Radon Transform and Sinogram



http://tech.snmjournals.org/cgi/content-nw/full/29/1/4/F3

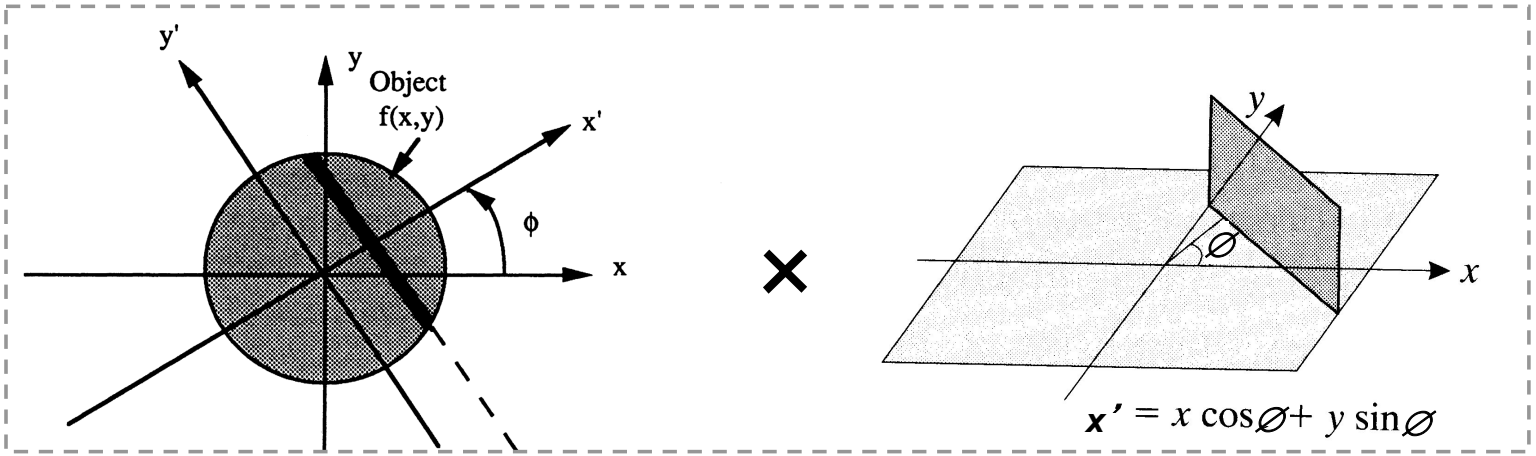


What Exactly Does Projection Data Tell Us?

The Central-Slice Theorem

(or Projection-Slice Theorem)

Radon Transform



2-D integral

The value of the projection function $p_\phi(x')$ at this point is the integral of the function of f(x,y) along the straight line: $x' = x\cos\phi + y\sin\phi$

The **Radon transform**, which produces the projection data, is defined as

$$p(\phi, x') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\phi + y\sin\phi - x')dxdy$$

Central Slice Theorem

Let's consider the 2D Fourier transform (FT) of an arbitrarily given function

$$F(u,v) = \int\int f(x,y) \cdot e^{-2\pi i (ux + vy)}\, dx\, dy\,. \qquad (1)$$

In polar coordinates and within the spatial-frequency domain, let

$$u = \rho \cos\phi \qquad \text{and} \qquad v = \rho \sin\phi,$$

Recall that
$\mathcal{F}[f(x-\tau)] = \mathcal{F}[f(x)]e^{-2\pi i\omega\tau}$,
and
$\mathcal{F}[f(x)] = 1$,
So
$\mathcal{F}[\delta(x-\tau)] = \mathrm{e}^{-2\pi i\omega\tau}$.

then (1) becomes

$$F(\rho, \phi) = \int\int f(x,y) \cdot e^{-2\pi i \rho (x\cos\phi + y\sin\phi)}\, dx\, dy\,. \qquad (2)$$

If we treat ($x \cos\phi + y \sin\phi$) as a constant, then $exp\,[\,-i\,2\pi\,\rho\,(x\cos\phi + y\sin\phi)]$ could be written as the Fourier transform of a shifted delta function. Let's write the complex exponential as the FT of a δ function,

$$F(\rho, \phi) = \int_y \int_x f(x,y) \cdot \mathcal{F}_{1d,x'}\{\delta[x' - (x\cos\phi + y\sin\phi]\}\; dx \cdot dy\,,$$

and

$$F(\rho, \phi) = \int_y \int_x f(x,y) \cdot \{\int_{x'} \delta[x' - (x\cos\phi + y\sin\phi] \cdot e^{-2\pi i \rho x'}\, dx'\} \cdot dx\, dy\,. \qquad (3)$$

Central Slice Theorem

$$F(\rho, \phi) = \int_y \int_x f(x,y) \cdot \{\int_{x'} \delta[x' - (x \cos\phi + y \sin\phi] \cdot e^{-2\pi i \rho x'} dx' \} \cdot dx\, dy . \quad (3)$$

At this point, we can change the order of the integration operators as

$$F(\rho, \phi) = \int_{x'} [\int_y \int_x f(x,y)\, \delta(x \cos\phi + y \sin\phi - x')\, dx\, dy]\, e^{-2\pi i \rho x'} dx' . \quad (4)$$

Recall that the Radon transform is defined as,

$$p(\phi, x') = \int_x \int_y [f(x,y) \cdot \delta(x \cos\phi + y \sin\phi - x')] dx\, dy . \quad (5)$$

We take the 1-D Fourier transform of $p(\phi, x')$ as

$$F\{p(\phi, x')\} = \int_{x'} [\int_y \int_x f(x,y)\, \delta(x \cos\phi + y \sin\phi - x')\, dx\, dy]\, e^{-2\pi i \rho x'} dx' , \quad (6)$$

which is exactly what we wrote for $F(\rho, \phi)$ above!

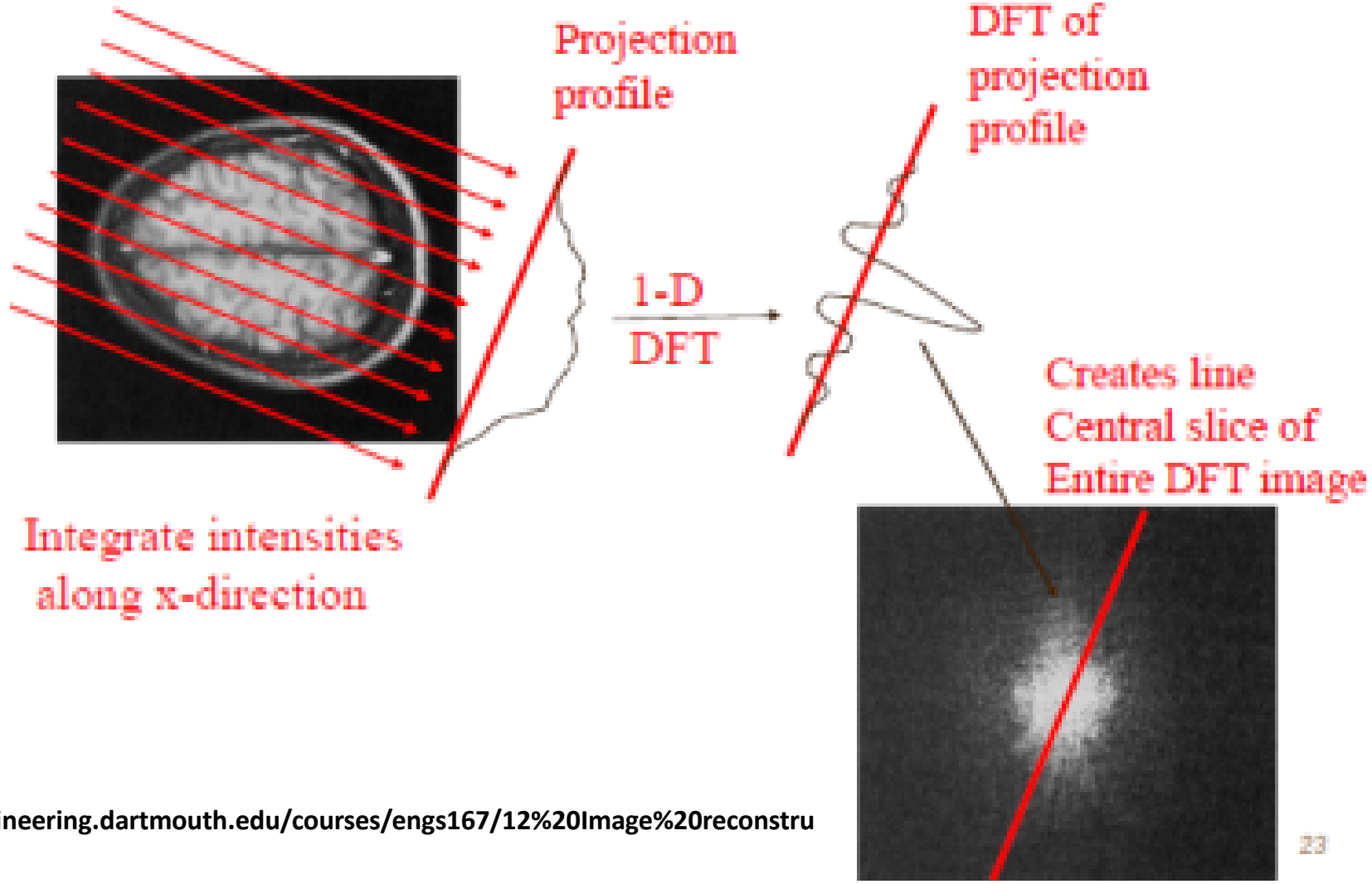
Therefore, we have proved that

$$\mathcal{F}_{1d,x'}[p(\phi, x')] = F(\rho, \phi).$$

$F(\rho, \phi)$ is defined in Eq. (2).

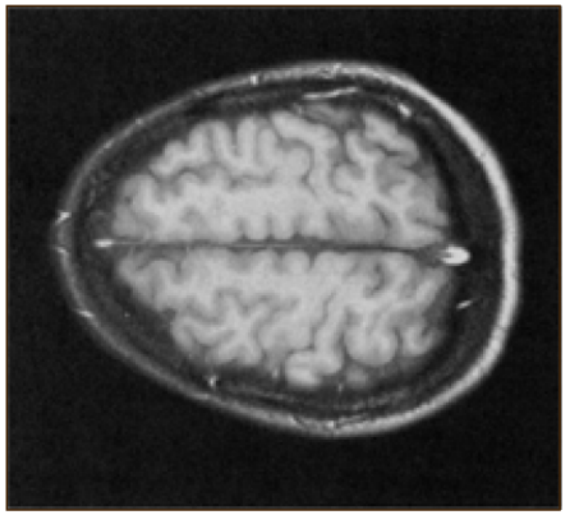
Central Slice Theorem

$$\mathcal{F}_{1d,x'}[p(\phi, x')] = F(\rho, \phi)$$



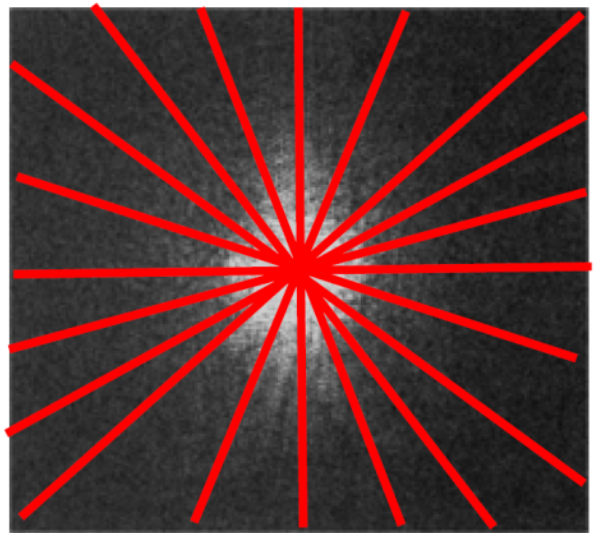
http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf

Central Slice Theorem



DFT image represents integration of original projections DFT transformed and summed together.

1-D DFTs at each projection



This is the fast way to create the DFT image from projection data. The more projections taken, the more complete the sampling.

http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf

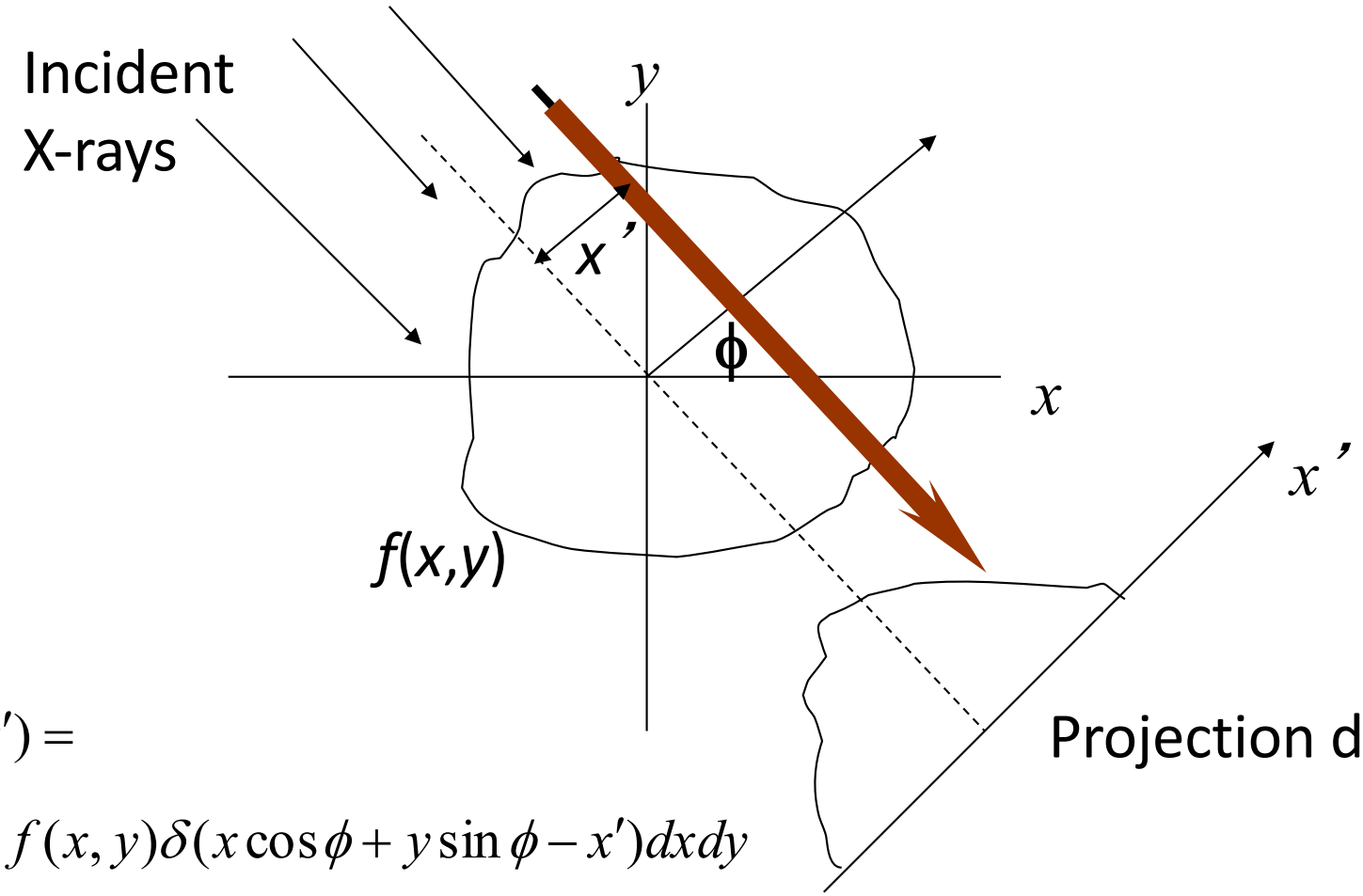
Central Section Or Projection Slice Theorem

$$\mathcal{F}_{1d,x'}[p(\phi, x')] = F(\rho, \phi).$$

So, in words, the Fourier transform of a projection at an angle ϕ gives us a line in the polar Fourier space at the same angle ϕ.

The central slice theorem is the key to understanding reconstructions from projection data.

Radon Transform and Projection Data (Revisited)

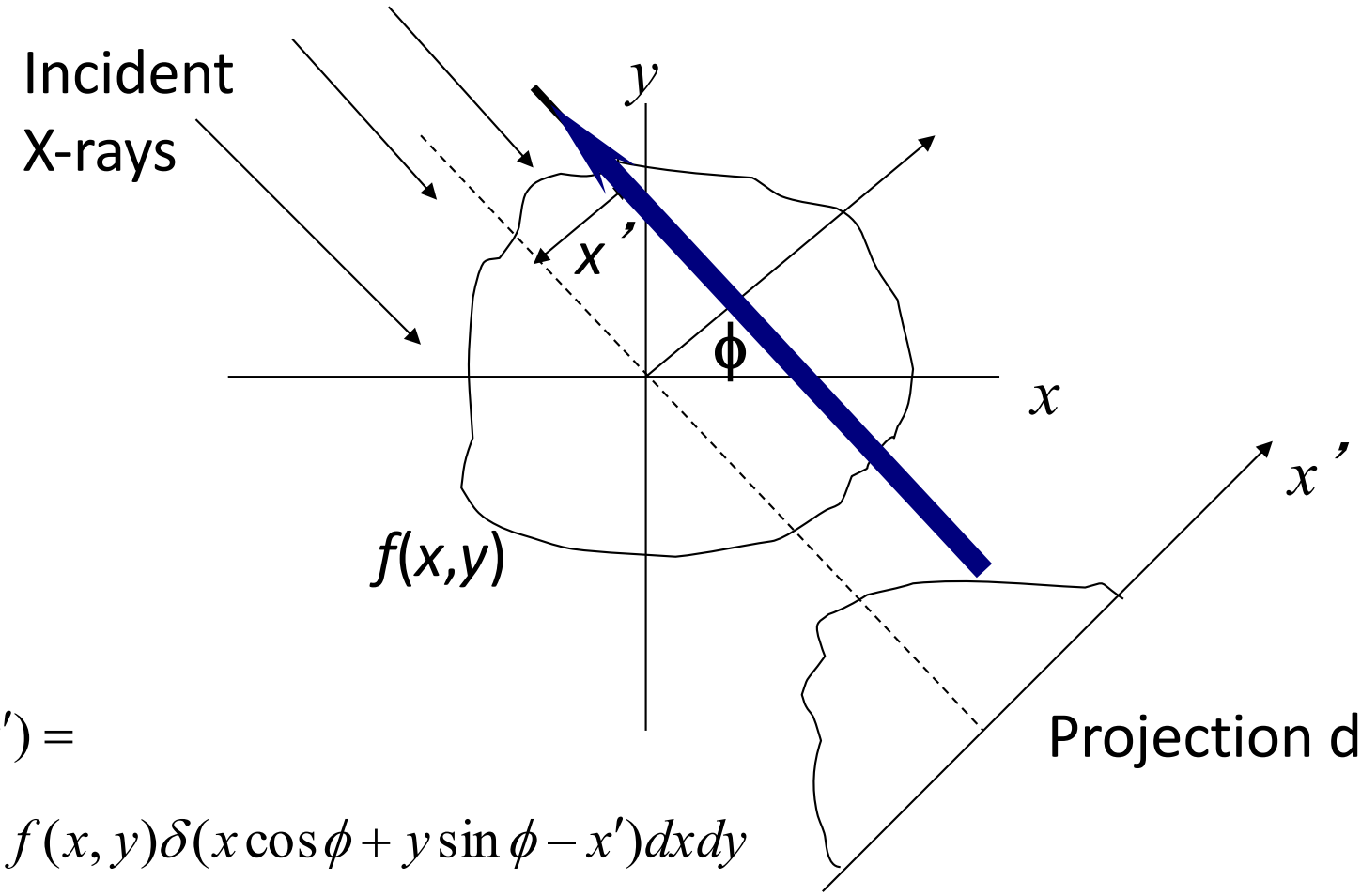


$$p(\phi, x') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\phi + y\sin\phi - x')dxdy$$



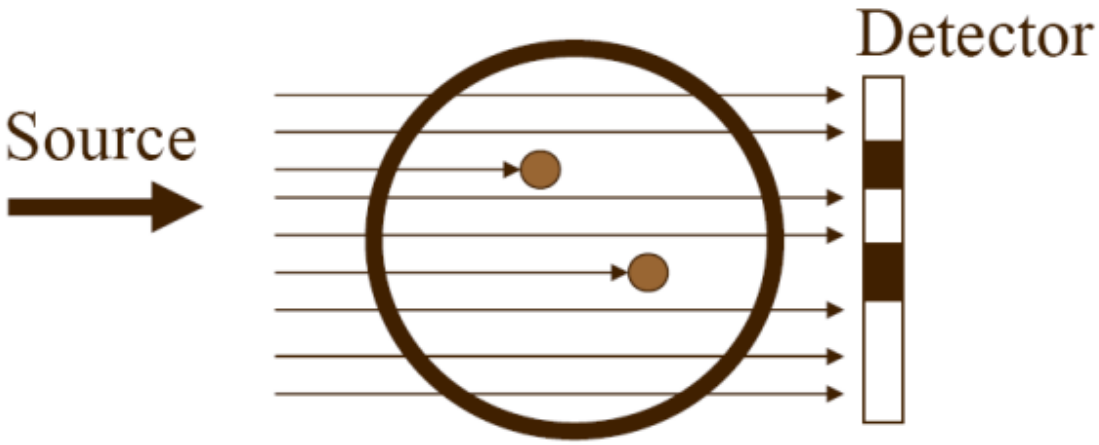
Simple Backprojection
as a Way to Restore the Original Image?

Simple Backprojection



$$p(\phi, x') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\phi + y\sin\phi - x')dxdy$$

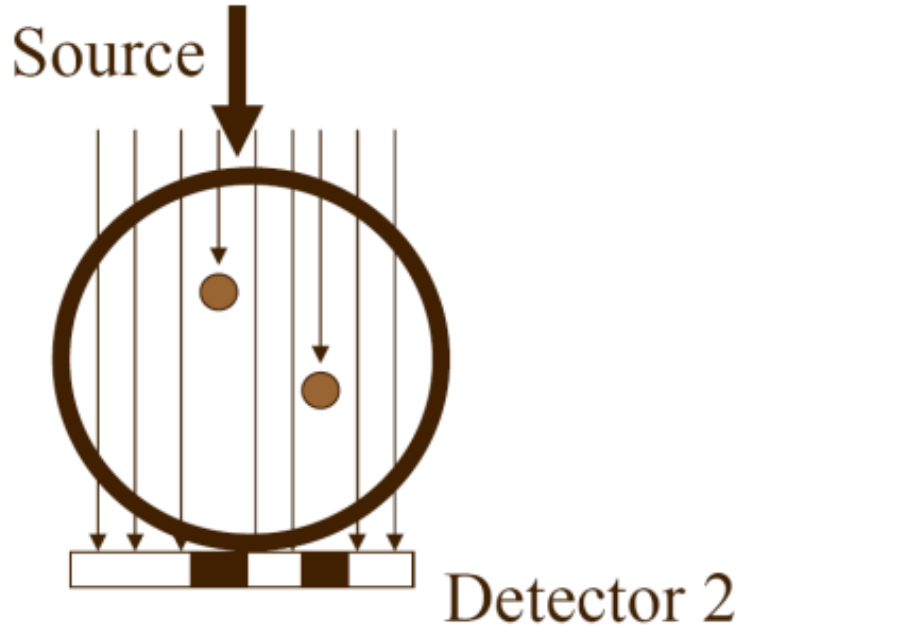
Simple Backprojection

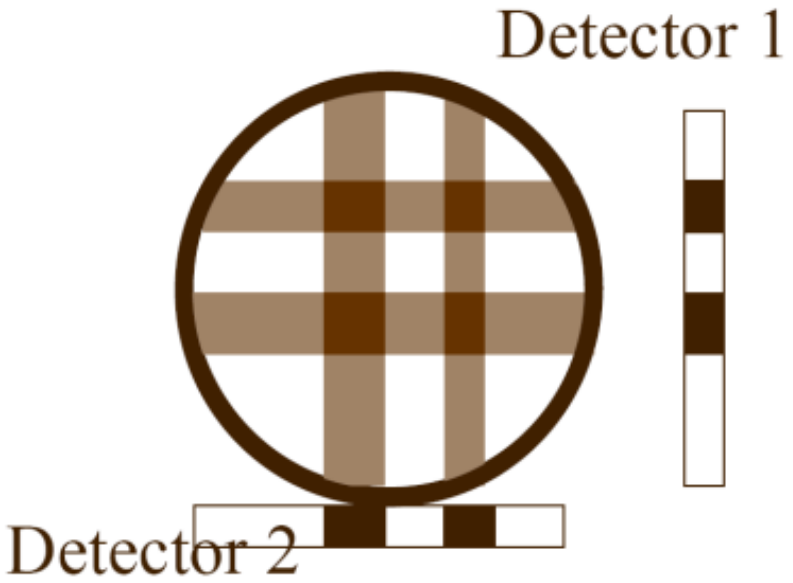


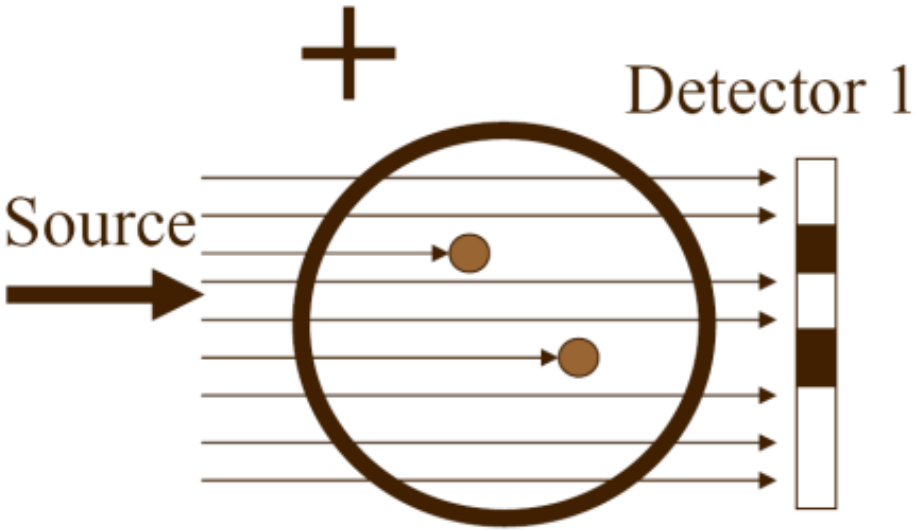
inverse = 1 back projection



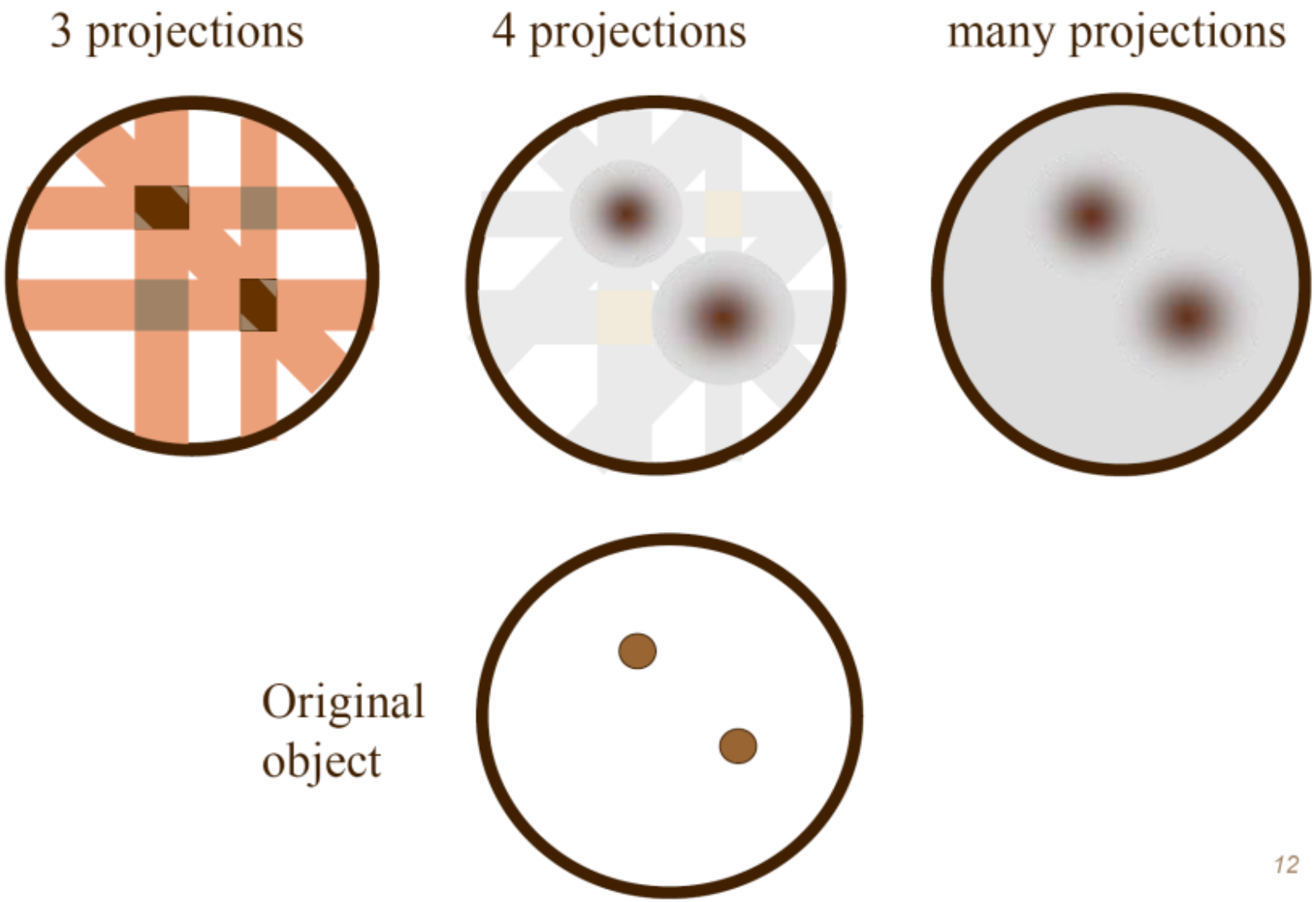
Simple Backprojection







Simple Backprojection



Simple Backprojection

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

$$b_\phi(x,y) = \int p_\phi(x') \, \delta(x \cos\phi + y \sin\phi - x') \, dx'$$

Adding up all the back projections from all the angles gives,

$$f_{\text{simple back-projection}}(x,y) = \int b_\phi(x,y) \, d\phi$$

$$f_{\text{simple back-projection}}(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} p_\phi(x') \, \delta(x \cos\phi + y \sin\phi - x') \, dx'$$

Simple Back-projection and the 1/r Blurring

MP
BE

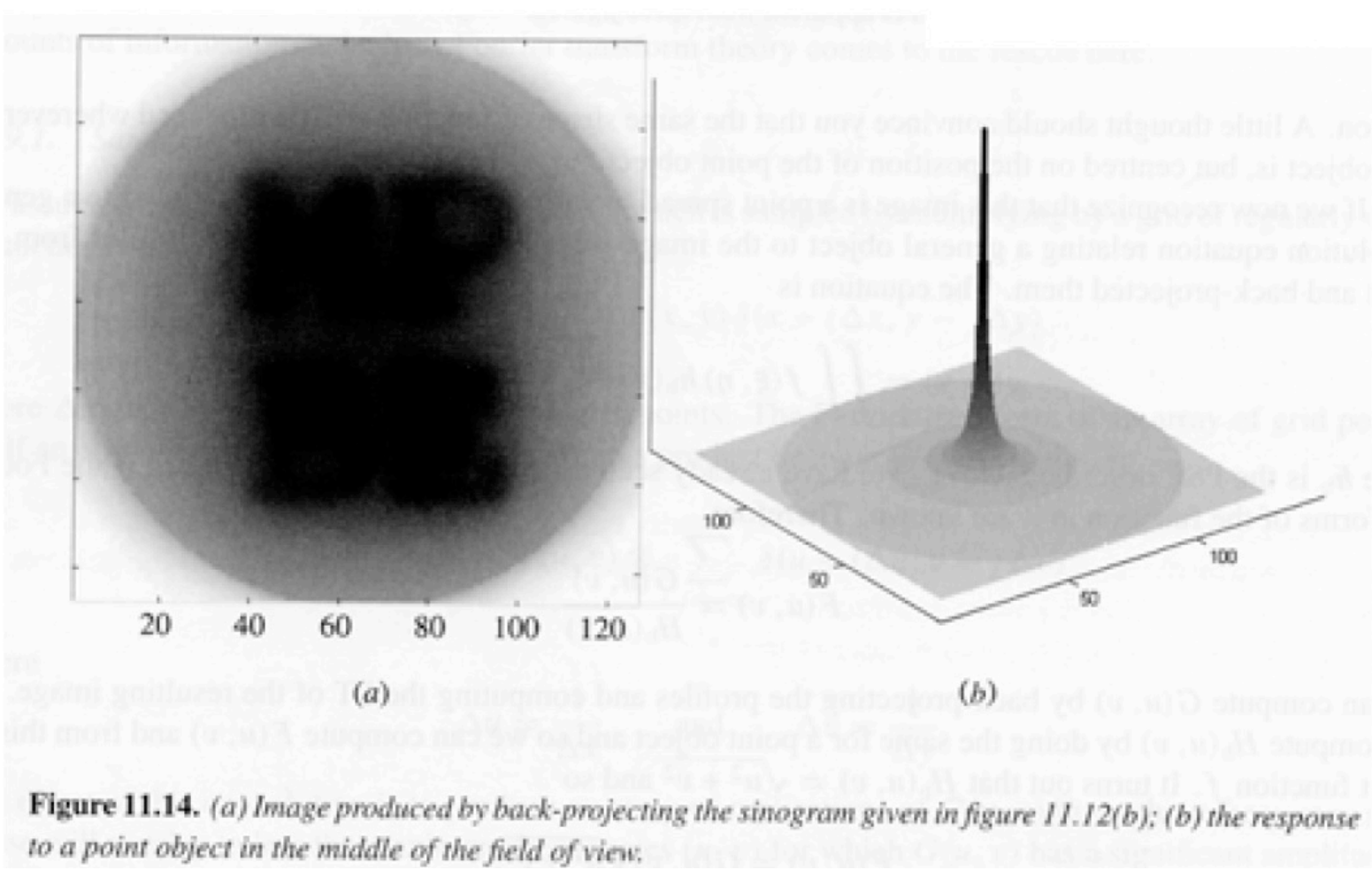


Figure 11.14. *(a) Image produced by back-projecting the sinogram given in figure 11.12(b); (b) the response to a point object in the middle of the field of view.*

From Medical Physics and Biomedical Engineering, Brown, IoP Publishing

Impulse Response Function of the Simple Backprojection Operator

$$h_b(r) = 1/r$$

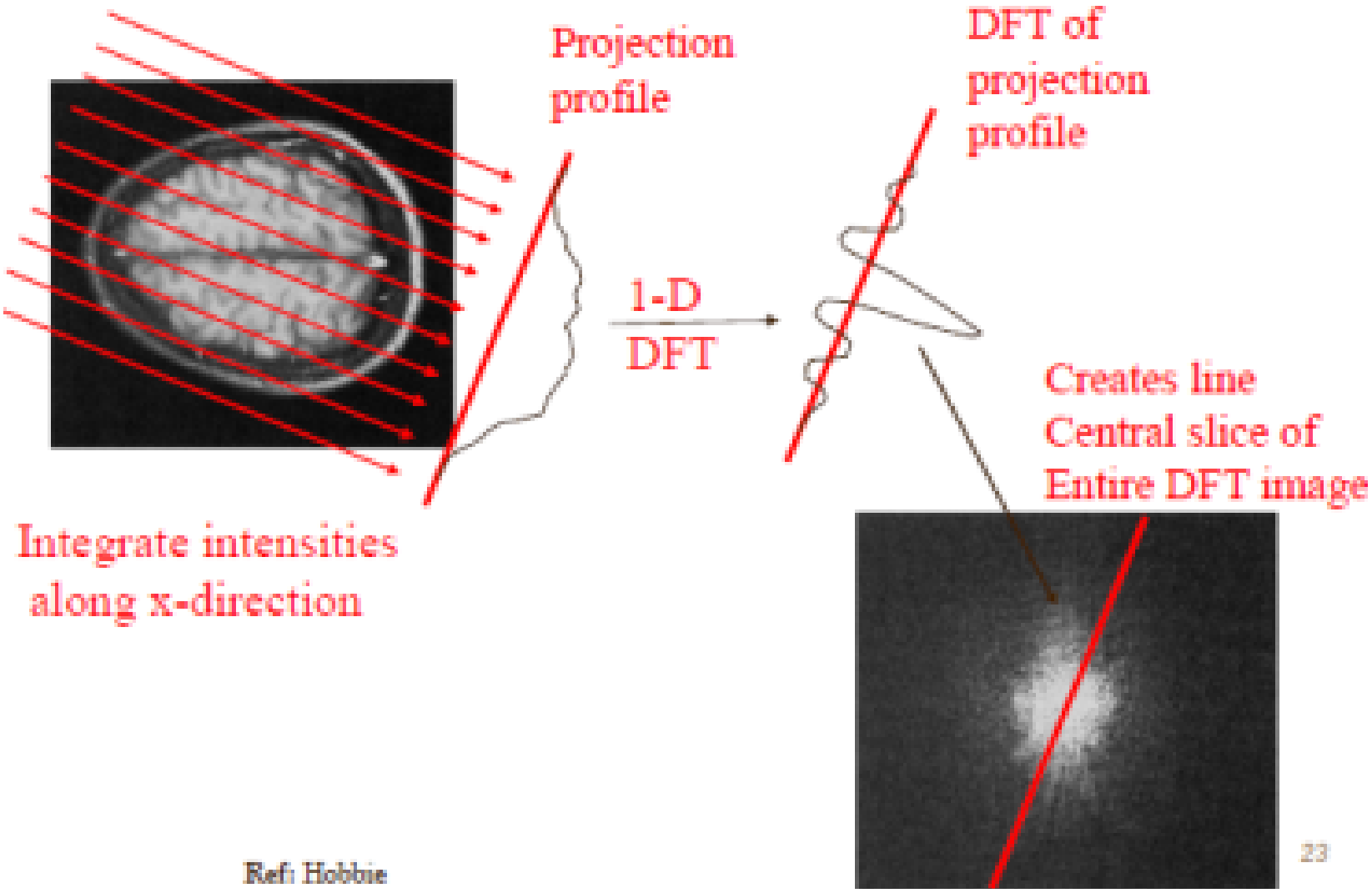
$$f_b(x,y) = f(x,y) * 1/r$$

$$F_b(\rho,\phi) = F(\rho, \phi) / \rho, \qquad \text{since } F\{1/r\} = 1/\rho$$

Backprojected image is blurred by convolution with $1/r$.

Central Slice Theorem

$$\mathcal{F}_{1d,x'}[p(\phi, x')] = F(\rho, \phi)$$



http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf

Simple Back-projection and the 1/r Blurring

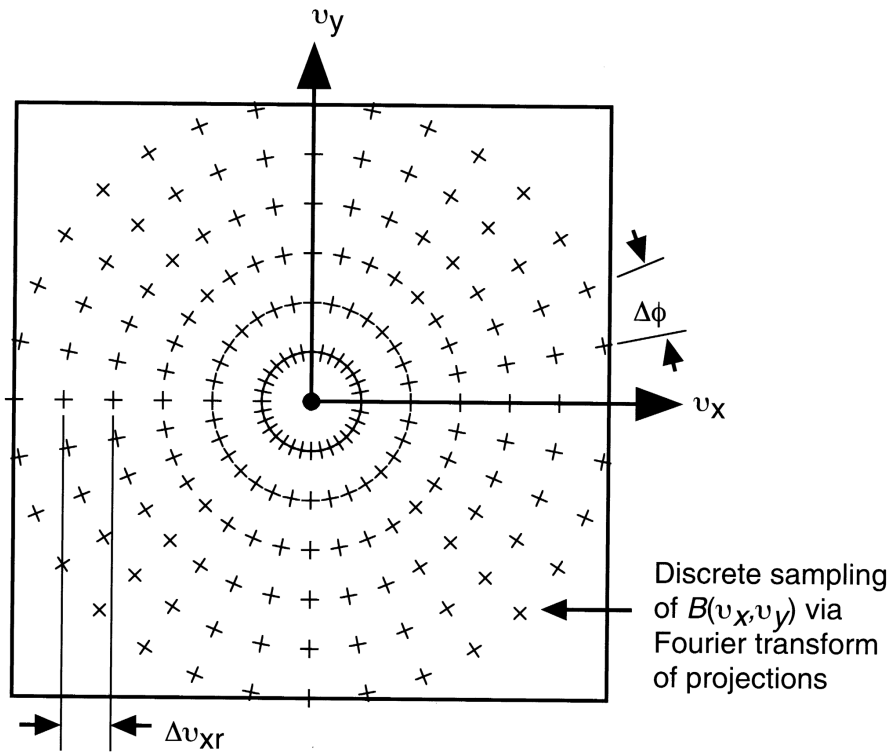


FIGURE 18 The discrete sampling pattern of $F(\upsilon_x,\upsilon_y)$ contained in $B(\upsilon_x,\upsilon_y)$, resulting from the use of discretely sampled projections.

The nature of the 1/r blurring:

Radon transform produces equally spaced radial sampling in the Fourier domain.

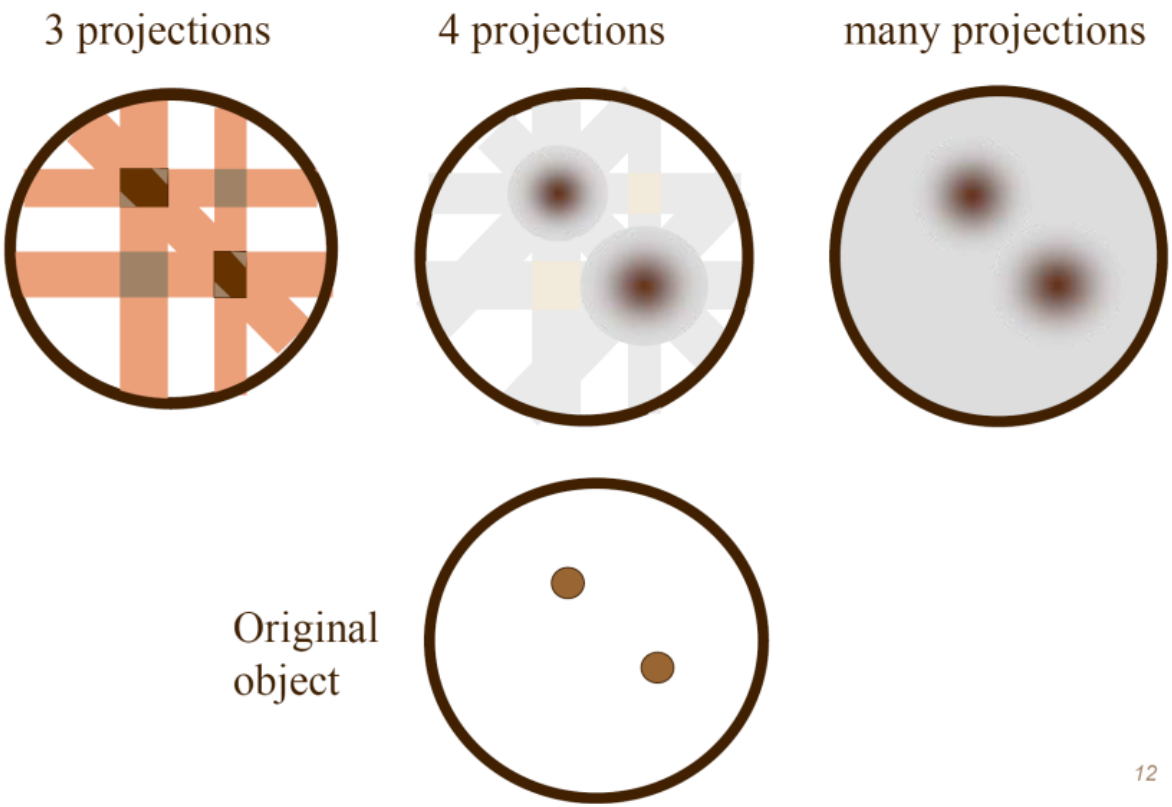


2-D Inverse Radon Transform

Simple Backprojection (Revisited)

Reconstruction with simple backprojection

$$\hat{f}_{\text{simple backprojection}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} p_\phi(x') \cdot \delta(x\cos\phi + y\sin\phi - x')dx'$$



From Simple Backprojection to Inverse Radon Transform

Simple backprojection

Result from a back projection from a single angle:

$$b_\phi(x,y) = \int p_\phi(x')\,\delta(x\cos\phi + y\sin\phi - x')\,dx'$$

Adding up all the back projections from all the angles gives,

$$\hat{f}_{\text{simple backprojection}}(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} p_\phi(x')\cdot\delta(x\cos\phi + y\sin\phi - x')dx'$$

Inverse Radon transform

$$\hat{f}_{\text{Inverse Radon transform}}(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} F^{-1}\{F_{1d,x'}[p_\phi(x')]\cdot|w|\}\cdot\delta(x\cos\phi + y\sin\phi - x')dx'$$

Inverse Radon Transform

Radon transform

$$\hat{f}_{\text{simple backprojection}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} p_\phi(x') \cdot \delta(x\cos\phi + y\sin\phi - x')dx'$$

Inverse Radon transform

$$\hat{f}_{\text{inverse Radon transform}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x')] \cdot |\omega|\} \cdot \delta(x\cos\phi + y\sin\phi - x')dx'$$

$$\hat{f}(x,y) = \hat{f}(r,\theta) = \beta\mathcal{H}\{p_\phi(x')\}$$

The simple back-projection operator

$$= \beta\mathcal{H}\{\mathcal{R}[f(x,y)]\}$$

$$= \mathcal{R}^{-1}\{\mathcal{R}[f(x,y)]\}$$

where

$$\mathcal{H}\{p_\phi(x')\} = \mathcal{F}_1^{-1}[|\omega|] * p_\phi(x')$$

The inverse Radon transform can be represented as a filtering process followed by a back-projection operation.

Is there something missing in this discussion?, such as

Noise in data?

Possible artifacts?

Filtered Back-projection – The Optimum Filter

Have we forgot something? What about the noise in the projection data?

Where is the noise coming from?

In reality, the true projection is

$$\dot{p}_\phi(x') = p_\phi(x') + n_\phi(x')$$

Review of Fourier Transform and Filtering Spectral Filtering

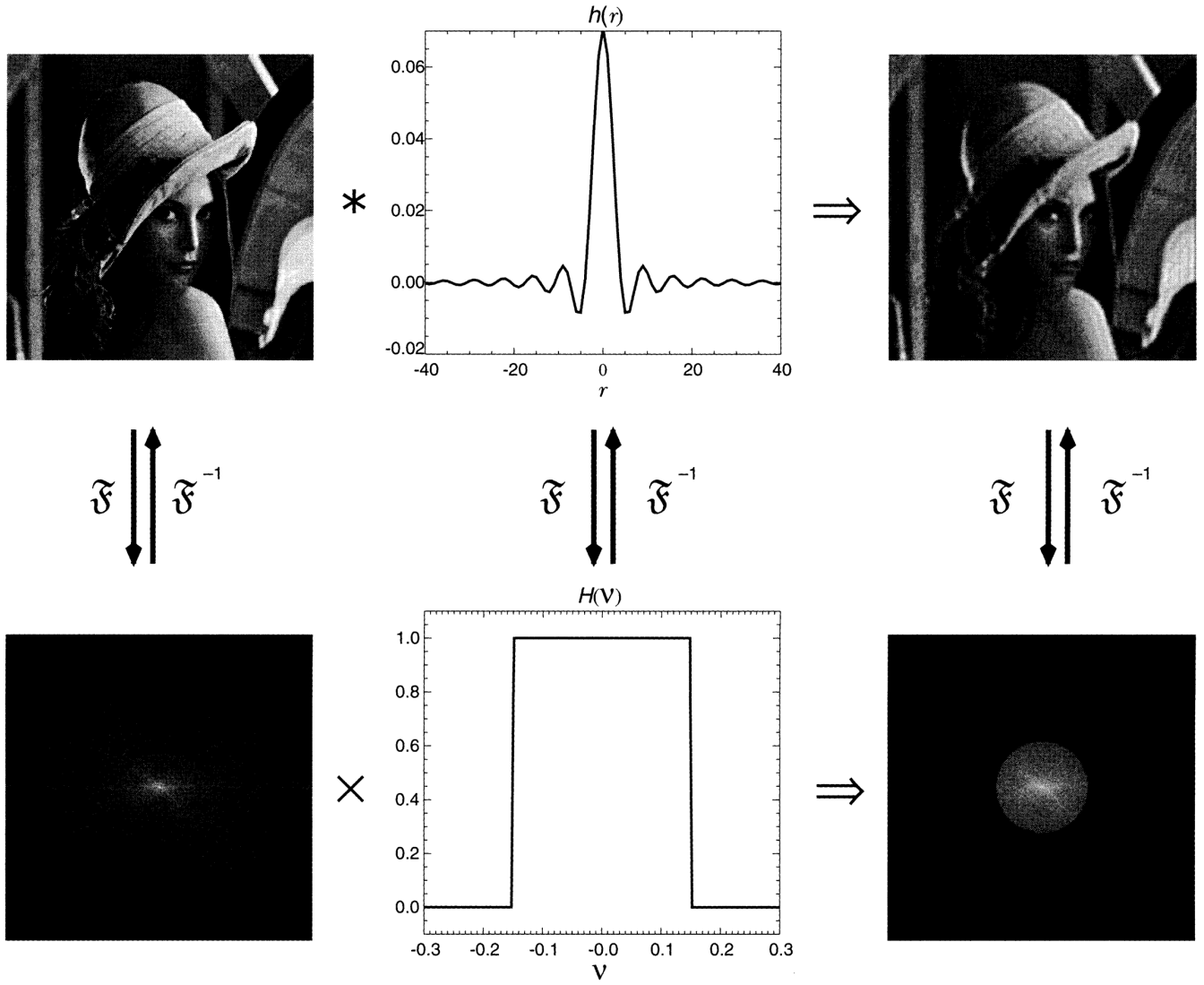


FIGURE 3 Ideal lowpass filtering of an image. Figures in the top and bottom rows demonstrate the filtering operation in the spatial and frequency domains, respectively. The left column shows the input image and the magnitude of its Fourier transform. The middle column are the circular symmetric PSF and its transfer function, where $\nu = \sqrt{\nu_x^2 + \nu_y^2}$ and $r = \sqrt{x^2 + y^2}$. Images in the right column are the output image and the magnitude of its Fourier transform.

Review of Fourier Transform and Filtering
Spectral Filtering

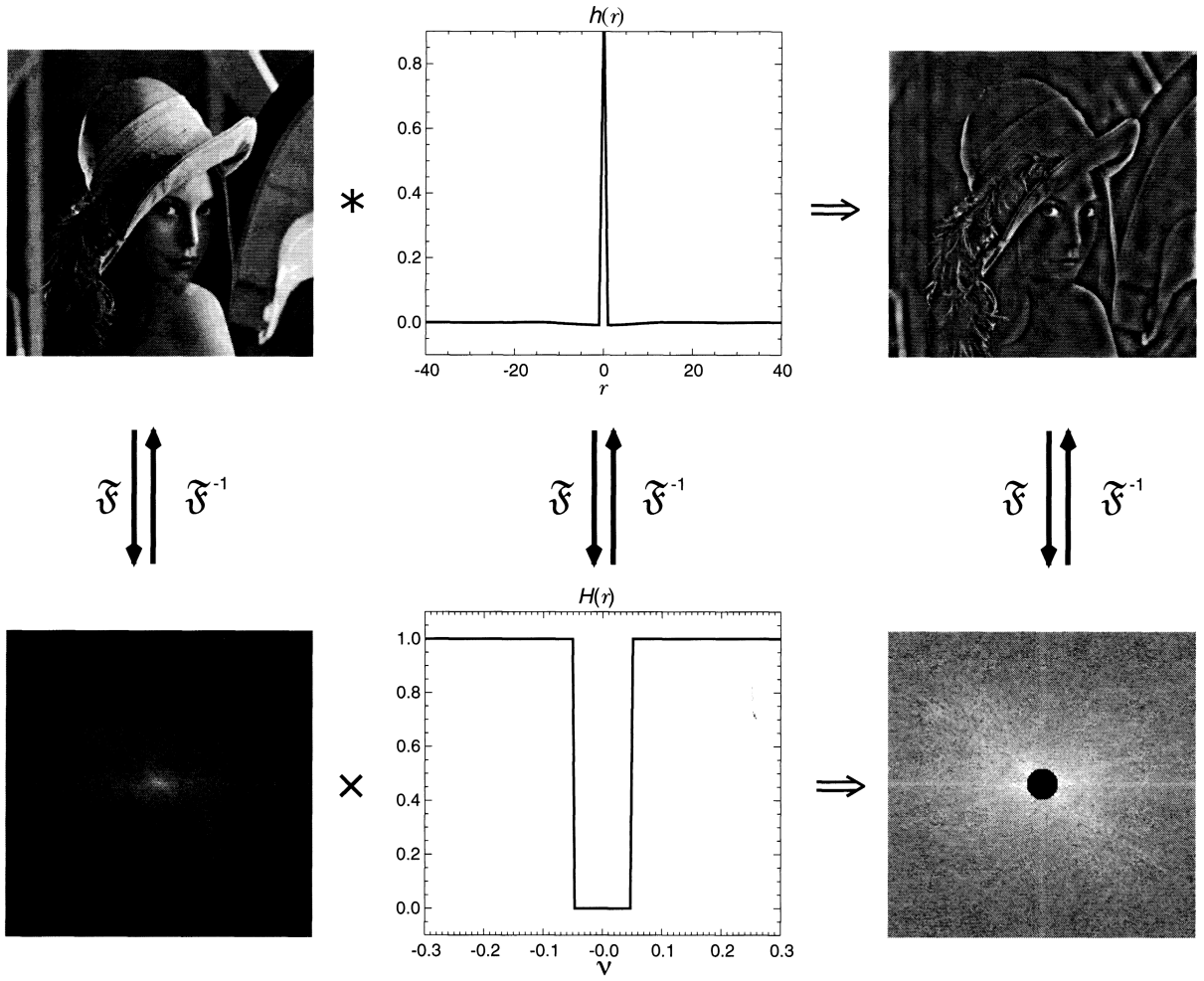
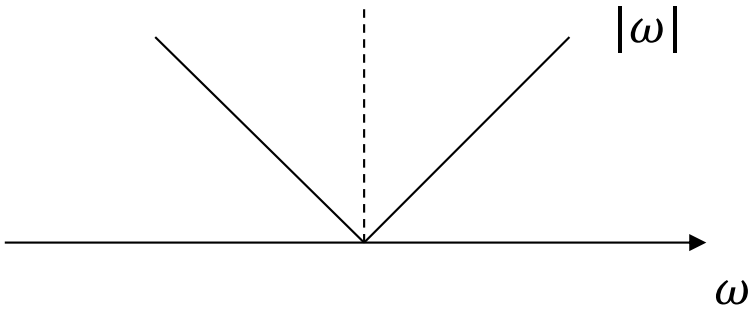


FIGURE 4 Ideal highpass filtering of an image. Figures in the top and bottom rows demonstrate the filtering operation in the spatial and frequency domains, respectively. The left column shows the input image and the magnitude of its Fourier transform. The middle column shows the circular symmetric PSF and its transfer function, where $\nu = \sqrt{\nu_x^2 + \nu_y^2}$ and $r = \sqrt{x^2 + y^2}$. Images in the right column are the output image and the magnitude of its Fourier transform.

Filtered Back-projection

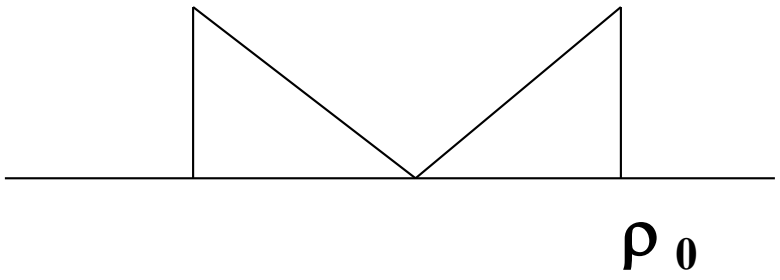
Ideal filter



The Ram-Lak filter

$$H_{\mathrm{RL}}(\omega) = \begin{cases} |\omega|, & (|\omega| \le 2\pi B) \\ 0, & (\text{otherwise}) \end{cases}$$

Ram-Lak filter



From Inverse Radon Transform to Filtered Backprojection (FBP)

Simple backprojection

$$\hat{f}_{\text{simple backprojection}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} p_\phi(x') \cdot \delta(x\cos\phi + y\sin\phi - x')dx'$$

Inverse Radon transform

$$\hat{f}_{\text{inverse Radon transform}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x')] \cdot |\omega|\} \cdot \delta(x\cos\phi + y\sin\phi - x')dx'$$

Filtered Backprojection (FBP)

$$\hat{f}_{\text{FBP}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x')] \cdot |\omega| \cdot H'(\omega)\} \cdot \delta(x\cos\phi + y\sin\phi - x')dx'$$

or

$$\hat{f}_{\text{FBP}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x')] \cdot H(\omega)\} \cdot \delta(x\cos\phi + y\sin\phi - x')dx'$$

From Inverse Radon Transform to Filtered Backprojection (FBP)

Filtered Backprojection (FBP)

$$\hat{f}_{\mathrm{FBP}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x')] \cdot |\omega| \cdot H'(\omega)\} \cdot \delta(x\cos\phi + y\sin\phi - x')dx' ,$$

or

$$H(\omega) = |\omega| \cdot H'(\omega)$$

$$\hat{f}_{\mathrm{FBP}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x')] \cdot H(\omega)\} \cdot \delta(x\cos\phi + y\sin\phi - x')dx'$$
$$= \int_0^{\pi} d\phi \int_{-\infty}^{\infty} p_\phi^*(x') \cdot \delta(x\cos\phi + y\sin\phi - x')dx' ,$$

where the **filtered projection**, $p_\phi^*(x')$, is defined as

$$p_\phi^*(x') \equiv F^{-1}\{F[p_\phi(x')] \cdot H(\omega)\} \equiv p_\phi(x') * h(x')$$

and $H(\omega) \equiv F[h(x')]$ is the filter defined in spatial frequency domain.

Filtered Backprojection (FBP) Algorithm (revisited)

The Filtered-backprojection (FBP) operator:

$$\hat{\boldsymbol{f}}_{\text{Filtered backprojection}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x')] \cdot H(\omega)\} \cdot \delta(x\cos\phi + y\sin\phi - x')dx',$$

where $H(\omega) = |\omega| \cdot H'(\omega)$.

The Filtered-backprojection (FBP) operator can also be expressed within the spatial domain as

$$\hat{\boldsymbol{f}}_{\text{Filtered backprojection}}(x,y) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} p_\phi(x') \cdot h(x\cos\phi + y\sin\phi - x')dx' ,$$

where $h(x\cos\phi + y\sin\phi - x') = F^{-1}[H(\omega)]$.

Simple and Filtered Back-projection

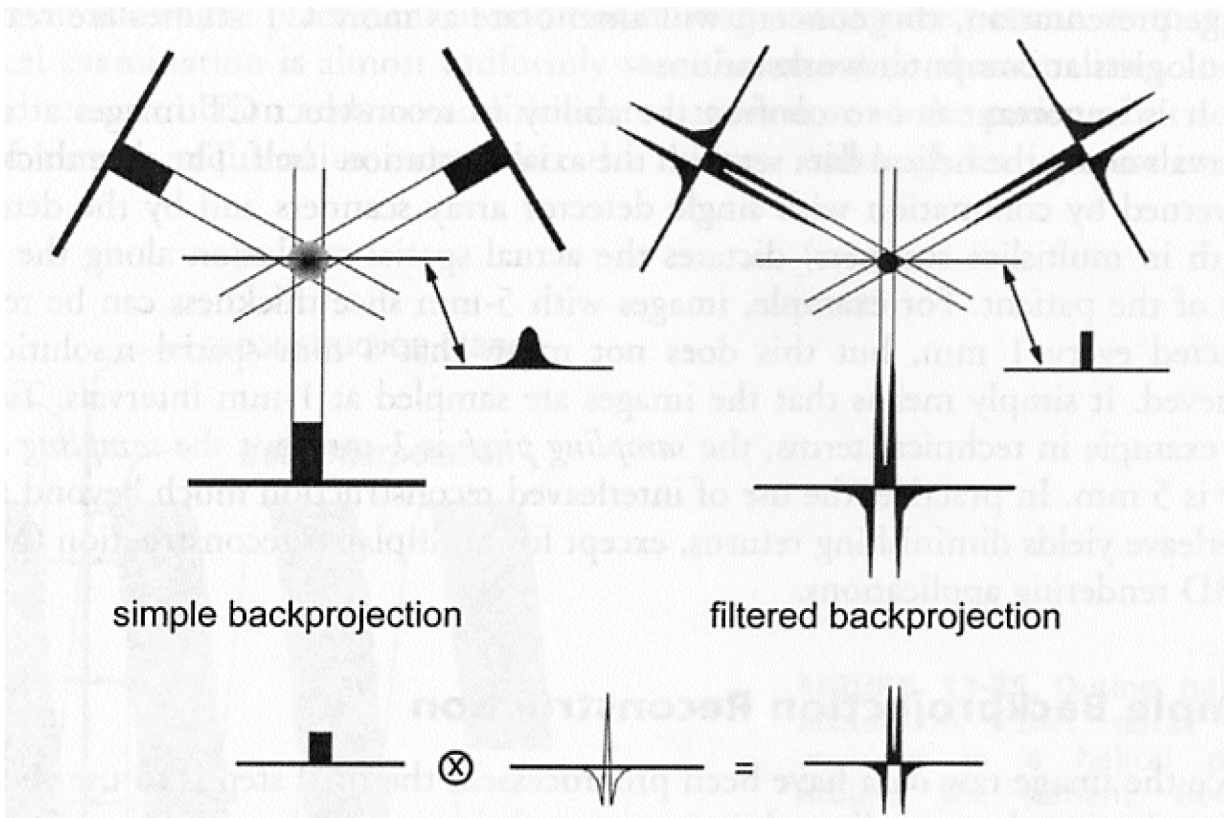
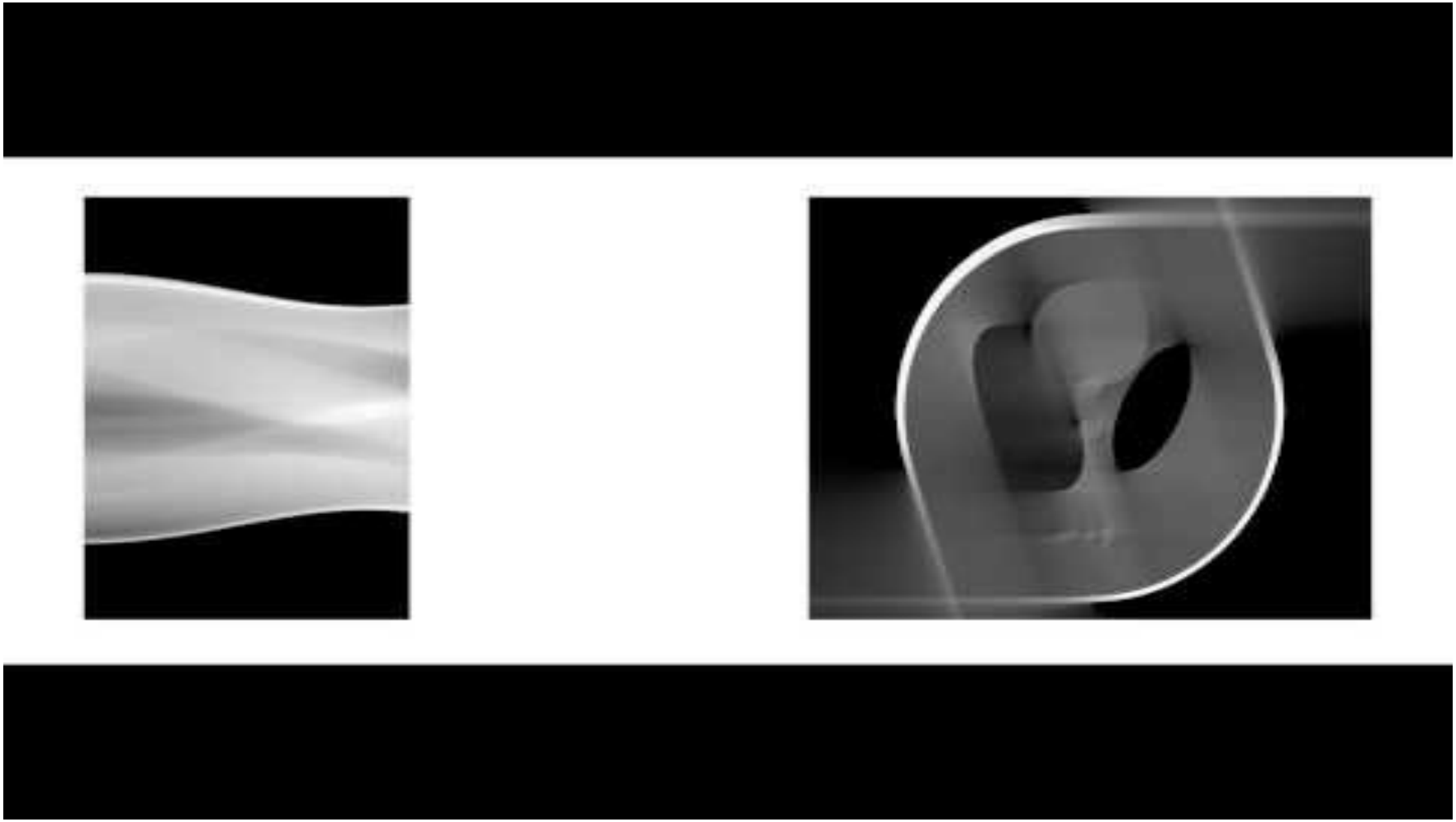


FIGURE 13-28. Simple backprojection is shown on the left; only three views are illustrated, but many views are actually used in computed tomography. A profile through the circular object, derived from simple backprojection, shows a characteristic 1/r blurring. With filtered backprojection, the raw projection data are convolved with a convolution kernel and the resulting projection data are used in the backprojection process. When this approach is used, the profile through the circular object demonstrates the crisp edges of the cylinder, which accurately reflects the object being scanned.

Chapters 12 & 13, The Essential Physics of Medical Imaging, Bushberg

Filtered Back-projection



https://www.youtube.com/watch?v=ddZeLNh9aac

Filtered Back-projection

The sharp boundary of the Ram-Lak filter often makes the spatial domain filter kernel oscillatory.

It sometimes introduces the "ringing" artifacts in reconstructed images.

This can be effectively resolved by the Shepp and Logan filter

$$H_{SL}(\omega) = \begin{cases} |\omega| \sin\left(\dfrac{\omega}{4B}\right), & |\omega| < 2\pi B \\ 0, & \text{otherwise} \end{cases}$$

$$h_{SL}(x) = \frac{B}{\pi^2}\left\{\frac{1-\cos 2\pi B[(1/4B)+x]}{(1/4B)+x} + \frac{1-\cos 2\pi B[(1/4B)-x]}{(1/4B)-x}\right\}$$

An Alternative View of the Filters

In the spatial frequency domain

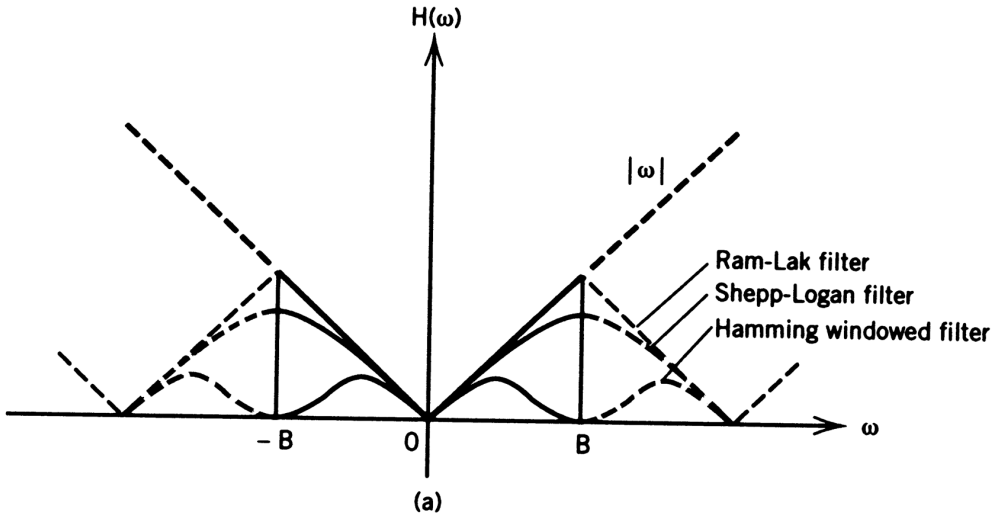


Figure 3-4 (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.

$$H(\omega)$$

In the spatial domain

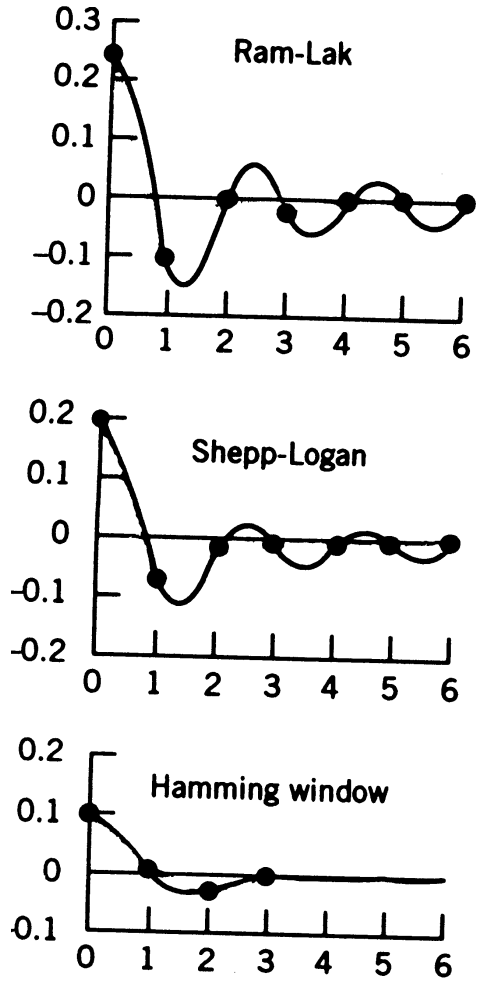


Figure 3-4 (b) Spatial domain filter kernals corresponding to the filter functions shown in the Ram-Lak filter is a high-pass filter with a sharp response but results in some noise enhancement, while the Shepp-Logan and the Hamming window filters are noise-smoothed filters and therefore have better SNR.

Chapter 2: Mathematical Preliminaries

Iterative Reconstruction Methods

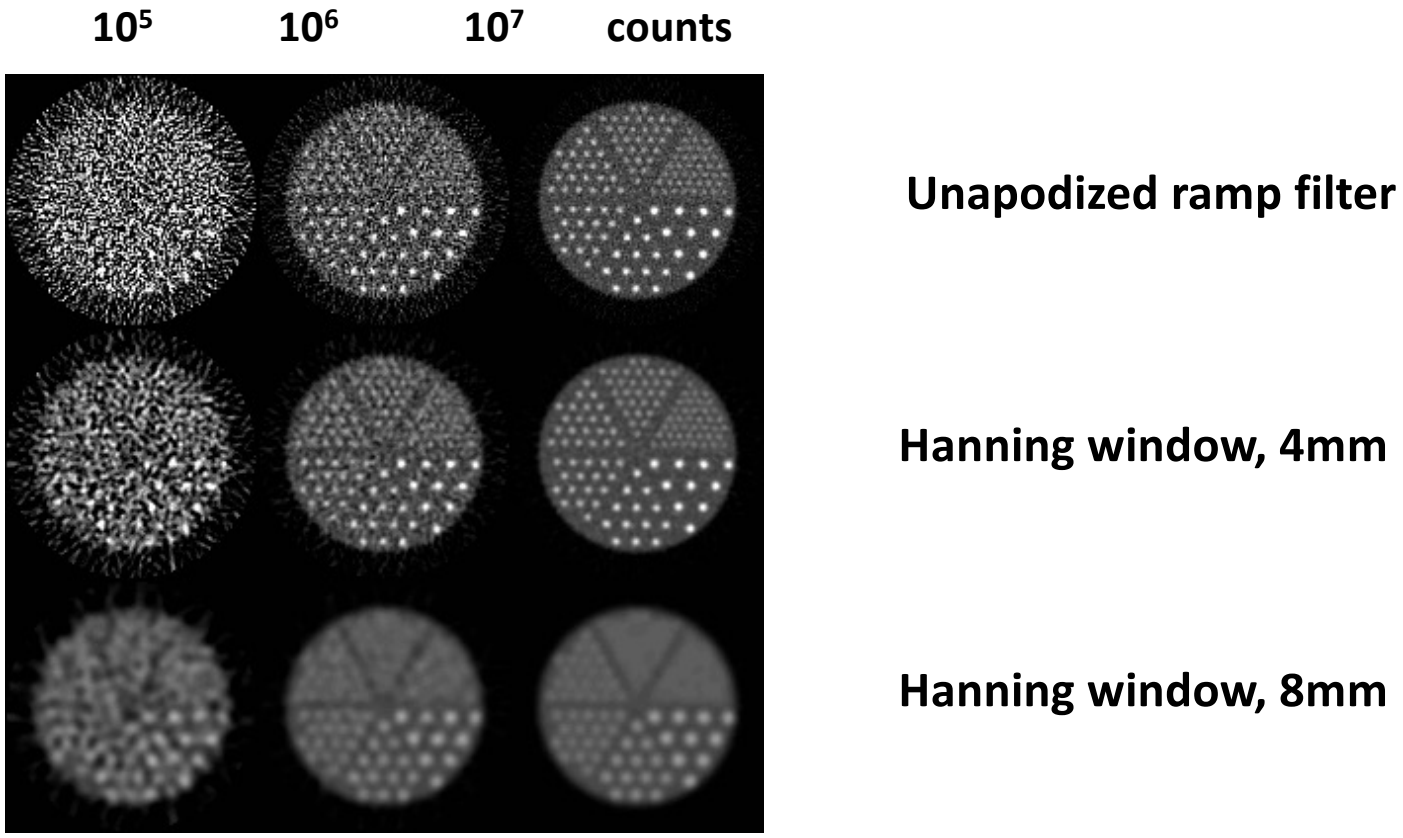
Reconstruct 2-D Images using the Maximum-Likelihood Estimation (MLE)

Learning objectives

1. The **maximum-likelihood estimate** (MLE)
   a. What is MLE?
   b. Key elements for **implementing MLE** for estimating unknown parameters
      a. Understanding the nature of the data
      b. Writing the likelihood function as an explicit function of the parameters to be estimated
      c. Having a computation strategy for finding the MLE solution.
2. MLE for PET image reconstruction? How about for X-ray CT?
3. MLE is only a starting point for image reconstruction. Why? and how to improve from MLE?

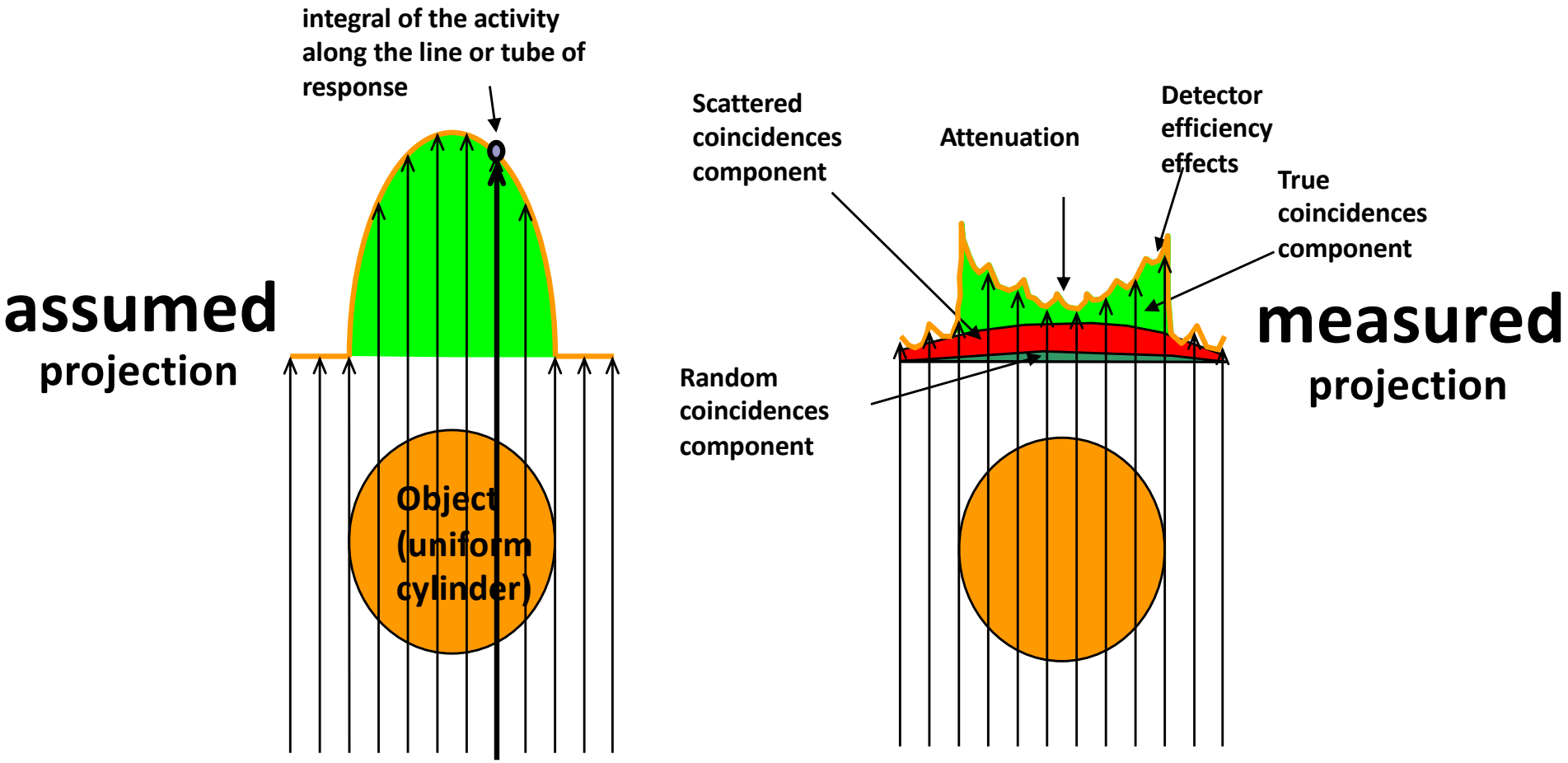
Noise In PET Images

- Noise in PET images is dominated by the counting statistics of the coincidence events detected.
- Noise can be reduced at the cost of image resolution by using an apodizing window on ramp filter in image reconstruction (FBP algorithm).



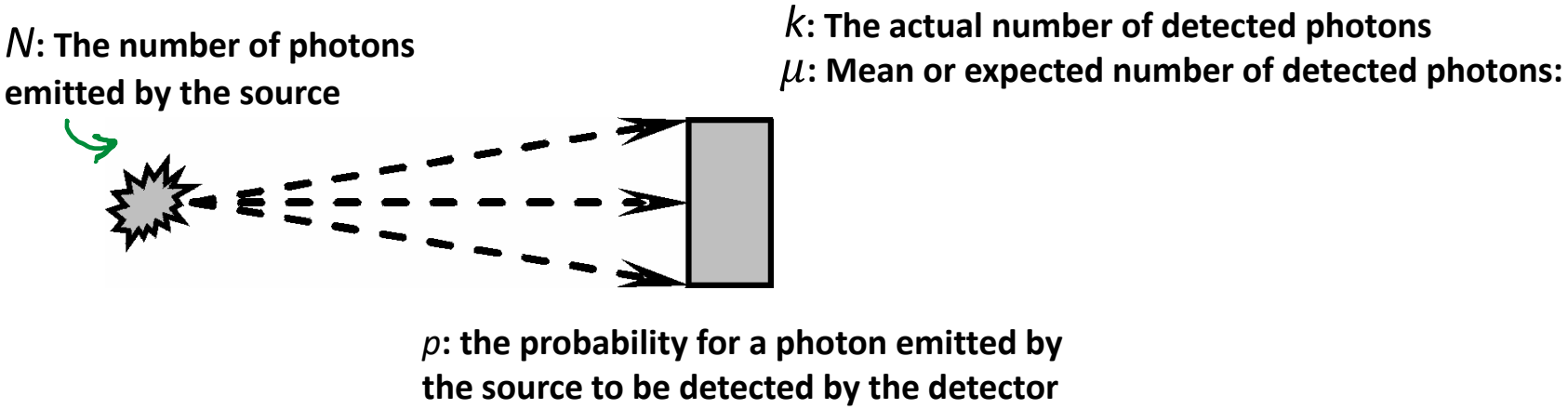
An Example of PET Image Reconstruction

- Data corrections are necessary
  - the measured projections are not the same as the projections assumed during image reconstruction



The Basic Idea of Statistical Reconstruction

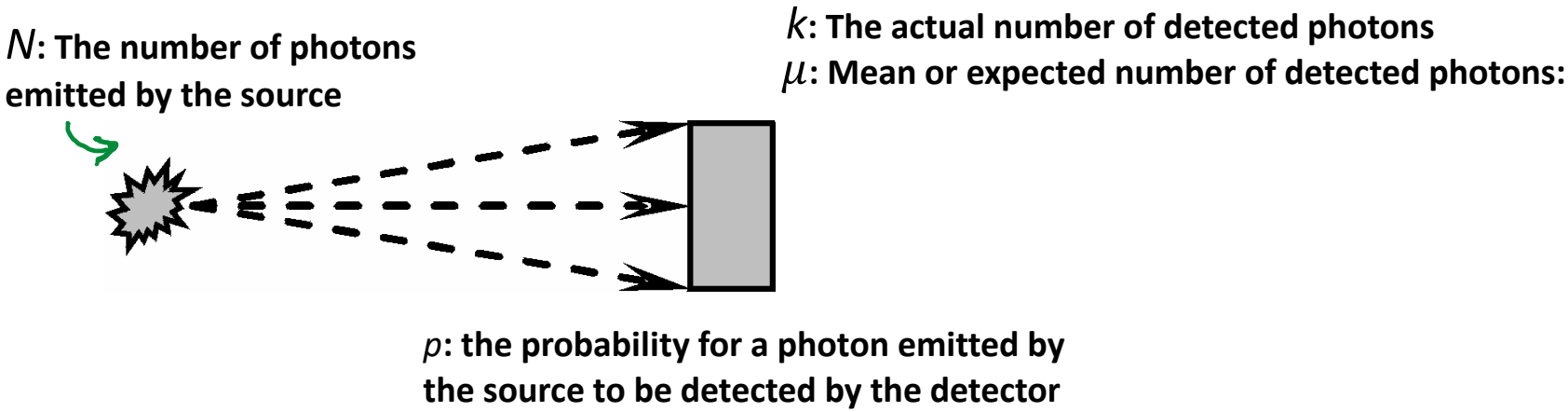
Consider the simplest imaging problem as illustrated below:



During an imaging study, we observed k counts, and we know that the probability for a photon emitted by the source to be detected by the detector is p, then

how do we estimate the number of photons being emitted by the source during the imaging period?

The Basic Idea of Statistical Reconstruction



We can estimate the underlying parameter *N* by the Maximum-Likelihood Estimation (MLE) strategy:

$$\widehat{N}_{Maximum\ Likelihood\ estimation} = \arg\max_{N}\{P[k|N]\}$$

$P[k(N)]$ is call the Likelihood function: the probability of detecting *k* counts given there were a total of *N* gamma rays emitted by the source.

The Basic Idea of Statistical Reconstruction

The likelihood function:

In very non-technical terms, the conditional probability,

$$\boldsymbol{L}(\boldsymbol{f}, \boldsymbol{g}) \equiv p(\boldsymbol{g}|\boldsymbol{f}),$$

of the underlying source function ($\boldsymbol{f}$) leading to the measurement ($\boldsymbol{g}$).

The basic idea of Maximum-Likelihood reconstruction is simple:
We choose the solution (estimated source function) that maximizes the likelihood function, or the solution that is the most likely to produce the measured data.

$$\hat{\boldsymbol{f}}_{Maximum\ Likelihood} = \arg\max_{\boldsymbol{f}}\{P[\boldsymbol{g}|\boldsymbol{f}]\}$$

Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- It consists of N trails.
- Each trail has a binary outcome: success or failure (decay or not).
- The probability of success (decay) is a constant from trail to trail – all atoms have equal probability to decay.
- The trails are independent.

In statistics, these four conditions characterize a Bernoulli process.

Binomial Distribution

The binomial distribution gives the discrete probability distribution of obtaining exactly k successes out of N Bernoulli trials (where the result of each Bernoulli trial is true with probability *p* and false with probability *1-p*). The binomial distribution is therefore given by

$$p(k|N,p) = \binom{N}{k} p^k (1-p)^{N-k}$$

where

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}$$

Binomial Distribution and Imaging Applications

Suppose we have *a point source* generated *N* gamma rays and we have a pixilated detector for detecting these photons

Each gamma ray photon has *a fixed probability p* of reaching a given detector element. *p* is defined by the relative distance between the source and the pixel and the collimation configuration used.

So the *probability of detecting k gamma rays* on the pixel is given as

$$p(k|N,p) = \binom{N}{k} p^k (1-p)^{N-k}$$

Furthermore, *the number of counts detected on different pixels is independent* …

Binomial Distribution

For a binomial distribution, the mean or the expectation of the number of disintegration in time t is given by

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

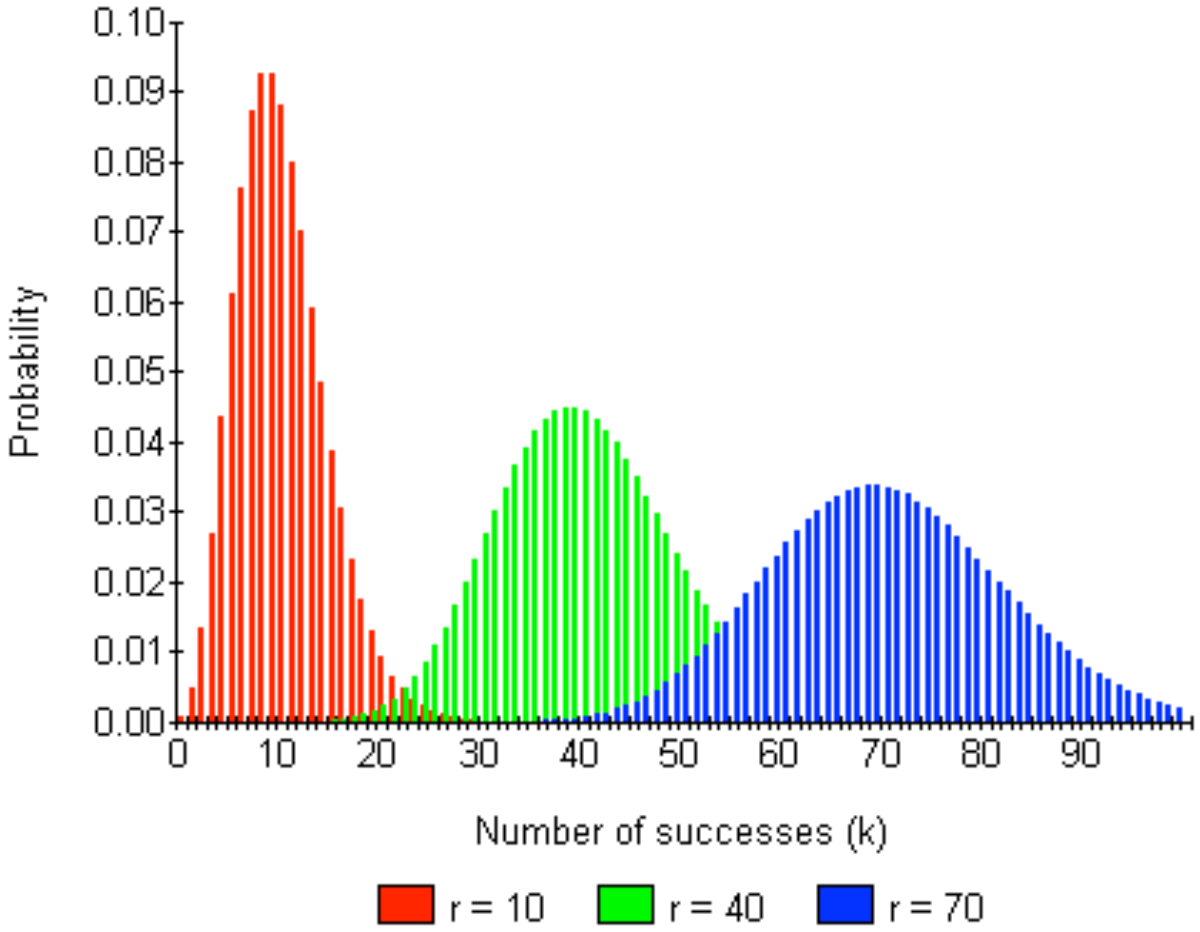
and the fluctuation on the number of disintegrations is given by the variance or the standard deviation of the

$$\sigma^2 \equiv \sum_{n=0}^{N} (n-\mu)^2 \cdot P_n = Npq$$

and

$$\sigma \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

Binomial Distribution



☞ **What are the mean and standard deviation of a Binomial distribution?**

Poisson Distribution

The counting statistics related to nuclear decay processes is often more conveniently described by the Poisson distribution, is related to situations that involves a collection of multiple trails that satisfy the following conditions:

1. The number of trials, N, is very large, e.g., $N > 1$.
2. Each trail is independent.
3. The probability that each single trial is successful is a constant and approaches zero, $p << 1$. Therefore, the number of successful trials fluctuates around a finite number.

Binomial and Poisson Random Variable

When $N \rightarrow \infty$ and $p \rightarrow 0$, k follows the Poisson distribution, whose probability function is

Binomial distribution

$$p(k|N,p) = \binom{N}{k} p^k (1-p)^{N-k} \Rightarrow p(k) = \frac{\mu^k}{k!} e^{-\mu}$$

Poisson distribution

where μ is the expected value of the Poission variable defined

$$\mu = \lim_{N \to \infty} N\,p$$

Poisson Distribution

The probability of having *n* successful trails can be approximated with the Poisson distribution.

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

and the mean and the variance of number of successful trail are given by

$$Mean(n) = \mu = N \cdot p$$

$$Std(n) \equiv \sigma = \sqrt{\mu} = \sqrt{Np}$$

Poisson Distribution

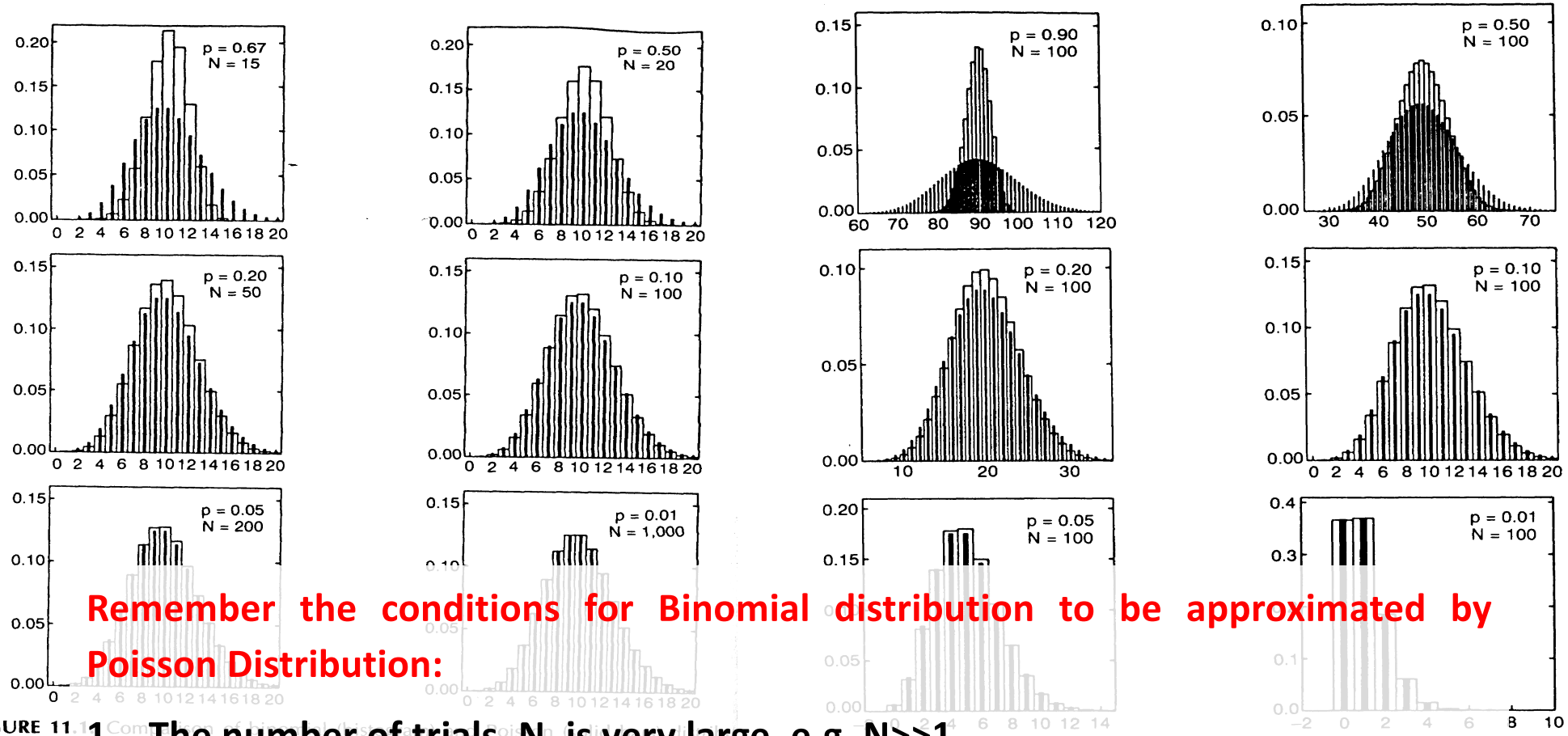


FIGURE 11. tions, having *p* and sample size *N*. The ordinate in each panel shows the probability P_n of exactly successes, shown on the abscissa. With fixed *p*, the Poisson distribution is the same throughout. (Courtesy James S. Bogard.)

FIGURE 11.2. Comparison of binomial (histogram) and Poisson (solid bars) distributions for fixed *N* and different *p*. The ordinate shows P_n and the abscissa, *n*. The mean of the two distributions in a given panel is the same. (Courtesy James S. Bogard.)

Remember the conditions for Binomial distribution to be approximated by Poisson Distribution:

1. **The number of trials, N, is very large, e.g. N>>1.**
2. **Each trial is independent.**
3. **The probability that each single trial is successful is a constant and approaching zero, p<<1. So the number of successful trials is fluctuating around a finite number.**

Image Reconstruction and General Linear Inverse Problem

An Emission Tomography System

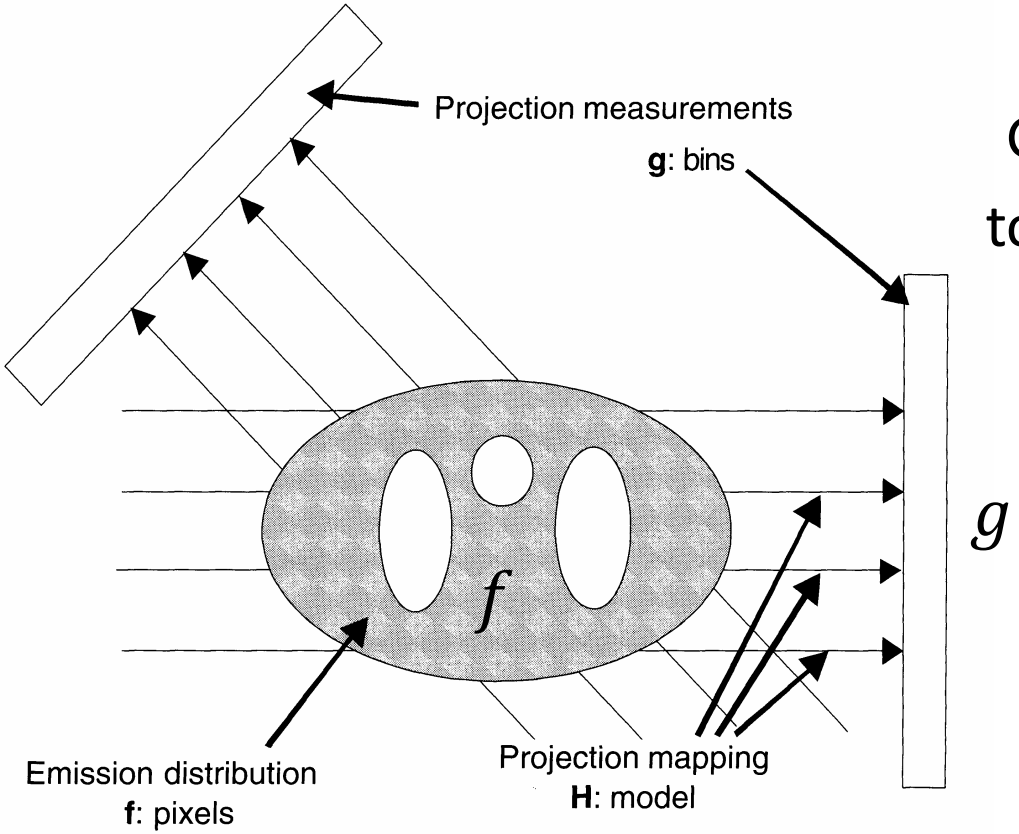


FIGURE 1 A general model of tomographic projection in which the measurements are given by weighted integrals of the emitting object distribution.

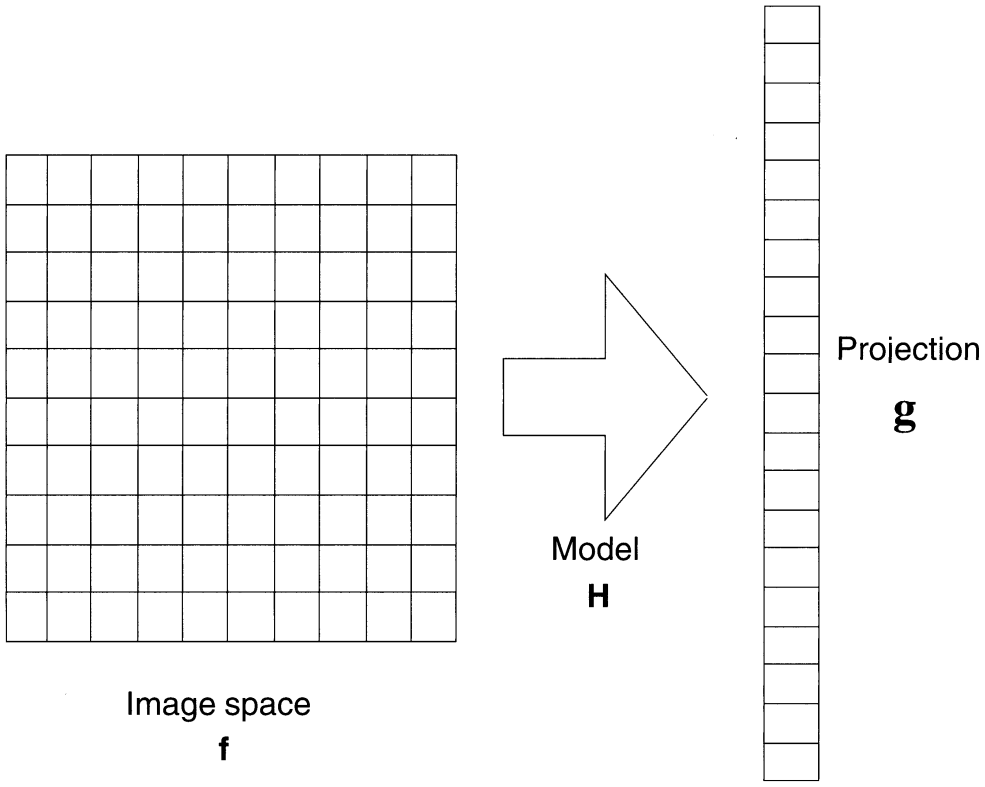
Given a measured data set g, how to formulate the ML estimator of f?

$$\hat{f}_{ML} = \arg\max P(\boldsymbol{g}|\boldsymbol{f})$$

Then how to evaluate $P(\boldsymbol{g}|\boldsymbol{f})$?

Image Reconstruction and General Linear Inverse Problem

Linear imaging systems – An alternative representation



$$\bar{\mathbf{g}} = \mathbf{Hf}$$

FIGURE 2 A discrete model of the projection process.

Poisson Statistics of the Projection Data

Considering that the measured projection data, $\boldsymbol{g}=(g_1, g_2, g_3,\ldots, g_M)$, is a collection of independent Poisson random numbers, the probability of the measured projection data $\boldsymbol{g}$ may be written as

$$p(\boldsymbol{g}|\boldsymbol{f}) = p(\boldsymbol{g}) = \prod_{m=1}^{M} p(g_m) = \prod_{m=1}^{M} \frac{\bar{g}_m{}^{g_m}}{g_m!} e^{-\bar{g}_m}$$

where $\bar{g}_m$ is the expected value for the number of counts on detector pixel no. m.

$$\bar{g}_m = \sum_{n=1}^{N} f_n p_{mn}$$

In the context of emission tomography, p_{mn} is the probability that a gamma ray generated at a source pixel n is detected by detector element m.

A Typical Imaging System Described in Matrix Form

$$\begin{pmatrix} \bar{g}_1 \\ \bar{g}_2 \\ \bar{g}_3 \\ \vdots \\ \bar{g}_M \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1N} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2N} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & p_{M3} & \cdots & p_{MN} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix}$$

Expected number of photons detected on each detector element

Everything we know about the imaging system – The system response function.

The source object.

The number of photons being emitted from each source voxel

p_{mn} : the probability for a gamma ray emitted by the n'th source voxel to be detected by the m'th detector pixel.

The Maximum Likelihood Reconstruction

$$\hat{\boldsymbol{f}}_{ML} = \underset{f}{\operatorname{argmax}}\, L(\boldsymbol{f}, \boldsymbol{g}) = \underset{f}{\operatorname{argmax}} \log[L(\boldsymbol{f}, \boldsymbol{g})]$$

$$= \underset{f}{\operatorname{argmax}} \left\{ log \left[\prod_{m=1}^{M} \left(\frac{\bar{g}_m{}^{g_m}}{g_m!} e^{-\bar{g}_m} \right) \right] \right\} = \underset{f}{\operatorname{argmax}} \left[\sum_{m=1}^{M} log \left(\frac{\bar{g}_m{}^{g_m}}{g_m!} e^{-\bar{g}_m} \right) \right]$$

Consider all detector elements working independently, So the ML reconstruction for Poisson-distributed data is

$$\hat{\boldsymbol{f}}_{\boldsymbol{ML}} = \underset{\boldsymbol{f}}{\operatorname{argmax}} l(\boldsymbol{f}, \boldsymbol{g}) = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[\sum_{m=1}^{M} (g_m \log \bar{g}_m - \bar{g}_m - \log g_m\,!) \right]$$

since g_m is not function of f, we get

$$\hat{\boldsymbol{f}}_{\boldsymbol{ML}} = \underset{\boldsymbol{f}}{\operatorname{argmax}}\ l\,(\boldsymbol{f}, \boldsymbol{g}) = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[\sum_{m=1}^{M} (g_m \log \bar{g}_m - \bar{g}_m) \right]$$

$$\bar{g}_m = \sum_{n=1}^{N} f_n p_{mn}$$

$$\hat{\boldsymbol{f}}_{\boldsymbol{ML}} = \underset{\boldsymbol{f}}{\operatorname{argmax}}\ l\,(\boldsymbol{f}, \boldsymbol{g}) = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[\sum_{m=1}^{M} \left(g_m \cdot \log \left(\sum_{n=1}^{N} f_n p_{mn} \right) - \left(\sum_{n=1}^{N} f_n p_{mn} \right) \right) \right]$$

The Maximum Likelihood Reconstruction

The ML estimate

Underlying/unknown image function to be estimated

$$\hat{\boldsymbol{f}}_{\boldsymbol{ML}} = \underset{\boldsymbol{f}}{\operatorname{argmax}}\ l(\boldsymbol{f}, \boldsymbol{g}) = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[\sum_{m=1}^{M} \left(g_m \cdot \log \left(\sum_{n=1}^{N} f_n p_{mn} \right) - \left(\sum_{n=1}^{N} f_n p_{mn} \right) \right) \right]$$

Measured data

The Maximum Likelihood Expectation Maximization (MLEM) Algorithm

The source (image) function ***f*** that has the maximum likelihood of giving rise to the observed data ***g*** can be found by the following iterative updating scheme

$$f_n^{(new)} = \frac{f_n^{(old)}}{\sum_{m=1}^{M} p_{nm}} \sum_{m=1}^{M} \frac{g_m}{\sum_{n'=1}^{N} p_{mn'} f_{n'}^{(old)}} p_{mn}\,, \qquad n = 1,2,\ldots,N$$

$where$

$m = 1,2,\ldots,M$ is the index of detector elements in the imaging system

$n = 1,2,\ldots,N$ is the index of source object pixels

$f_n^{(old)}$: the current estimate of the source function in pixel n

$f_n^{(new)}$: the updated estimate of the source pixel n

p_{nm}: the probability of a gamma ray generated in source pixel n and detected by the detector element m

g_m: the observed number of counts in detector pixel m

Shepp & Vardi, IEEE TMI, 1982.

The Maximum Likelihood Expectation Maximization (MLEM) Algorithm

$$f_n^{(new)} = \frac{f_n^{(old)}}{\sum_{m=1}^{M} p_{nm}} \sum_{m=1}^{M} \frac{g_m}{\sum_{n'=1}^{N} p_{mn'} f_{n'}^{(old)}} p_{mn}\,, \qquad n = 1,2,\ldots,N$$

Each iteration updates all elements in the source function *f* sequentially.

Each iteration is guaranteed to provide a new image function that has an *increased likelihood* compared to the old image function (unless the maximum likelihood solution has been achieved).

IEEE Trans Med Imaging, Maximum likelihood reconstruction for emission tomography L A Shepp, Y Vardi, 1982.

An Example of PET Image Reconstruction

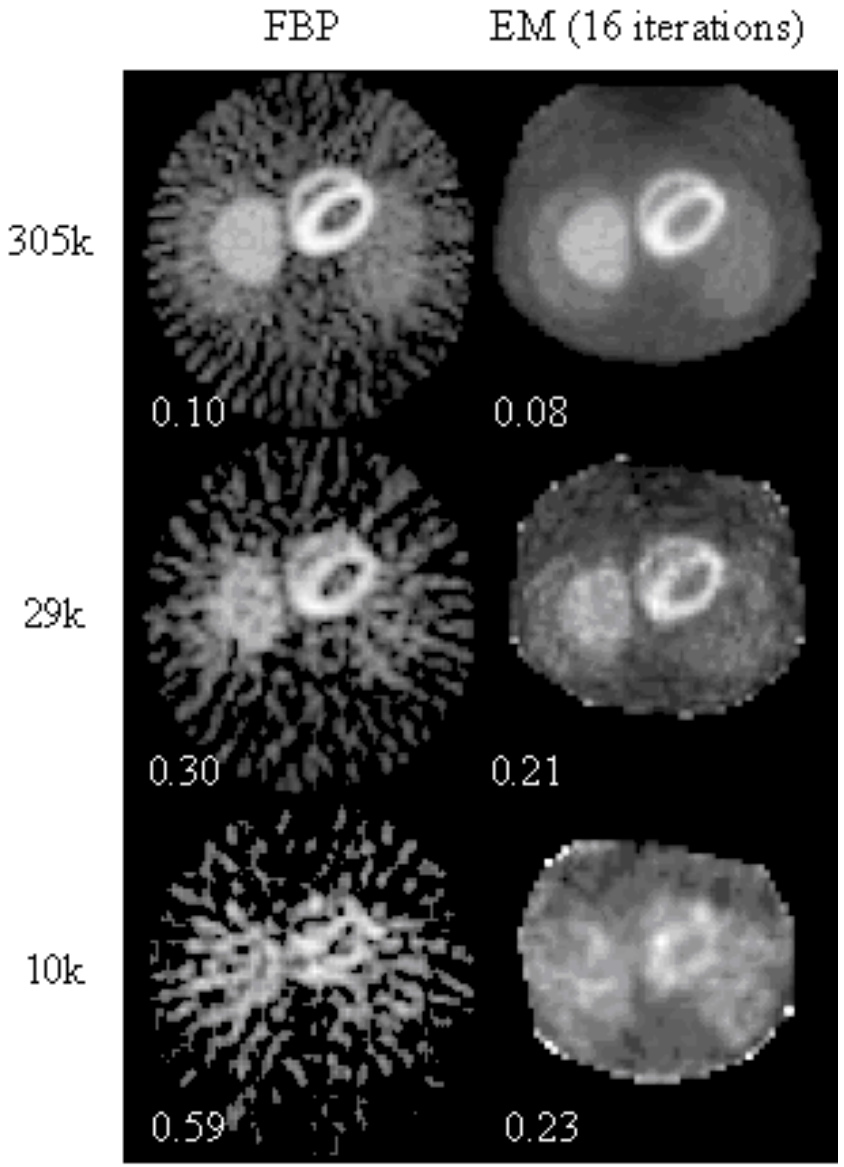


Figure 3: Comparison of EM reconstruction with FBP for different total counts. Note particular the streaking and noise appearance at low counts using FBP. Figures indicate the noise (standard deviation / mean) for a region in the liver.

http://web.uchile.cl/vignette/borrar3/alasbimn/CDA/sec_a/0,1205,SCID%253D470%2526PRT%253D455%2526LNID%253D10,00.html

Further Improvements – Penalized ML Algorithms

The basic idea:

Instead of finding the original image by maximizing the likelihood function:

$$\hat{\boldsymbol{f}} = \underset{\boldsymbol{f}}{\operatorname{argmax}}\, p(\boldsymbol{g} \mid \boldsymbol{f}) = \underset{\boldsymbol{f}}{\operatorname{argmax}}\, l(\boldsymbol{f}, \boldsymbol{g})$$

Probability of a measurement *g* given the underlying image *f*

The so-called log-likelihood function

We can use a different selection criterion

$$\hat{\boldsymbol{f}} = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[l(\boldsymbol{f}, \boldsymbol{g}) - \beta U(\boldsymbol{f}) \right]$$

The so-called penalty function that has increased value when an undesired imaging feature presents in the solution.

β is a scaling constant that controls "how strong the penalty is for the presence of a given feature"

So by minimizing, we are selecting an image by dis-encouraging the undesired features to be presented in the image.

Penalized Maximum Likelihood (PML) reconstruction …

An Example Penalized ML Algorithms

An example is to use the so-called **roughness penalty** function that measures the differences between neighboring pixels in the image, as follows

$$\hat{\boldsymbol{f}}_{\boldsymbol{PML}} = \underset{\boldsymbol{f}}{\operatorname{argmax}}\ l\,(\boldsymbol{f}, \boldsymbol{g}) = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[l(\boldsymbol{f}, \boldsymbol{g}) - \beta \sum_{j=1}^{N} \sum_{k=\mathcal{N}_j} \phi(f_j - f_k) \right],$$

where

$\phi(f_j - f_k)$ may be defined as $a_{jk}(f_j - f_k)^2$,

$\mathcal{N}_j$ refers to the collection of neighboring pixels of pixel *j*, and

a_{jk} is a user-defined weighting factor to reflect different preferences on how the user would like to penalize the differences between the target pixel j and its neighboring pixels.

So the above Penalized ML Algorithm becomes

$$\hat{\boldsymbol{f}}_{\boldsymbol{PML}} = \underset{\boldsymbol{f}}{\operatorname{argmax}}\ l\,(\boldsymbol{f}, \boldsymbol{g}) = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[l(\boldsymbol{f}, \boldsymbol{g}) - \beta \sum_{j=1}^{N} \sum_{k=\mathcal{N}_j} a_{jk}(f_j - f_k)^2 \right],$$

Further Improvements – Maximum A Posteriori (MAP) Algorithms

Alternatively, we may consider the unknown image, $\boldsymbol{f}$, as a random variable that follows a statistical law $p(\boldsymbol{f})$.

The Bayis's law:

if the underlying image is *f*, the conditional probability of observing the measurement *g* is

a priori probability

$$p(\boldsymbol{f} \mid \boldsymbol{g}) = \frac{p(\boldsymbol{g} \mid \boldsymbol{f})\, p(\boldsymbol{f})}{p(\boldsymbol{g})}$$

a posteriori probability

This leads to the maximum a posteriori (MAP) estimation …

Further Improvements –
Bayesian Estimation and Maximum A Posteriori (MAP) Algorithms

The maximum a posteriori (MAP) approach…

$$\hat{\boldsymbol{f}}_{\boldsymbol{MAP}} = \underset{\boldsymbol{f}}{\operatorname{argmax}} \log p(\boldsymbol{f} \mid \boldsymbol{g}) = \underset{\boldsymbol{f}}{\operatorname{argmax}} \log \frac{p(\boldsymbol{g} \mid \boldsymbol{f})\, p(\boldsymbol{f})}{p(\boldsymbol{g})}$$

$$= \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[\log p(\boldsymbol{g} \mid \boldsymbol{f}) + \log p(\boldsymbol{f}) \right]$$

The likelihood function

The *a priori* probability of the data. This is where you can fold in your prior knowledge about the underlying image … for example, the image "should" be relatively smooth …

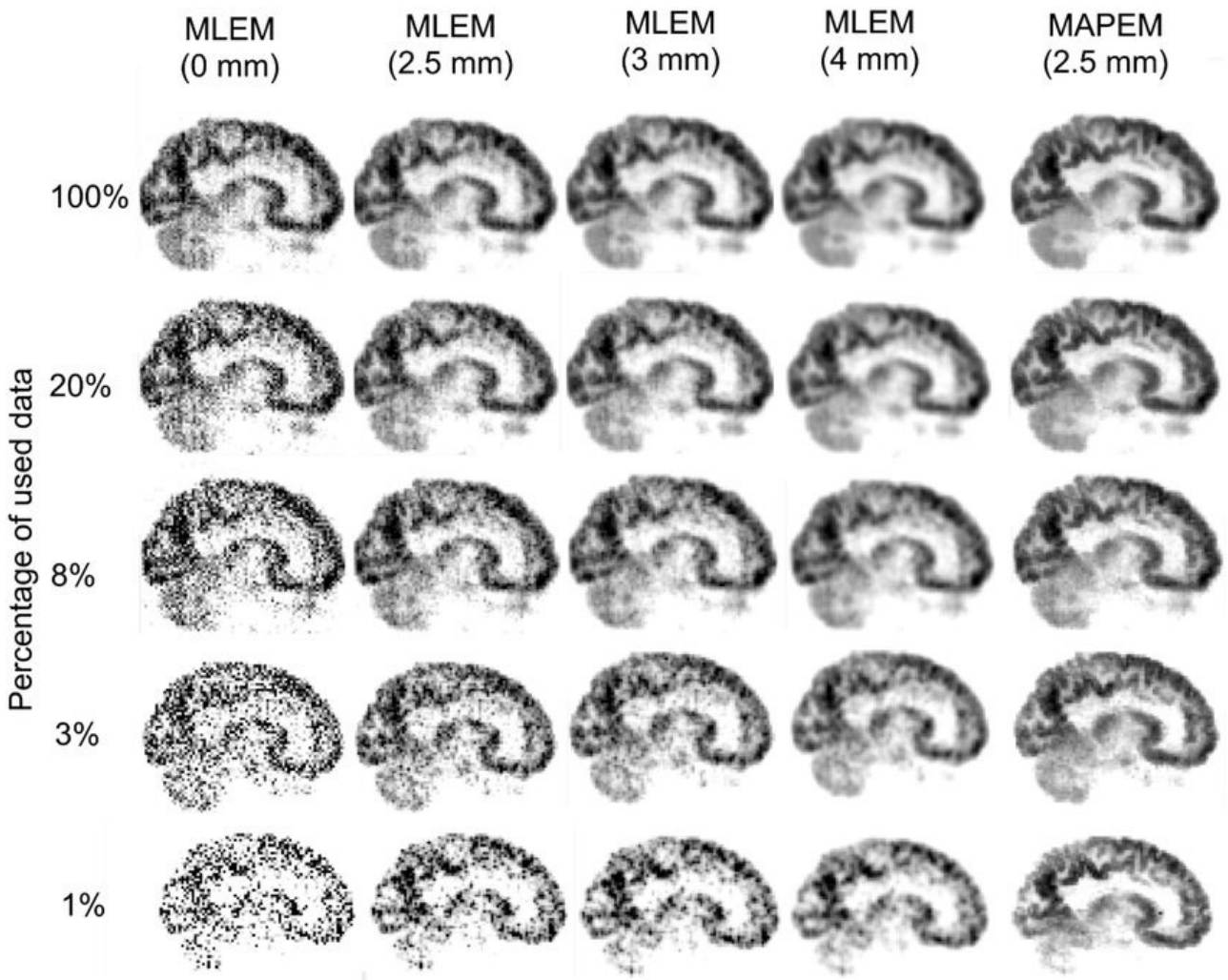
This is numerically equivalent to PML approach if we write

$$p(\boldsymbol{f}) = -\beta U(\boldsymbol{f}),$$

therefore

$$\hat{\boldsymbol{f}}_{\mathrm{PML}} = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[\log p(\boldsymbol{g}|\boldsymbol{f}) - \beta U(\boldsymbol{f}) \right] \Leftrightarrow \hat{\boldsymbol{f}}_{\mathrm{MAP}} = \underset{\boldsymbol{f}}{\operatorname{argmax}} \left[\log p(\boldsymbol{g}|\boldsymbol{f}) + \log p\,(\boldsymbol{f}) \right].$$

Example PET Images Reconstructed with MLEM and PML Algorithms



The reconstruction of various percentages of the FDG PET data using the MLEM and MAP-EM algorithms. The MLEM reconstructions have each been post-smoothed using a Gaussian kernel with 0, 2.5, 3, and also 4 mm FWHM. All PET images are shown with the same grey scale intensity range.

"PET image reconstruction using multi-parametric anato-functional priors", *Mehranian, et al., PMB, 2017.*

Chapter 3: Mathematical Preliminaries

Image Quality

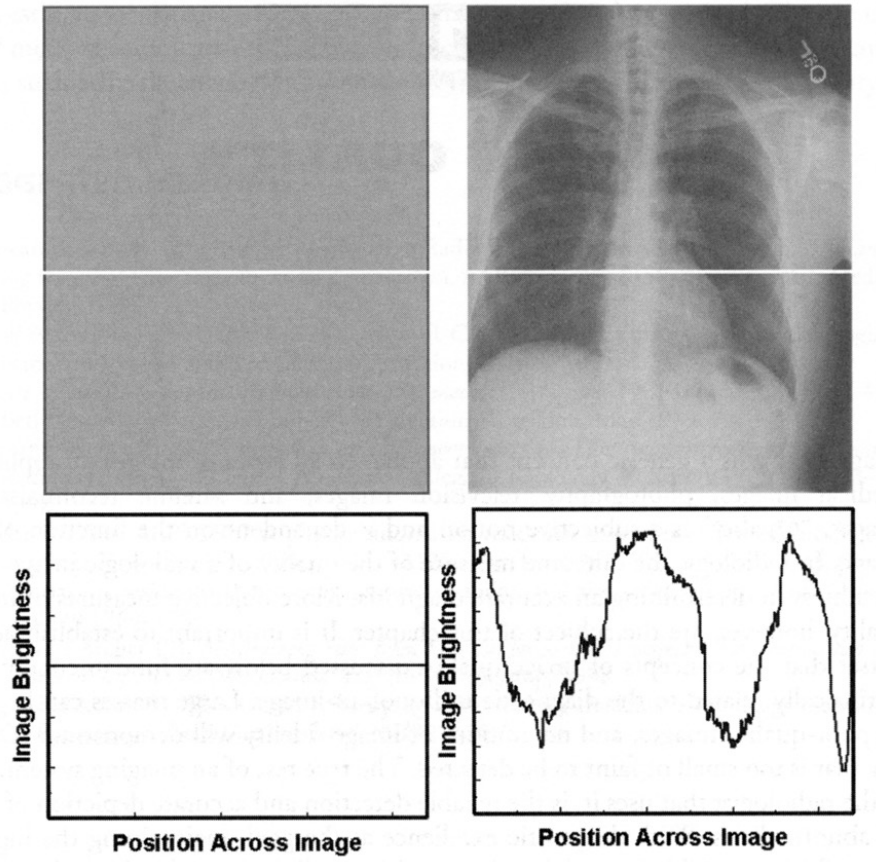
Image Quality

Learning objectives

1. **Contrast** in images
2. Modulation Transfer Function (MTF)
   a. What is **modulation**?
   b. What is the **modulation transfer function**?
   c. What exactly does MTF tell us about an imaging system?
3. Spatial resolution
   a. How do we measure the spatial resolution of an imaging system?
   b. Relationship between spatial resolution and MTF.

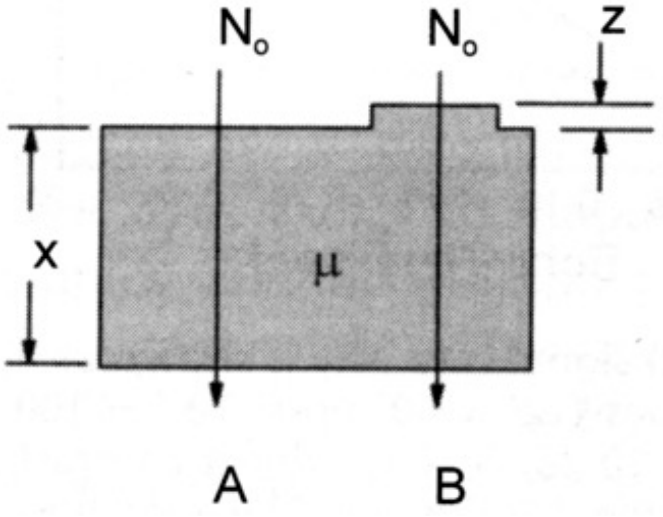
Contrast

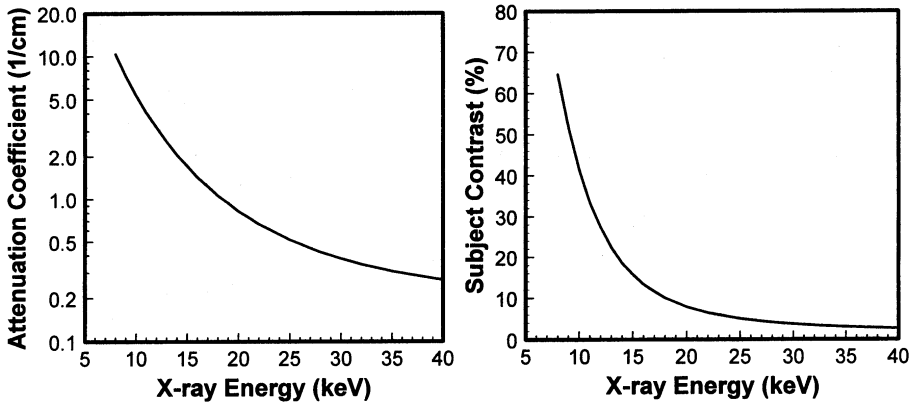
- What is contrast?
- The difference in the image gray scale between closely adjacent regions of the image.
- Medical image contrast is the result of many steps during image acquisition, processing and display



Subject (or Object) Contrast

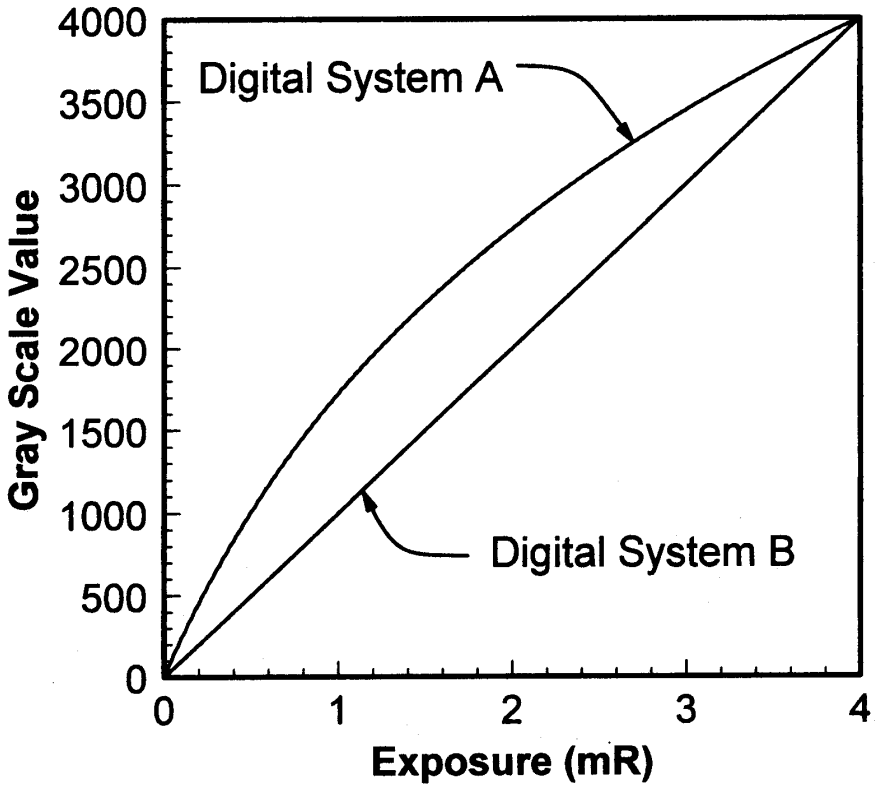
- Difference in some aspects of the signal prior to it being recorded
- Consequence of fundamental differences in the object, e.g., in x-ray intensity based on attenuation
- C = (A-B)/A





Detector Contrast

- Detector Contrast (C_d)
- A detector's characteristics play an important role in producing contrast in the final image
- C_d determined principally by how the detector 'maps' detected energy into the output signal
- Characteristic curve: input radiation exposure to output value (analog or digital)



Modulation

Recall that an arbitrary signal can be decomposed into the weighted sum of periodic signals, such as sinusoidal waves, so study the response of an imaging system to such signals provide a effective approach for quantifying the image quality.

Modulation

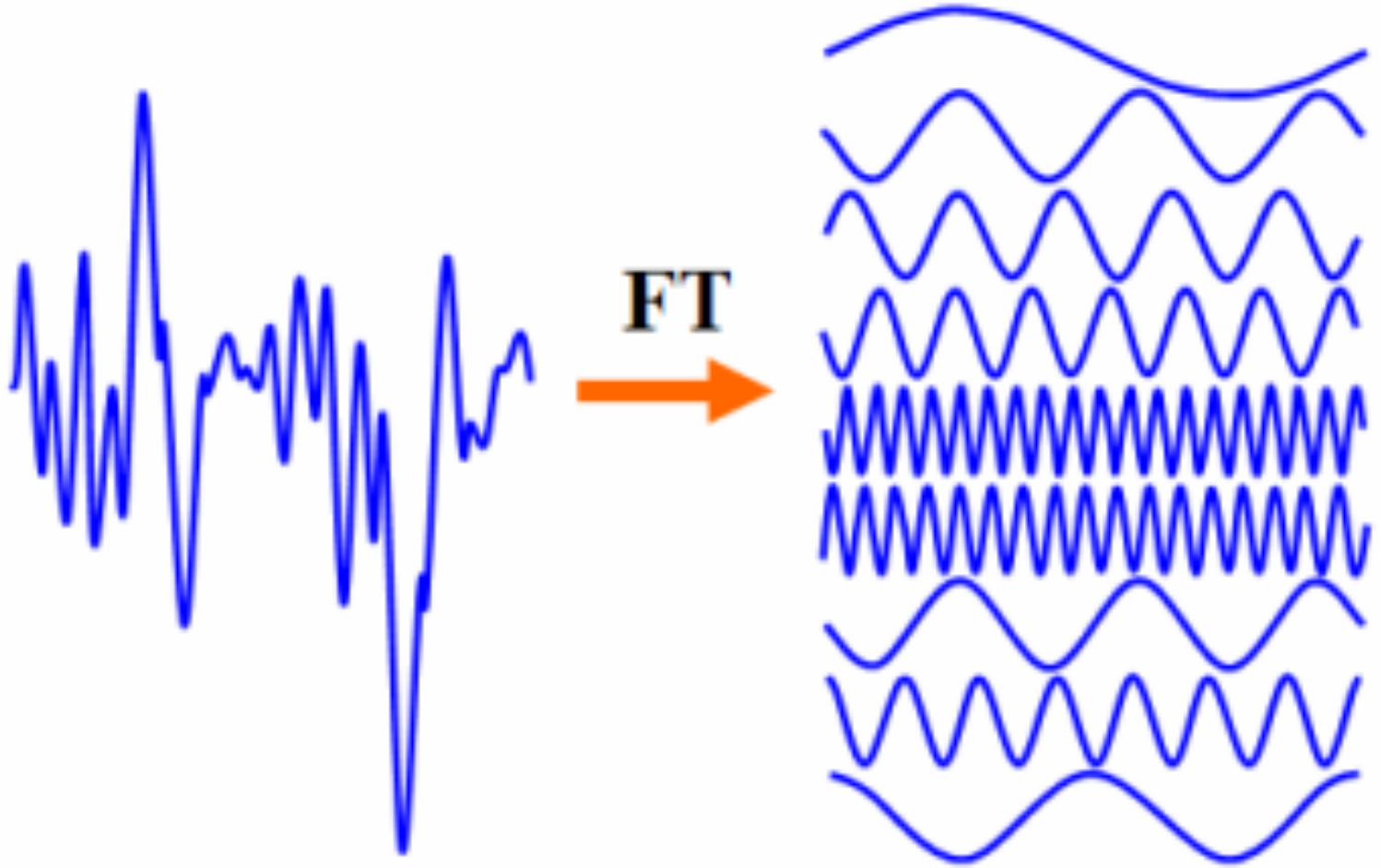
- The modulation m_f is an effective way to quantify the contrast of a periodic signal

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}}.$$

- In general, m_f is referred to as the contrast of a periodic signal f(x,y) relative to its average value.

- So within two signals, f(x,y) and g(x,y), with the same average value , f(x,y) is said to have more contrast if $m_f > m_g$.

Continuous Fourier Transform



www.revisemri.com

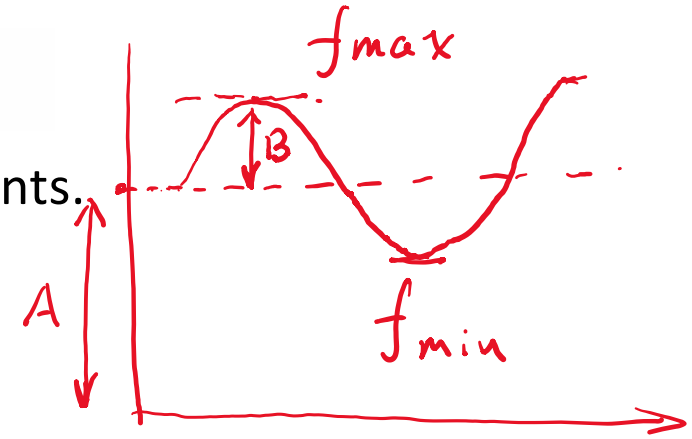
Modulation

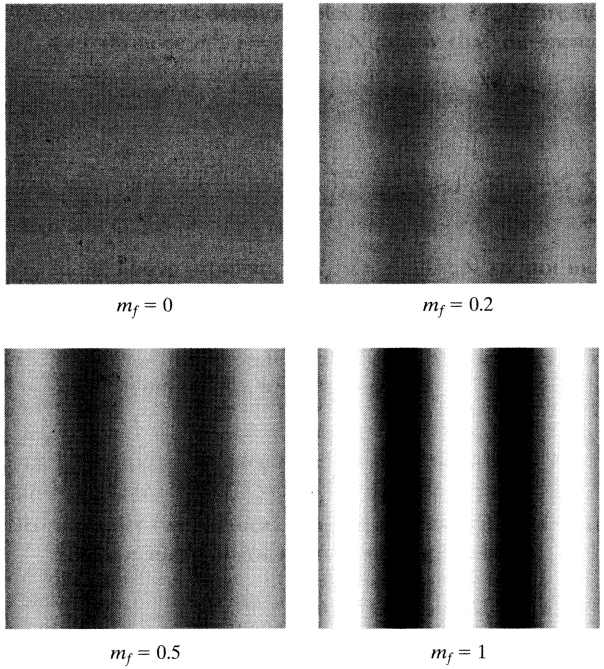
- Suppose an input signal function

$$f(x, y) = A + B\sin(2\pi u_0 x),$$

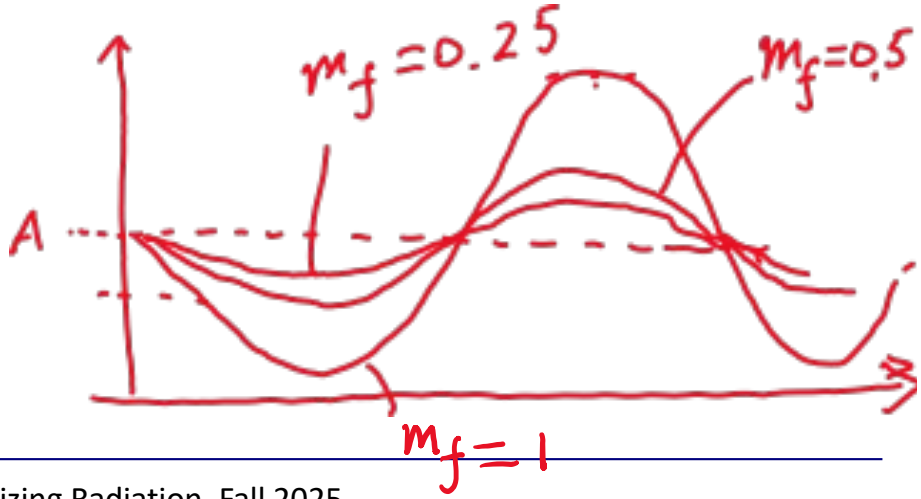
where A>B and both are non-negative constants.

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}}. \quad \Rightarrow \quad m_f = \frac{B}{A}.$$





Greater m_f, more contrast



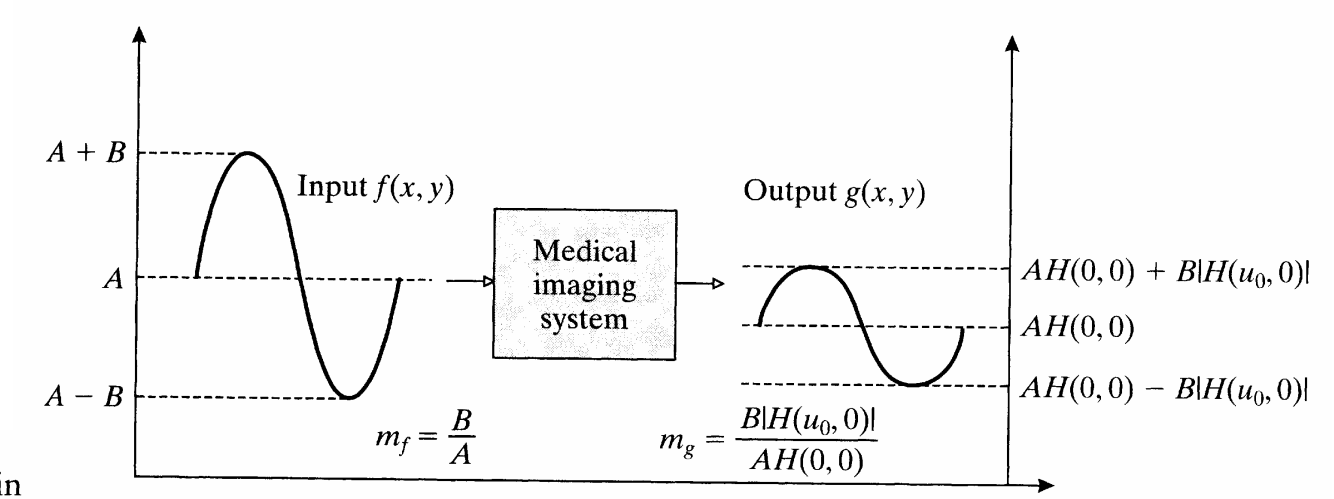
Modulation

Now let the following signal pass through an LSI imaging system,

$$f(x,y) = A + B\sin(2\pi u_0 x) = A + \frac{B}{2j}\left[e^{j2\pi u_0 x} - e^{-j2\pi u_0 x}\right],$$

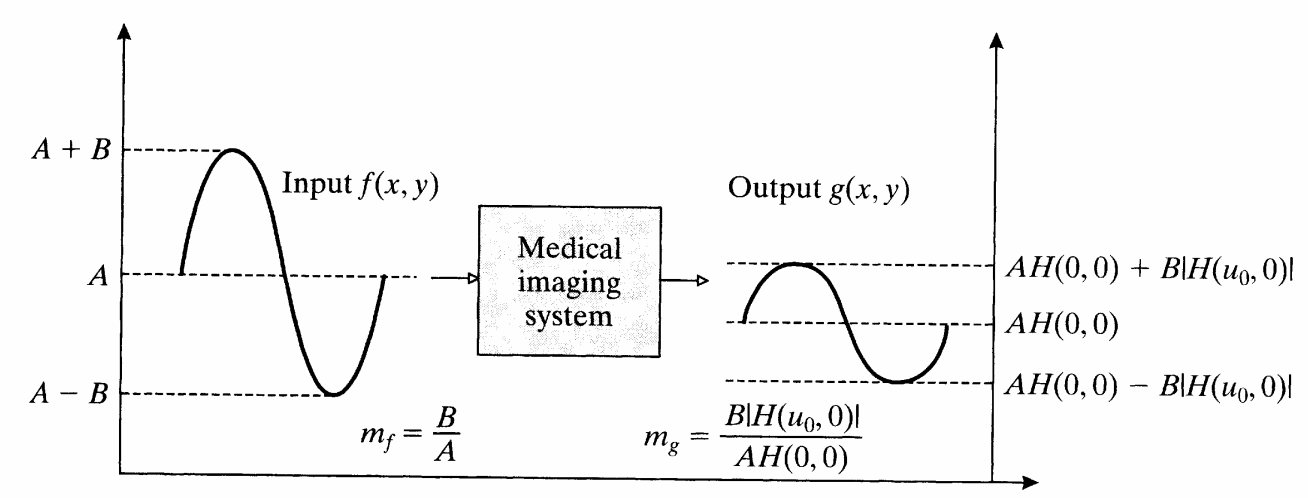
and consider the system impulse response function h(x,y) as circularly symmetric, then

$$g(x,y) = AH(0,0) + B\,|H(u_0,0)|\sin(2\pi u_0 x)\,.$$



$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}}$$

Modulation



Since

$$g_{min} = AH(0,0) - B|H(u_0,0)| \quad \text{and} \quad g_{\max} = AH(0,0) + B|H(u_0,0)|,$$

then the modulation of the input and output signals are

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}} = \frac{B}{A}, \quad \text{and} \quad m_g = \frac{g_{\max} - g_{\min}}{g_{\max} + g_{\min}} = \frac{B|H(u_0,0)|}{AH(0,0)} = m_f \frac{|H(u_0,0)|}{H(0,0)}$$

$$\frac{m_g}{m_f} = \frac{H(u_0,0)}{H(0,0)} \leq 1$$

Modulation Transfer Function (MTF)

- In case of non-isotropic impulse response function (h(x,y) is not circularly symmetric), the MTF can be defined as

$$\mathrm{MTF}(u,v) = \frac{m_g}{m_f} = \frac{|H(u,v)|}{H(0,0)},$$

- A typical MTF of an imaging system

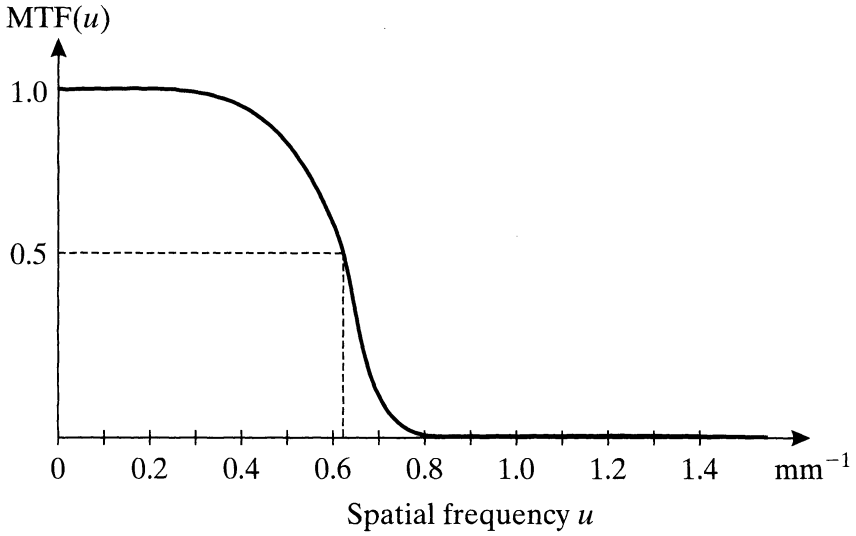
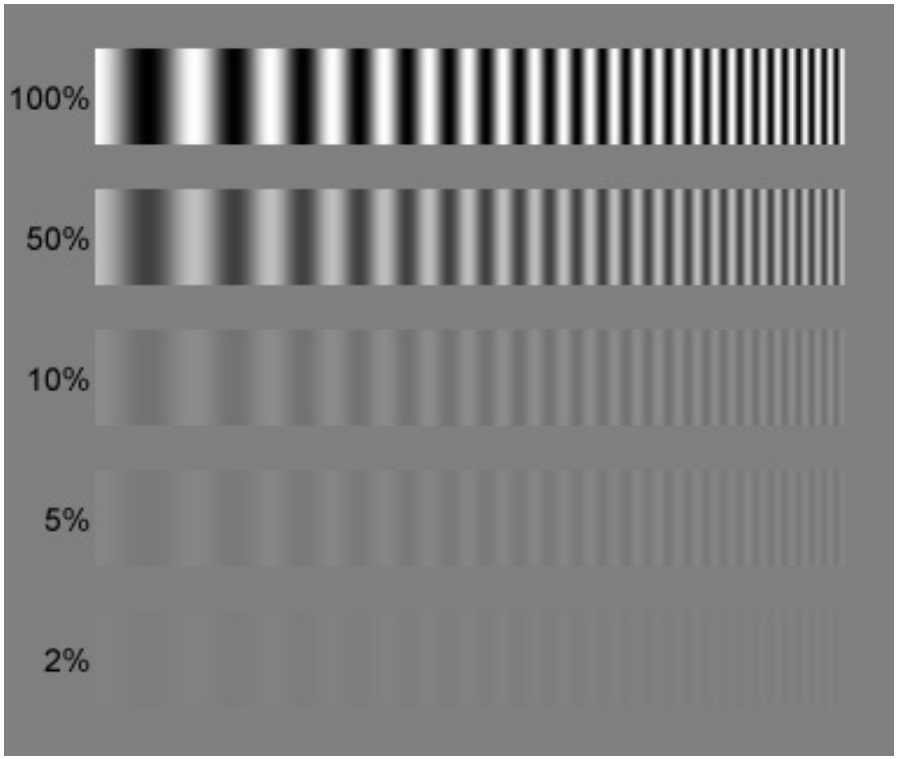


Figure 3.3
A typical MTF of a medical imaging system.

Modulation Transfer Function (MTF)

In case of non-isotropic impulse response function (h(x,y) is not circularly symmetric), the MTF can be defined as

$$\mathrm{MTF}(u,v) = \frac{m_g}{m_f} = \frac{|H(u,v)|}{H(0,0)},$$



Uniform MTF

Basic Fourier Transform Pairs

Basic Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0+vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\text{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0x+v_0y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0x + v_0y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0x + v_0y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\text{rect}(x, y)$	$\text{sinc}(u, v)$
$\text{sinc}(x, y)$	$\text{rect}(u, v)$
$\text{comb}(x, y)$	$\text{comb}(u, v)$
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$

Modulation Transfer Function (MTF)

$$MTF(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)}$$

If an imaging system is designed to faithfully reproduce a DC signal, then ((0,0)=1. So, the MTF becomes

$$MTF(u) = \frac{m_g}{m_f} = |H(u,0)| \ .$$

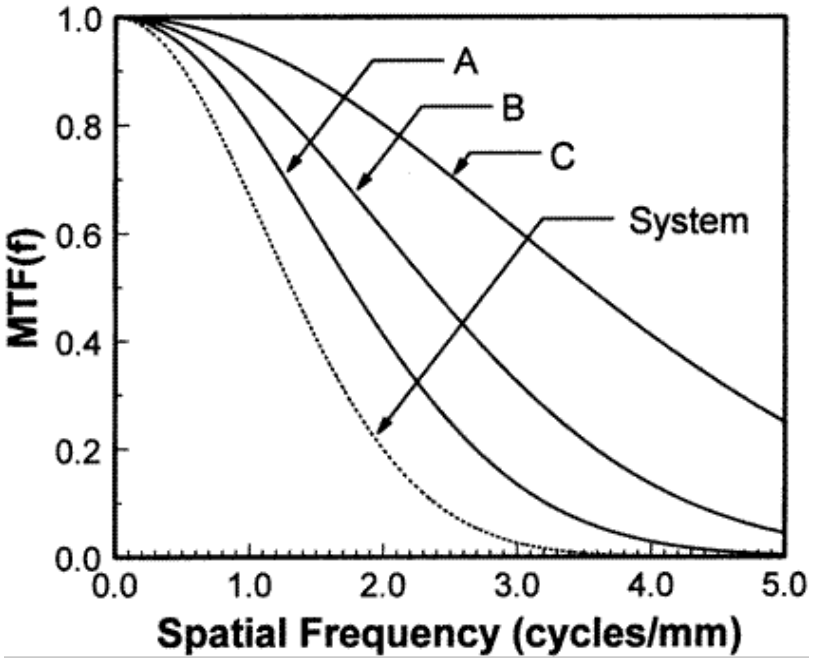
System Cascade

Given a cascade of LSI systems, the output of a signal through the overall system is given by

$$g(x,y) = h_K(x,y) * \cdots * (h_2(x,y) * (h_1(x,y) * f(x,y))) .$$

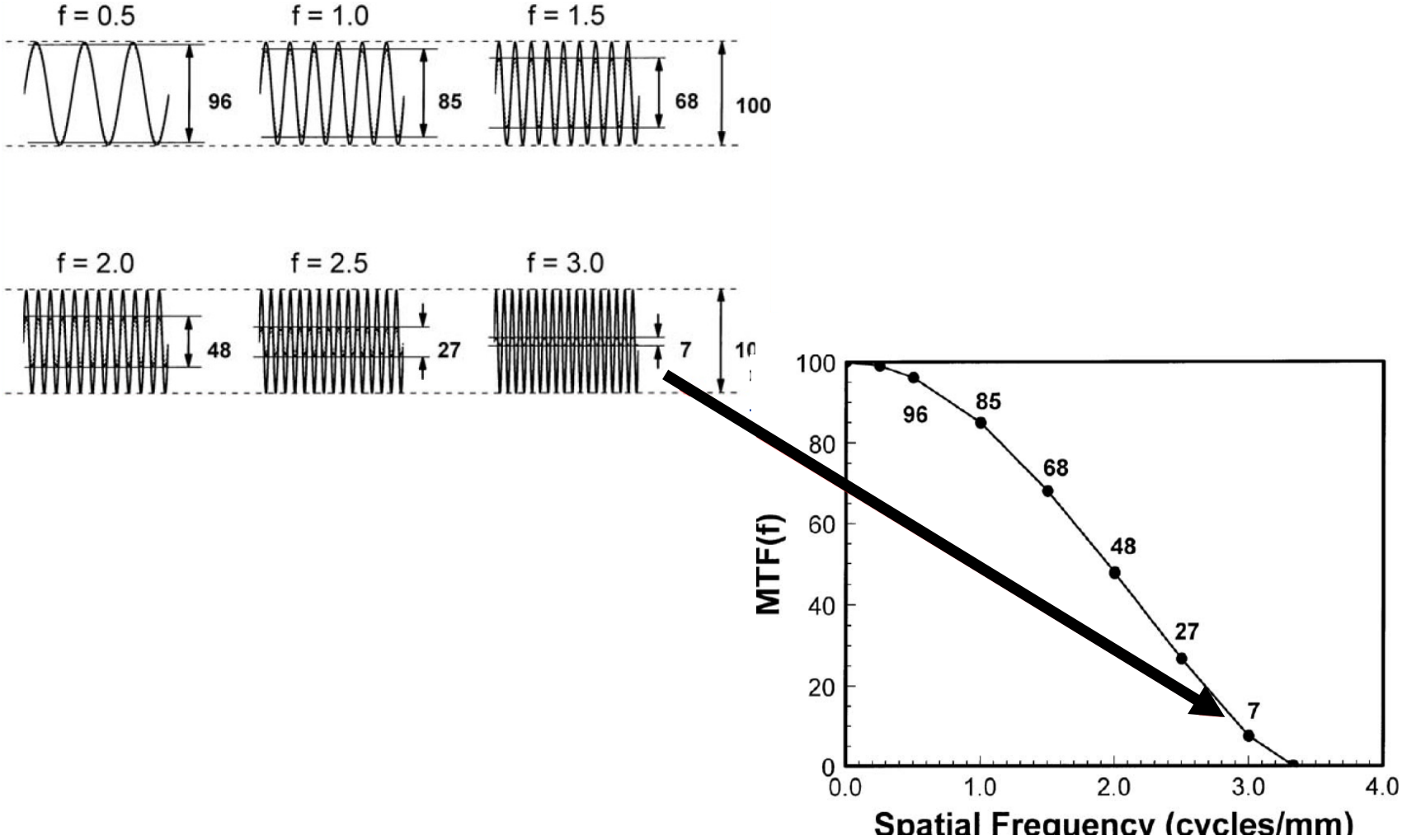
The overall MTF is the product of the MTF for sub-systems:

$$MTF_{total}(u) = \prod_{i=1}^{i=I} MTF_i(u)$$



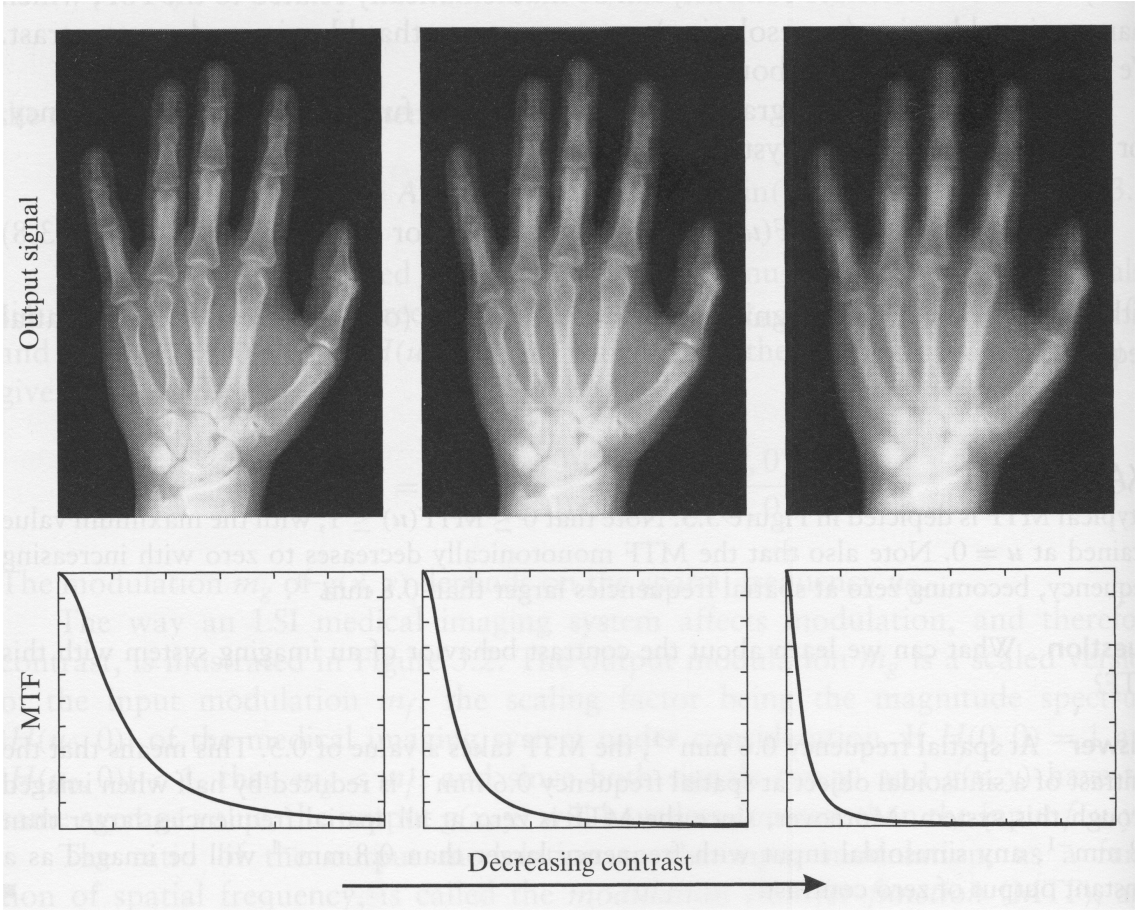
How is the Modulation Transfer Function Measured?

- Modulation Transfer Function (MTF).



Example of the Modulation Transfer Function

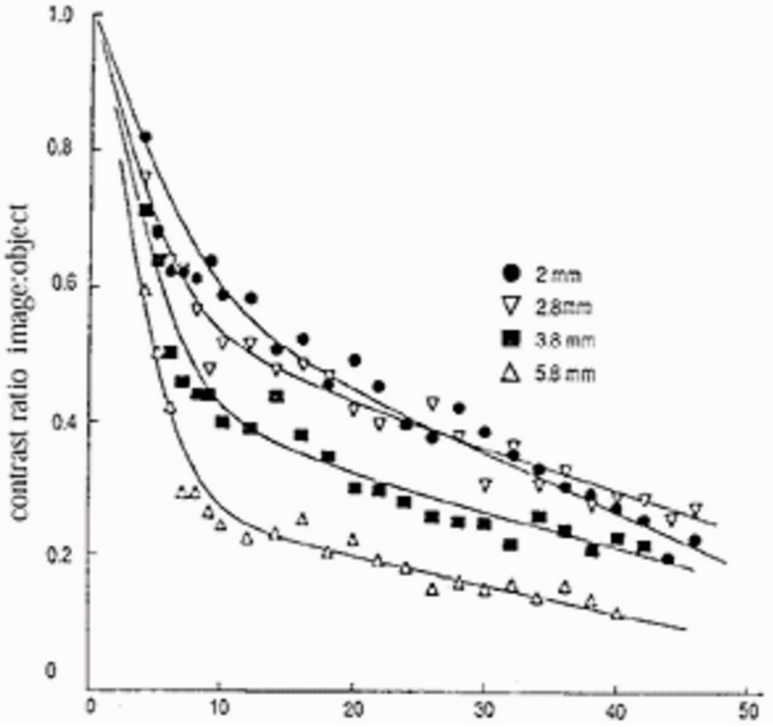
- An example of the same signal passing through three imaging systems with decreasingly poor MTF, which leads to decreasing contrast in images



Example of the Modulation Transfer Function

Optical modulation transfer function (MTF) of the human eye

- MTF is measured directly with sinewave gratings.
- The optical modulation transfer function (MTF) can be interpreted as Fourier transform of the optical LSF.



https://www.normankoren.com/Tutorials/MTF.html

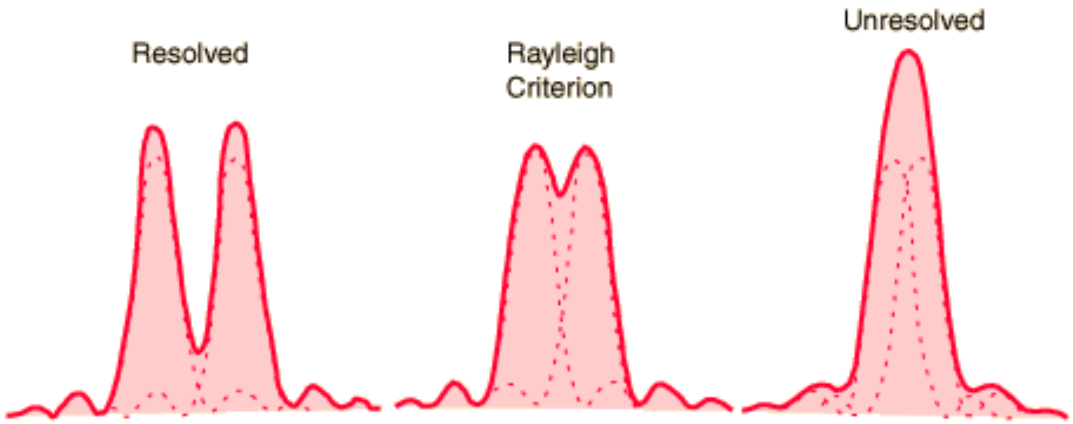
A Revisit to Key Image Quality Measures

Spatial Resolution

- How to quantify spatial resolution?
- How to measure spatial resolution?
- The relationship between spatial resolution and modulation transfer function (MTF)?

Spatial Resolution

- Resolution: the ability of a given imaging system to accurately depict two distinct events in space, time, or frequency, respectively.
- Resolution can also be considered as the degree of blurring or smearing.



- Spatial resolution is fully described by the point-spread function or impulse response function, h(x,y).

Full-width at Half Maximum (FWHM) of The Point Spread Function (PSF)

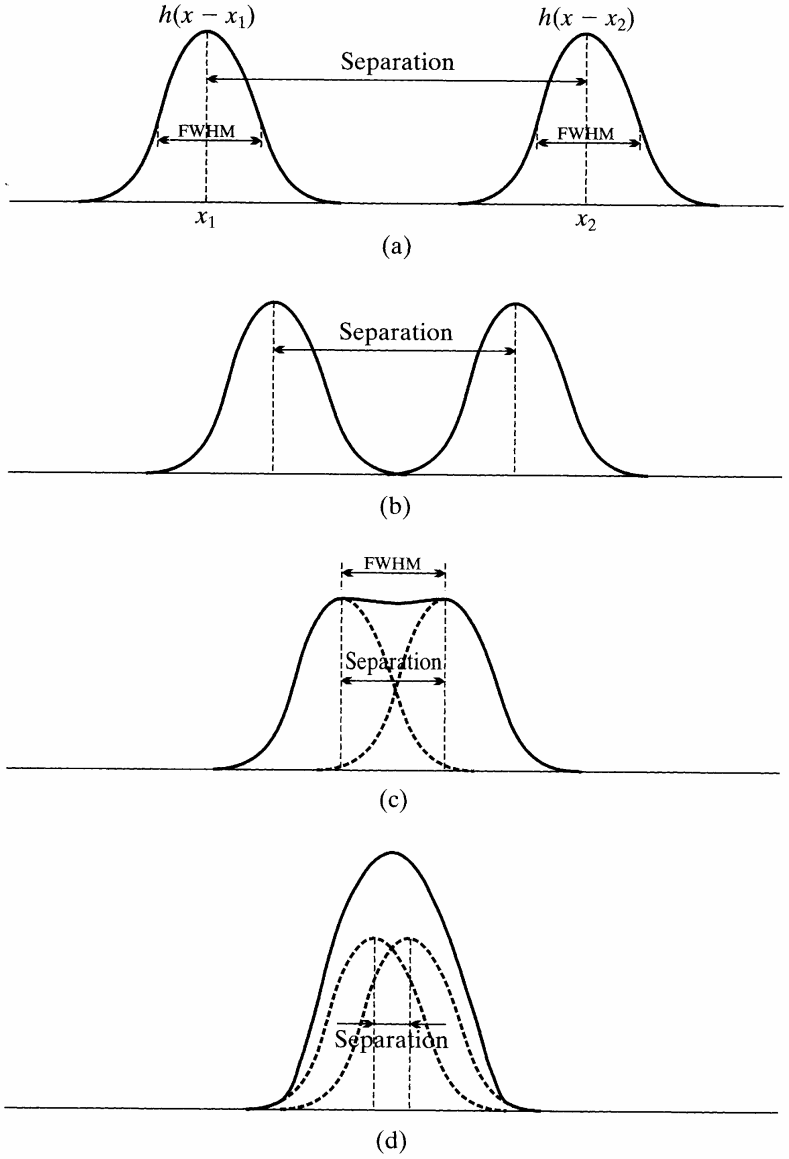
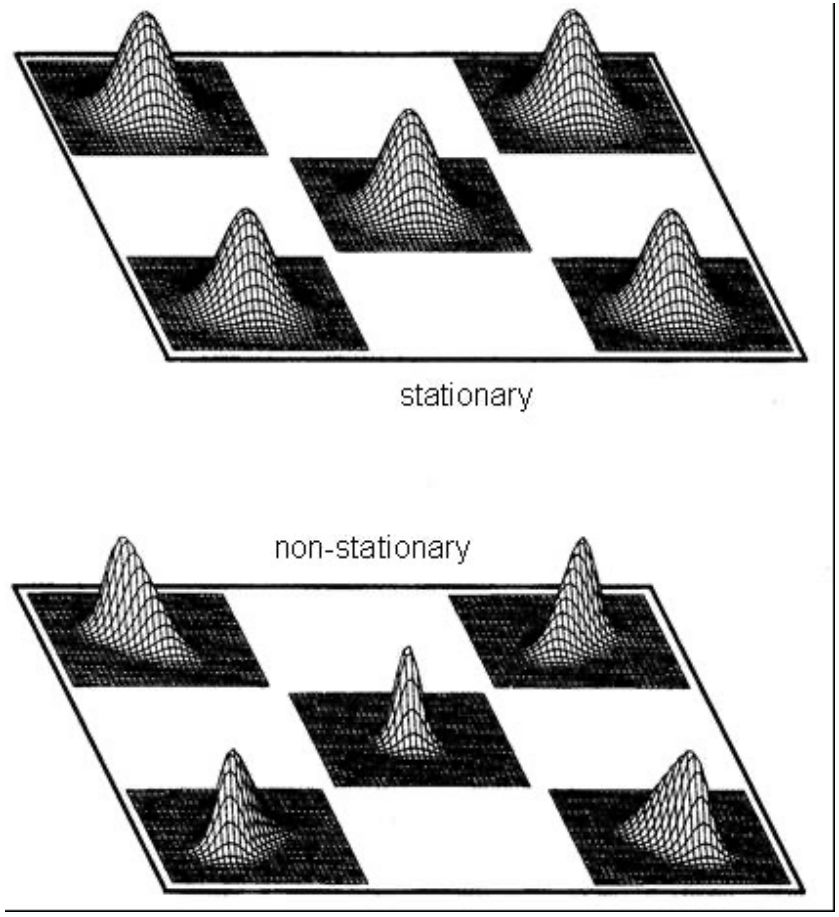


Figure 3.6
An example of the effect of system resolution on the ability to differentiate two points. The FWHM equals the minimum distance that the two points must be separated in order to be distinguishable.

Stationary and Non-stationary PSF

Spatial variation of the PSF is another important aspect of a given imaging system.



Other Ways to Measure the Spatial Resolution
Line Response Function

The resolution of an imaging system can also be estimated with the line-spread function

$$line\ impulse\ f(x,y) = \delta_\ell(x,y) = \delta(x\cos\theta + y\sin\theta - \ell)$$

If we assume the impulse response function of the imaging system *h*(*x*,*y*) is isotropic, it is sufficient to consider the response of the system to a vertical line through the origin,

$$f(x,y) = \delta(x).$$

For this case, line response function (the response of the system to the line impulse) is given by

$$\begin{aligned} g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta) f(x-\xi, y-\eta)\, d\xi\, d\eta, \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\xi,\eta)\delta(x-\xi)\, d\xi\right] d\eta, \\ &= \int_{-\infty}^{\infty} h(x,\eta)\, d\eta, \end{aligned}$$

Line Response Function (or Line-Spread Function, LSF)

$$Line\ Response\ Function\colon\ l(x) \equiv \int_{-\infty}^{\infty} h(x,\eta)d\eta$$

The 1-D Fourier transform of the line spread function is

$$\begin{aligned} L(u) &= \mathcal{F}_{1D}[l](u)\,, \\ &= \int_{-\infty}^{\infty} l(x)e^{-j2\pi ux}\,dx\,, \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,\eta)e^{-j2\pi ux}\,dx\,d\eta\,, \\ &= H(u,0). \end{aligned}$$

Remember that

$$MTF(u) = \frac{m_g}{m_f} = |H(u,0)|$$

So the values of the Fourier transform of the LSF crossing a horizontal line passing through the origin is numerically equivalent to the MTF,

$$L(u) = F_1[l(x)] = |H(u,0)| = \frac{m_g}{m_f} = MTF(u)$$

LSF and MTF

- Modulation Transfer Function (MTF).

$$\mathrm{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)} = \frac{|L(u)|}{L(0)}, \qquad \text{for every } u.$$

- For a "reasonable" imaging system, the L(0)=1, so that

$$\mathrm{MTF}(u) = L(u)$$

- MTF is an effective way to compare two imaging systems in terms of spatial resolution and contrast.

