NPRE435: Radiological Imaging, Fall, 2024
Solutions for Homework 2

2.1 Determine whether the following signals are separable. Fully justify your answers.

(a) $\delta_s(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n)$

(b) $\delta_l(x,y) = \delta(x\cos(\theta) + y\sin(\theta) + l)$

(c) $e(x,y) = e^{j2\pi(u_o x + v_o y)}$

(d) $s(x,y) = \sin[2\pi(u_o x + v_o y)]$

Solutions:

(a) $\delta_s(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n) = \sum_{m=-\infty}^{\infty} \delta(x-m) \cdot \sum_{n=-\infty}^{\infty} \delta(y-n)$, therefore it is a separable signal.

(b) $\delta_l(x,y)$ is separable if $\sin(2\theta) = 0$. In this case, either $\sin\theta = 0$ or $\cos\theta = 0$, $\delta_l(x,y)$ is a product of a constant function in one axis and a 1-D *delta* function in another. But in general, $\delta_l(x,y)$ is not separable.

(c) $e(x,y) = \exp[j2\pi(u_0 x + v_0 y)] = \exp(j2\pi u_0 x) \cdot \exp(j2\pi v_0 y) = e_{1D}(x; u_0) \cdot e_{1D}(y; v_0)$, where $e_{1D}(t;\omega) = \exp(j2\pi\omega t)$. Therefore, $e(x,y)$ is a separable signal.

(d) $s(x,y)$ is a separable signal when $u_0 v_0 = 0$. For example, if $u_0 = 0$, $s(x,y) = \sin(2\pi v_0 y)$ is the product of a constant signal in x and a 1-D sinusoidal signal in y. But in general, when both u_0 and v_0 are nonzero, $s(x,y)$ is not separable.

2.6 Determine whether the system g(x, y) = f(x, −1) + f(0, y) is

(a) linear?

(b) shift-invariant?

Solutions:

(a) If $g'(x,y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x,y)$, then

$$\begin{aligned} g'(x,y) &= \sum_{k=1}^{K} w_k f_k(x,-1) + \sum_{k=1}^{K} w_k f_k(0,y) \\ &= \sum_{k=1}^{K} w_k [f_k(x,-1) + f_k(0,y)] \\ &= \sum_{k=1}^{K} w_k g_k(x,y) \end{aligned}$$

where $g_k(x,y)$ is the response of the system to input $f_k(x,y)$. Therefore, the system is linear.

(b) If $g'(x,y)$ is the response of the system to input $f(x-x_0, y-y_0)$, then

$$g'(x,y) = f(x-x_0, -1-y_0) + f(-x_0, y-y_0);$$

while

$$g(x-x_0, y-y_0) = f(x-x_0, -1) + f(0, y-y_0).$$

Since $g'(x,y) \neq g(x-x_0, y-y_0)$, the system is not shift-invariant.

2.7 For each system with the following input-output equation, determine whether the system is (1) linear and (2) shift-invariant.

(a) $g(x, y) = f(x, y)f(x - x_0, y)$.

(b) $g(x,y) = \int_{-\infty}^{\infty} f(x,\eta)d\eta$

Solution:

(a) If $g'(x,y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x,y)$, then

$$\begin{aligned} g'(x,y) &= \left(\sum_{k=1}^{K} w_k f_k(x,y)\right)\left(\sum_{k=1}^{K} w_k f_k(x-x_0, y-y_0)\right) \\ &= \sum_{i=1}^{K}\sum_{j=1}^{K} w_i w_j f_i(x,y) f_j(x-x_0, y-y_0), \end{aligned}$$

while

$$\sum_{k=1}^{K} w_k g_k(x,y) = \sum_{k=1}^{K} w_k f_k(x,y) f_k(x-x_0, y-y_0).$$

Since $g'(x,y) \neq \sum_{k=1}^{K} g_k(x,y)$, the system is nonlinear.

On the other hand, if $g'(x,y)$ is the response of the system to input $f(x-a, y-b)$, then

$$\begin{aligned} g'(x,y) &= f(x-a, y-b) f(x-a-x_0, y-b-y_0) \\ &= g(x-a, y-b) \end{aligned}$$

and the system is thus shift-invariant.

(b) If $g'(x,y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x,y)$, then

$$\begin{aligned} g'(x,y) &= \int_{-\infty}^{\infty} \sum_{k=1}^{K} w_k f_k(x,\eta)\, d\eta \\ &= \sum_{k=1}^{K} w_k \left(\int_{-\infty}^{\infty} f_k(x,\eta)\, d\eta \right) \\ &= \sum_{k=1}^{K} w_k g_k(x,y), \end{aligned}$$

where $g_k(x,y)$ is the response of the system to input $f_k(x,y)$. Therefore, the system is linear.

2.9 Consider the 1-D system whose input-output equation is given by

$$g(x) = f(x) * f(x),$$

where * denotes convolution.

(**a**) Write an integral expression that gives g(x) as a function of f(x).

(**b**) Determine whether the system is linear.

(**c**) Determine whether the system is shift-invariant.

Solution:

(a) $g(x) = \int_{-\infty}^{\infty} f(x-t)f(t)dt$.

(b) Given an input as $af_1(x) + bf_2(x)$, where a, b are some constant, the output is

$$\begin{aligned} g'(x) &= [af_1(x) + bf_2(x)] * [af_1(x) + bf_2(x)] \\ &= a^2 f_1(x) * f_1(x) + 2ab f_1(x) * f_2(x) + b^2 f_2(x) * f_2(x) \\ &\neq ag_1(x) + bg_2(x), \end{aligned}$$

where $g_1(x)$ and $g_2(x)$ are the output corresponding to an input of $f_1(x)$ and $f_2(x)$ respectively. Hence, the system is nonlinear.

(c) Given a shifted input $f_1(x) = f(x - x_0)$, the corresponding output is

$$\begin{aligned} g_1(x) &= f_1(x) * f_1(x) \\ &= \int_{-\infty}^{\infty} f_1(x-t) f_1(t) dt \\ &= \int_{-\infty}^{\infty} f(x - t - x_0) f_1(t - x_0) dt. \end{aligned}$$

Changing variable $t' = t - x_0$ in the above integration, we get

$$\begin{aligned} g_1(x) &= \int_{-\infty}^{\infty} f(x - 2x_0 - t') f_1(t') dt' \\ &= g(x - 2x_0). \end{aligned}$$

Thus, if the input is shifted by x_0, the output is shifted by $2x_0$. Hence, the system is not shift-invariant.

2.10 (20 pts) Given a continuous signal f(x, y) = x + y^2, evaluate the following:

(a) $f(x, y)\delta(x - 1, y - 2)$

(b) $f(x, y) * \delta(x - 1, y - 2)$.

(c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x - 1, y - 2)f(x, 3)dx\,dy$

(d) $\delta(x - 1, y - 2) * f(x + 1, y + 2)$

Solution:

(a)

$$\begin{aligned} f(x,y)\delta(x-1,y-2) &= f(1,2)\delta(x-1,y-2) \\ &= (1+2^2)\delta(x-1,y-2) \\ &= 5\delta(x-1,y-2) \end{aligned}$$

(b)

$$\begin{aligned} f(x,y) * \delta(x-1,y-2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(x-\xi-1,y-\eta-2)\,d\xi\,d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-1,y-2)\delta(x-\xi-1,y-\eta-2)\,d\xi\,d\eta \\ &= f(x-1,y-2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\xi-1,y-\eta-2)\,d\xi\,d\eta \\ &= f(x-1,y-2) \\ &= (x-1)+(y-2)^2 \end{aligned}$$

(c)

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-1,y-2)f(x,3)dx\,dy &\overset{①}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-1,y-2)f(1,3)dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-1,y-2)(1+3^2)dx\,dy \\ &= 10\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-1,y-2)dx\,dy \\ &\overset{②}{=} 10 \end{aligned}$$

Equality ① comes from the Eq. (2.7) in the text. Equality ② comes from the fact:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-1,y-2)dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dx\,dy = 1.$$

(d)

$$\begin{aligned}\delta(x-1,y-2)*f(x+1,y+2) &\overset{③}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-\xi-1,y-\eta-2)f(\xi+1,\eta+2)d\xi\, d\eta\\ &\overset{④}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-\xi-1,y-\eta-2)f((x-1)+1,(y-2)+2)d\xi\, d\eta\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-\xi-1,y-\eta-2)f(x,y)d\xi\, d\eta\\ &\overset{⑤}{=} f(x,y)=x+y^2\end{aligned}$$

③ comes from the definition of convolution; ④ comes from the Eq. (2.7) in text; ⑤ is the same as ② in part (c). Alternatively, by using the sifting property of $\delta(x,y)$ and defining $g(x,y)=f(x+1,y+2)$, we have

$$\begin{aligned}\delta(x-1,y-2)*g(x,y) &= g(x-1,y-2)\\ &= f(x-1+1,y-2+2)\\ &= f(x,y)\\ &= x+y^2\,.\end{aligned}$$

2.19 (15 pt) Find the Fourier Transforms of the following continuous functions:

(a) $\delta_s(x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n)$.

(b) $\delta_s(x,y;\Delta x,\Delta^y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x,y-n\Delta y)$.

Solutions:

(a) See the solution to part **(b)** below. The Fourier transform is

$$\mathcal{F}_2\{\delta_s(x,y)\}=\delta_s(u,v)$$

(b)

$$\mathcal{F}_2\{\delta_s(x,y;\Delta x,\Delta y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_s(x,y;\Delta x,\Delta y)e^{-j2\pi(ux+vy)}\,dx\,dy$$

$\delta_s(x,y;\Delta x,\Delta y)$ is a periodic signal with periods Δx and Δy in x and y axes. Therefore it can be written as a Fourier series expansion. (Please review Oppenheim, Willsky, and Nawad, *Signals and Systems* for the definition of *Fourier series expansion* of periodic signals.)

$$\delta_s(x,y;\Delta x,\Delta y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}C_{mn}e^{j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)}\,,$$

where

$$\begin{aligned} C_{mn} &= \frac{1}{\Delta x \Delta y} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \delta_s(x, y; \Delta x, \Delta y) e^{-j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)}\, dx\, dy \\ &= \frac{1}{\Delta x \Delta y} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y) e^{-j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)}\, dx\, dy. \end{aligned}$$

In the integration region $-\frac{\Delta x}{2} < x < \frac{\Delta x}{2}$ and $-\frac{\Delta y}{2} < y < \frac{\Delta y}{2}$ there is only one impulse corresponding to $m = 0$, $n = 0$. Therefore, we have

$$\begin{aligned} C_{mn} &= \frac{1}{\Delta x \Delta y} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \delta(x, y) e^{-j2\pi\left(\frac{0 \cdot x}{\Delta x} + \frac{0 \cdot y}{\Delta y}\right)}\, dx\, dy \\ &= \frac{1}{\Delta x \Delta y}. \end{aligned}$$

We have:

$$\delta_s(x, y; \Delta x, \Delta y) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)}.$$

Therefore,

$$\begin{aligned} \mathcal{F}_2\{\delta_s\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_s(x, y; \Delta x, \Delta y) e^{-j2\pi(ux+vy)} dx\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)} e^{-j2\pi(ux+vy)}\, dx\, dy \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)} e^{-j2\pi(ux+vy)}\, dx\, dy \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \mathcal{F}_2\left\{e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)}\right\} \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \delta\left(u - \frac{m}{\Delta x}, v - \frac{n}{\Delta y}\right) \\ &\overset{⑤}{=} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \cdot \Delta x \Delta y \delta(u\Delta x - m, v\Delta y - n) \\ \mathcal{F}_2\{\delta_s\} &= \delta_s(u\Delta x, v\Delta y) \end{aligned}$$

Equality ⑤ comes from the property $\delta(ax) = \frac{1}{|a|}\delta(x)$.

2.23 (10 pt) The PSF of a medical imaging system is given by

$$h(x, y) = e^{(-|x|+|y|)}$$

where x and y are in millimeters.

(a) Is the system separable? Explain.
(b) What is the response of the system to the line impulse $f(x, y) = \delta(x)$?
(c) What is the response o the system to the line impulse $f(x, y) = \delta(x - y)$?

Solution:

(a) The system is separable because $h(x, y) = e^{-(|x|+|y|)} = e^{-|x|}e^{-|y|}$.

(b) The system is not isotropic since $h(x, y)$ is not a function of $r = \sqrt{x^2 + y^2}$.

Additional comments: An easy check is to plug in $x = 1$, $y = 1$ and $x = 0$, $y = \sqrt{2}$ into $h(x, y)$. By noticing that $h(1, 1) \neq h(0, \sqrt{2})$, we can conclude that $h(x, y)$ is not rotationally invariant, and hence not isotropic.

Isotropy is rotational symmetry around the origin, not just symmetry about a few axes, e.g., the x- and y-axes. $h(x, y) = e^{-(|x|+|y|)}$ is symmetric about a few lines, but it is not rotationally invariant.

When we studied the properties of Fourier transform, we learned that if a signal is isotropic then its Fourier transform has a certain symmetry. Note that the symmetry of the Fourier transform is only a necessary, but not sufficient, condition for the signal to be isotropic.

(c) The response is

$$\begin{aligned}
g(x, y) &= h(x, y) * f(x, y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi, \eta) f(x - \xi, y - \eta) d\xi\, d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(|\xi|+|\eta|)} \delta(x - \xi) d\xi\, d\eta \\
&= \int_{-\infty}^{\infty} e^{-(|x|+|\eta|)} d\eta \\
&= e^{-|x|} \int_{-\infty}^{\infty} e^{-|\eta|} d\eta \\
&= e^{-|x|} \left[\int_{-\infty}^{0} e^{\eta} d\eta + \int_{0}^{\infty} e^{-\eta} d\eta \right] \\
&= 2e^{-|x|}\,.
\end{aligned}$$