Chapter 3: X-ray Radiography and Computer Tomography

X-ray CT
Principles for CT Image Formation
Basic Principle of Planar X-ray

\[ I = I_o e^{-\mu \Delta x} \]

\[ I = I_o e^{-\mu_1 \Delta x} e^{-\mu_2 \Delta x} \ldots e^{-\mu_n} \]

\[ = I_o e^{-(\mu_1 + \mu_2 + \ldots + \mu_n) \Delta x} \]

\[ \Delta x \to 0, \quad P = - \ln \left( \frac{I}{I_o} \right) = \int_{-\infty}^{\infty} \mu(x) \, dx \]
0.2 % attenuation change detectable in CT Images
Projection Data from Early X-ray CT Systems

Data measured by translating the detector is a typical projection data.

The intensity of the beam after passing through the object:

\[ I = I_0 e^{-\int_a^b \mu(t) \cdot dt} \]

\[ \Leftrightarrow \ln\left(\frac{I_0}{I}\right) = \int_a^b \mu(t) \cdot dt \]

From Computed Tomography, Kalender, 2000.
The value of the projection function $p_\phi(x')$ at this point is the integral of the function of $f(x,y)$ along the straight line:

$$x' = x \cos \phi + y \sin \phi$$

The integral of a line impulse function and a given 2-D signal gives the projection data from a given view ... 

$$p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy$$
Central Slice Theorem

\[ F\{p(\phi, x')\} = F(r, \phi) \]

http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf
Central Slice Theorem

DFT image represents integration of original projections DFT transformed and summed together.

This is the fast way to create the DFT image from projection data. The more projections taken, the more complete the sampling.

http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf
The nature of the 1/r blurring:
Radon transform produced equally spaced radial sampling in Fourier domain.
Review of 2-D Analytical Reconstruction Methods

Projection Data

Projection data $p(\phi, x')$

Incident X-rays

Detected $p(\phi, x')$

\[
p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy
\]
Simple Back-projection and the 1/r Blurring

Figure 11.14. (a) Image produced by back-projecting the sinogram given in figure 11.12(b); (b) the response to a point object in the middle of the field of view.

From Medical Physics and Biomedical Engineering, Brown, IoP Publishing
Simple and Filtered Back-projection

**FIGURE 13-28.** Simple backprojection is shown on the left; only three views are illustrated, but many views are actually used in computed tomography. A profile through the circular object, derived from simple backprojection, shows a characteristic 1/r blurring. With filtered backprojection, the raw projection data are convolved with a convolution kernel and the resulting projection data are used in the backprojection process. When this approach is used, the profile through the circular object demonstrates the crisp edges of the cylinder, which accurately reflects the object being scanned.

Chapters 12 & 13, *The Essential Physics of Medical Imaging*, Bushberg
Simple Backprojection and Inverse Radon Transform

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

\[ b_\phi (x,y) = \int p_\phi (x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

Adding up all the back projections from all the angles gives,

\[ f_{\text{back-projection}} (x,y) = \int b_\phi (x,y) \, d\phi \]

\[ \hat{f}_{\text{simple backprojection}} (x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} p_\phi (x') \cdot \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

\[ \hat{f}_{\text{Inverse Radon transform}} (x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi (x')] \cdot |w| \} \cdot \delta(x \cos \phi + y \sin \phi - x') \, dx' \]
Filtered Back-projection

Figure 3-4  (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.
Filtered Back-Projection

\[ \Delta x = 1/(2B), \]

Sampled version

\[
\begin{align*}
  h_{RL}(0) &= B^2 = \frac{1}{4\Delta x^2} \quad \text{(if } k = 0) \\
  h_{RL}(k) &= 0 \quad \text{(if } k \text{ even)} \\
  h_{RL}(k) &= \frac{-4B^2}{\pi^2k^2} = \frac{-1}{\pi^2k^2\Delta x^2} \quad \text{(if } k \text{ odd)}
\end{align*}
\]

\[
  h_{SL}(k) = \frac{-2}{\pi^2\Delta x^2(4k^2 - 1)}
  = \frac{-8B^2}{\pi^2(4k^2 - 1)}
\]

Figure 3-4  (b) Spatial domain filter kernels corresponding to the filter functions shown in the Ram-Lak filter is a high-pass filter with a sharp response but results in some noise enhancement, while the Shepp-Logan and the Hamming window filters are noise-smoothed filters and therefore have better SNR.
Simple Backprojection and Inverse Radon Transform

Result from a back projection from a single view angle:

\[ b_\phi (x,y) = \int p_\phi (x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

Adding up all the back projections from all the angles gives,

\[ f_{\text{back-projection}} (x,y) = \int b_\phi (x,y) \, d\phi \]

\[ \hat{f}_{\text{simple backprojection}}(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} p_\phi (x') \cdot \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

\[ \hat{f}_{\text{Inverse Radon transform}}(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi (x')] \cdot |w|\} \cdot \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

\[ \hat{f}_{\text{Filtered backprojection}}(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi (x')] \cdot W(w)\} \cdot \delta(x \cos \phi + y \sin \phi - x') \, dx' \]
Evolution of X-Ray CT Instrumentation
Advancements in X-ray CT

One of the first X-ray CT brain images taken in early 1970’s


0.2 % attenuation change detectable in CT Images
Advancements in X-ray CT

• Scan time/slice for consecutive volume coverage has reduced by a factor of 15,000 over the last 30+ years.
• This is a reduction of factor 1.36/year.
• Data acquisition speed doubles every 2.2 years!
Clinical Applications

Reveals nice anatomical details!
Clinical Applications

Always something missing! (?)
Generations of X-ray CT Systems
Electron Beam scanner was built between 1980 and 1984 for cardiac applications.
In helical scanning, the patient is translated at a constant speed while the gantry rotates.

Helical pitch:

\[ h = \frac{q}{d} \]

- \( h \): distance gantry travel in one rotation
- \( q \): collimator aperture

\[ q \]
Generations of X-ray CT Systems

- Multi-slice CT contains multiple detector rows.
- For each gantry rotation, multiple slices of projections are acquired.
- Similar to the single slice configuration, the scan can be taken in either the step-and-shoot mode or helical mode.
- Unlike the single slice, the slice thickness is defined by detector aperture.
Multi-slice Helical Scanning CT

- When acquiring data in a helical mode, the N (4 or higher) detector rows form N interweaving helixes.
- Because multiple detector rows are used in the data acquisition, the acquisition speed is typically higher.
- Similar to the single slice helical, the projection data are inherently inconsistent.
Multi-slice Helical Scanning

- Large coverage and faster scan speed
- Better contrast utilization
- Less patient motion artifacts
- Near-isotropic spatial resolution
Inside an X-ray CT System

- HV tank
- Controller
- Multi-slice detector
- X-ray tube
- Controller
- HV tank
- Data acquisition system
Inside an X-ray CT System

Over 18 millions of samples (400 MBAUD) need to be passed through the slip-ring each second for 8-slice scanner.
For 16-slice, data rate exceeds 800 MBAUD.
Typical Dose Received in CT Scanning

To understand relative radiation exposures and risk, consider common radiation exposures:

| Natural (earth, radon, cosmic rays) in Boston | 3 |
| In Denver | 6 |
| Flying across country one way | .06 |
| Live next door to nuclear power plant | .02 |
| Watching color TV | .03 |

The average person in the U.S. receives a yearly radiation dose of 3 mSv from natural radiation sources (55% of which is from radon) and cosmic radiation. Compare this natural background radiation exposure to typical radiation doses for diagnostic studies.

<table>
<thead>
<tr>
<th>Type X-Ray</th>
<th>Exposure (mSv)</th>
<th>Comparable to natural background radiation for:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chest – PA and Lat</td>
<td>0.06</td>
<td>6 days</td>
</tr>
<tr>
<td>Extremity</td>
<td>0.06</td>
<td>6 days</td>
</tr>
<tr>
<td>Thoracic Spine</td>
<td>0.7 - 1.4</td>
<td>3 months</td>
</tr>
<tr>
<td>Lumbar Spine</td>
<td>1.0 - 1.8</td>
<td>3 months</td>
</tr>
<tr>
<td>Pelvis AP</td>
<td>0.7</td>
<td>2 months</td>
</tr>
<tr>
<td>IVP</td>
<td>1.5-2.5</td>
<td>6 months</td>
</tr>
<tr>
<td>Upper GI</td>
<td>2 - 3</td>
<td>8 months</td>
</tr>
<tr>
<td>Barium Enema</td>
<td>4 - 7</td>
<td>16 months</td>
</tr>
<tr>
<td>Mammogram</td>
<td>0.13</td>
<td>1 month</td>
</tr>
<tr>
<td>CT Head</td>
<td>2</td>
<td>8 months</td>
</tr>
<tr>
<td>CT Chest</td>
<td>8</td>
<td>3 years</td>
</tr>
<tr>
<td>CT Abdomen</td>
<td>10</td>
<td>3 years</td>
</tr>
</tbody>
</table>

http://www.permenate.net/homepage/kaiser/pages/d4933-40810.html
CT & Radiatien

- CT ~ 10% of Xray Examinations
- CT → source of ~ 50% medical related radiation
- Increases stochastic risk for cancer development:
  - 1mSv Totalbody irradiation → risk for lethal cancer is increased by 5%
  - USA: maybe about 500 children dying a year by cancer due to CT
Key Elements of X-ray CT Physics

Part 1: X-ray Source
X-ray Generation – Characteristic X-rays

**Figure 5.5**
Relative intensity of x-ray photons. (Adapted from Webster, 1998. This material is used by permission of John Wiley & Sons, Inc.)
X-ray Generation – X-ray Tube

Figure 5.3
An x-ray tube.

Figure 5.4
Schematic diagram of an x-ray tube.

Motor, Why?
Electron beam? How are electrons generated?

Anode assembly
High voltage
Stator
Cathode assembly
Electrons
Filament
ground
Filament circuit
Rotating target
X-rays

X-ray Generation – Anode Design

**FIGURE 5-13.** The anode (target) angle, $\theta$, is defined as the angle of the target surface in relation to the central ray. The focal spot length, as projected down the central axis, is foreshortened, according to the line focus principle (lower right).

References:
Bushberg text
The Physics of Medical Imaging, Webb, IOP Publ.
Power output of the X-ray tube is crucial for CT applications, due to the stringent need for speed!
To increase the target loading, the anode surface is at a shallow angle with respect to the scanning plane.
Key Elements of X-ray CT Physics

Part 2: X-ray Interactions
**X-ray Linear Attenuation Coefficient**

**Monoenergetic and Narrow Beam Case**

- Suppose the slab is not homogeneous $\Rightarrow \mu$ dependents on $x$ (along the beam direction)

$$\mu = \frac{n/N}{\Delta x},$$

- Assuming monoenergetic

\[ N = N_0 e^{-\mu \Delta x}, \quad \quad I = I_0 e^{-\mu \Delta x}, \]
X-ray Linear Attenuation Coefficient
Polyenergetic and Narrow Beam Case

• The attenuation of a poly energetic X-ray beam by a thin slab of material can be decomposed as x-ray flux at different energies.

\[ S(E) = S_0(E) e^{-\mu(E) \Delta x} \, . \]

• The attenuation of a poly energetic X-ray beam by a heterogeneous slab is then

\[ S(x; E) = S_0(E) \exp \left\{ - \int_0^x \mu(x'; E) \, dx' \right\} \, . \]
X-ray Linear Attenuation Coefficient
Polyenergetic and Narrow Beam Case

- The energy dependent X-ray flux can be converted to the intensity as

\[
I = \int_0^\infty S_0(E') E' \exp \left\{-\mu(E') \Delta x\right\} dE'
\]

- Taking into account the spatial variation of the energy-dependent attenuation coefficient, the remaining beam intensity at a given position \(x\) is

\[
I(x) = \int_0^\infty S_0(E') E' \exp \left\{-\int_0^x \mu(x'; E') \, dx'\right\} dE'.
\]
Key Performance Measures in X-Ray CT

There are many important performance parameters for x-ray computed tomography.

The most important parameters are:

- CT number accuracy
- Spatial resolution
- Signal-to-Noise Ratio in CT Images
- Image artifacts
CT Number is defined/measured as:

\[ h = 1000 \times \frac{\mu - \mu_{\text{water}}}{\mu_{\text{water}}} (\text{HU}), \]  

where the HU is the Hounsfield unit.

CT number \( \rightarrow \) relative attenuation coefficient respect to that of water.

CT number is widely used for quality control purpose, basically,

\( \rightarrow \) The same object should appear with the same CT number in CT images taken with different scanners.

\( \rightarrow \) An uniform object should appear to have homogeneous CT numbers across the entire reconstructed image.

\[ h_{\text{air}} = -1000, \ h_{\text{water}} = 1 \text{ and } h_{\text{bone}} \approx 1000. \]
CT- Resolution – Modulation Transfer Function

• The modulation $m_f$ is an effective way to quantify the contrast of a periodic signal

$$m_f = \frac{f_{\text{max}} - f_{\text{min}}}{f_{\text{max}} + f_{\text{min}}}.$$

• In general, $m_f$ is refer to as the contrast of a periodic signal $f(x,y)$ relative to its average value.

• So within two signals, $f(x,y)$ and $g(x,y)$, with the same average value, $f(x,y)$ is said to have more contrast if $m_f > m_g$. 
Modulation

- Suppose an input signal function

\[ f(x, y) = A + B \sin(2\pi u_0 x), \]

where \( A > B \) and both are non-negative constants.

\[ m_f = \frac{B}{A}. \]

Greater \( m_f \), more contrast
In Plane Spatial Resolution

Resolution of an CT system is measured with the modulation transfer function (MTF):

CT number $\rightarrow$ relative attenuation coefficient respect to that of water.
In Plane Spatial Resolution

Resolution of an CT system is measured with the modulation transfer function (MTF):

$$MTF(u, v) = \frac{m_g}{m_f} = \frac{|H(u, v)|}{H(0, 0)},$$

- A typical MTF of an imaging system

![MTF Graph](image-url)

**Figure 3.3**
A typical MTF of a medical imaging system.
LSF and MTF

- Modulation Transfer Function (MTF).

\[
MTF(u) = \frac{mg}{mf} = \frac{|H(u, 0)|}{H(0, 0)} = \frac{|L(u)|}{L(0)}, \quad \text{for every } u.
\]

- For a “reasonable” imaging system, the \( L(0)=1 \), so that

\[
MTF(u) = L(u)
\]

- MTF is an effective way to compare two imaging systems in terms of spatial resolution and contrast.

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) f(x - \xi, y - \eta) \, d\xi \, d\eta,
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\xi, \eta) \delta(x - \xi) \, d\xi \right] \, d\eta,
\]

\[
= \int_{-\infty}^{\infty} h(x, \eta) \, d\eta, \quad \Xi L(\infty), \quad 2\omega = \mathcal{F}_{1,0}[L(\infty)]
\]
LSF and MTF

- Modulation Transfer Function (MTF).

\[
\text{MTF}(u) = \frac{m_g}{m_f} = \left| \frac{H(u, 0)}{H(0, 0)} \right| = \left| \frac{L(u)}{L(0)} \right|, \quad \text{for every } u. 
\]

- For a “reasonable” imaging system, the \( L(0) = 1 \), so that

\[
\text{MTF}(u) = L(u) 
\]

- MTF is an effective way to compare two imaging systems in terms of spatial resolution and contrast.

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) f(x - \xi, y - \eta) \, d\xi \, d\eta, \\
\quad = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\xi, \eta) \delta(x - \xi) \, d\xi \right] \, d\eta, \\
\quad = \int_{-\infty}^{\infty} h(x, \eta) \, d\eta, \quad \equiv l(\infty), \quad \mathcal{L} \omega = \mathcal{F}_{1,0}[l(\infty)]
\]
System Cascade

- Modulation Transfer Function (MTF).

\[ g(x, y) = h_K(x, y) \ast \cdots \ast (h_2(x, y) \ast (h_1(x, y) \ast f(x, y))) \]

- The overall MTF is the product of the MTF for sub-systems:

\[ MTF_{\text{total}}(u) = \prod_{i=1}^{i=L} MTF_i(u) \]

- MTF of an imaging system that can be modeled as a chain if systems is often determined by the MTF of the worst system in the cascade.

The noise-suppressing filter function sets the upper limit for the MTF attainable with an X-ray CT system!
Effect of The Reconstruction Kernel Function

By modifying the cutoff frequency and the window function of the reconstruction kernel, spatial resolution of the image can be changed.
Helical Scanning

In helical scanning, the patient is translated at a constant speed while the gantry rotates.

Helical pitch:

\[ h = \frac{q}{d} \]

- \( q \) — distance gantry travel in one rotation
- \( d \) — collimator aperture

\[ \text{q} \]
Major Influence Factor for Radial Resolution

- X-ray focal spot size
- Detector size
- System geometry
- Sampling density
- Mechanical alignment
- Stability
The detector and the X-ray tube are moving **continuously**. Extra resolution loss occurs due to imperfection in detector technology.

Detector temporal response
- View sampling rate
- Distance from isocenter

The projection data acquired is the sum of multiple projections corresponding to multiple system angular positions.
Major Influence Factor for Resolution in Z Axis

For step-and-shoot mode, the slice sensitivity profile (SSP) can be approximated by the convolution of the projected focal spot function with the projected detector response.

\[ s(z) = \int f(z - z') d(z') dz' \]

For helical mode, additional table motion and reconstruction weighting need to be considered.
Low Contrast Resolution

Resolution is Meaningful Only With Sufficient SNR!

So what is the SNR in X-ray CT images?
Noise in CT Images is originated from the error in the measured projected data.

How do we write the noise in final images as a function of system design parameters?
SNR in Projection Data

Suppose a first-generation CT system that consists $N$ linear scanning steps and $M$ angular scanning steps. Given an X-ray beam intensity $N_0$ from the source, the average measured intensity is

\[ N_i = N_0 e^{-\int \mu \, dz} \]

$n_0 =$ Incoming photon density

$N_0 = n_0 A$

$i^{th}$ line integral
Error Propagation

In some situations, the variable of interest \((Q)\) is not measured directly, but derived as a function of more than one independent random variables whose values are directly measured. The error on the measured values is propagated into the uncertainty on the resultant quantity \(Q\).

Suppose a quantity \(Q(x,y)\) that depends on two independent random variables \(x\) and \(y\).

The sample mean and variance of variables \(x\) and \(y\) are derived as \(\sigma_x\) and \(\sigma_y\), by repeating measurements.

The standard deviation of the indirect quantity \(Q\) is approximately given by

\[
\sigma_Q^2 \approx \left( \frac{\partial Q}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial Q}{\partial y} \right)^2 \sigma_y^2
\]

\[
\sigma_Q^2 \approx \sum_i \left( \frac{\partial Q}{\partial x_i} \right)^2 \sigma_{x_i}^2
\]
SNR in X-Ray Projection Data

Let \( N_i = n_0 A \exp \int_i -\mu \, dl = N_0 \exp \int_i -\mu \, dl \) where \( A \) is area of detector and \( N_0 = n_0 A \) is the average number of x-ray photons emitting from the source and traveling towards the detector.

The basic measured CT measure is

\[
g_{ij} = \int \mu \, dl = -\ln \left( \frac{N_{ij}}{N_0} \right)
\]

Noting that the \( N_{ij} \) is a Poisson random number, we could use the following approximations:

\[
\text{Mean}(g_{ij}) = \mu \approx -\ln \left( \frac{N_{ij}}{N_0} \right)
\]

\[
\text{Var}(g_{ij}) \approx \frac{1}{\text{Mean}(N_{ij})}
\]
SNR in Projection Data

Noting that the $N_{ij}$ is a Poisson random number, so

$$\text{Mean}(g_{ij}) = \mu \approx -\ln \left( \frac{N_{ij}}{N_0} \right)$$

$$\text{Var}(g_{ij}) \approx 1/\text{Mean}(N_{ij})$$

The projection data is used in the Filtered back-projection reconstruction, let's see how the noise in the projection data is propagated through this process

$$\text{SNR} = C \frac{\mu(x,y)}{\sigma(x,y)}$$
Filtered Backprojection

Remember that the inverse Radon transform is defined as

\[ f(x, y) = \mathcal{R}^{-1}[f(x, y)] = \frac{1}{\pi} \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} |\omega| P_{\phi}(\omega)e^{j2\pi x'x} d\omega \]  

(1)

It (in principle) reproduces the original function \( f(x,y) \).

Compare the inverse Radon transform (above) with the Filtered backprojection operation (below)

\[ \hat{f}(x, y) = \frac{1}{\pi} \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} |\omega| W(\omega)P_{\phi}(\omega)e^{j2\pi x'x} d\omega \]  

(2)

where \( x' = x \cos \phi + y \sin \phi \)
Discrete Filtered Backprojection

\[ f(x, y) = \frac{1}{\pi} \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} \left| \omega \right| W(\omega) P_{\phi}(\omega) e^{j2\pi x'} d\omega \]

\[ f(x, y) = \frac{1}{\pi} \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} dx' \ p_{\phi}(x') h(x \cos \phi + y \sin \phi - x') \]

\[ \hat{f}(x, y) = \sum_{j=1}^{M} \left[ \sum_{i=-N/2}^{N/2} g_{ij} c(x \cos \phi_{j} + y \sin \phi_{j} - iT) \cdot T \right] \Delta \phi \]

\[ = \frac{\pi T}{M} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} g_{ij} c(x \cos \phi_{j} + y \sin \phi_{j} - iT) \]

M: # of projections taken from 0 → π

T: Physical spacing between detectors

c(·) is the discretized version of the filter function h(x'), whose FT is |\omega|W(\omega)
SNR in Projection Data

Noting that the $N_{ij}$ is a Poisson random number, so

$$\text{Mean}(g_{ij}) = \mu \approx -\ln \left( \frac{N_{ij}}{N_0} \right)$$
$$\text{Var}(g_{ij}) \approx 1/\text{Mean}(N_{ij})$$

The projection data is used in the Filtered back-projection reconstruction, lets see how the noise in the projection data is propagated through this process

$$\text{SNR} = C \frac{\mu(f_i)}{\sigma(f_i)}$$
Discrete Filtered Backprojection

Note that this equation is a linear function of the projection data $g_{ij}$.

Therefore, the mean of the estimated attenuation coefficient at a given location $(x,y)$ is simply

$$\text{mean}[\hat{\mu}(x, y)] = \frac{\pi T}{M} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} \bar{g}_{ij} c(x\cos\phi_j + y\sin\phi_j - iT)$$

Noting that the $N_{ij}$ is a Poisson random number, so

$$\text{Mean}(g_{ij}) = \mu \approx \ln\left(\frac{N_{ij}}{N_0}\right)$$

$$\text{Var}(g_{ij}) \approx \frac{1}{\text{Mean}(N_{ij})}$$
Discrete Filtered Backprojection

Now, let’s consider the variance on the estimated attenuation coefficients.

The first thing to notice is that all measured data \( g_{ij} \) are assumed to be independent Poisson variables and \( c(\cdot) \) terms are deterministic. So the above-mentioned estimation process is essentially the sum of multiple independent random variables

\[
\hat{\mu}(x, y) = \frac{\pi T}{M} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} g_{ij} c(x \cos \phi_j + y \sin \phi_j - iT)
\]

The variance on the sum of a number of discrete random variables are the sum of individual variables
Discrete Filtered Backprojection

Note that

1. The variance of the sum of independent random variables is the sum of individual variances for each variables.

\[ \text{Var}\left[ \sum x_i \right] = \sum \text{Var}[x_i], \quad \text{when } x_i \text{s are independent.} \]

2. The variance of a random variable multiplied by a constant is the variance of the random variable times the square of the constant

\[ \text{Var}[ax] = a^2 \text{Var}[x] \]

So

\[ \text{Var}[\hat{\mu}(x, y)] = \text{Var}\left[ \frac{\pi T}{M} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} g_{ij} c(x \cos \varphi_j + y \sin \varphi_j - iT) \right] \]

\[ = \left( \frac{\pi T}{M} \right)^2 \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} \text{Var}(g_{ij}) \cdot \left[ c(x \cos \varphi_j + y \sin \varphi_j - iT) \right]^2 \]
Discrete Filtered Backprojection

Remember that

\[
\text{Mean}(g_{ij}) = \mu \approx -\ln \left( \frac{N_{ij}}{N_0} \right)
\]

\[
\text{Var}(g_{ij}) \approx 1/\text{Mean}(N_{ij})
\]

So

\[
\text{Var}[\hat{\mu}(x, y)] = \left( \frac{\pi T}{M} \right)^2 \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} \frac{1}{\overline{N}_{ij}} \cdot [c(x \cos \varphi_j + y \sin \varphi_j - iT)]^2
\]

To proceed, we assume that the object is uniform, so that the number of photons detected by all detector elements are also uniform. Therefore, we would make a “bad” assumption

\[
\overline{N}_{ij} = \overline{N}
\]

This is clearly wrong mathematically, since most of objects do not satisfy this assumption. But in practice, this is valid since one is interested in local variation of the attenuation of the object ...
Discrete Filtered Backprojection

With this assumption, we get

\[ \text{Var}[\hat{\mu}(x, y)] = \left( \frac{\pi T}{M} \right)^2 \frac{1}{N} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} [c(x \cos \phi_j + y \sin \phi_j - iT)]^2 \]

\[ = \frac{\pi T}{M} \frac{1}{N} \sum_{j=1}^{M} \left\{ \sum_{i=-N/2}^{N/2} [c(x \cos \phi_j + y \sin \phi_j - iT)]^2 T \right\} \Delta \phi \]

\[ = \frac{\pi T}{M} \frac{1}{N} \int_0^\pi d\phi \int_{-\infty}^{\infty} \left[ c(x \cos \phi_j + y \sin \phi_j - x') \right]^2 dx' \]

\[ = \frac{\pi T}{M} \frac{1}{N} \int_0^\pi d\phi \int_{-\infty}^{\infty} \left[ c(x') \right]^2 dx' = \frac{\pi^2 T}{M} \frac{1}{N} \int_{-\infty}^{\infty} \left[ c(x') \right]^2 dx' \]

Using the Parseval’s theorem

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u, v)|^2 \, du \, dv \]
Discrete Filtered Backprojection

\[
\hat{f}(x, y) = \frac{1}{\pi} \int_0^\pi d\phi \int_{-\infty}^{\infty} \omega |W(\omega)P_\phi(\omega)e^{-j2\pi x'}| d\omega
\]

\[
\hat{f}(x, y) = \frac{1}{\pi} \int_0^\pi d\phi \int_{-\infty}^{\infty} dx' p_\phi(x') h(x \cos \phi + y \sin \phi - x')
\]

\[
\hat{f}(x, y) = \sum_{j=1}^{M} \left[ \sum_{i=-N/2}^{N/2} g_{ij} c(x \cos \phi_j + y \sin \phi_j - iT) \cdot T \right] \Delta \phi
\]

\[
= \frac{\pi T}{M} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} g_{ij} c(x \cos \phi_j + y \sin \phi_j - iT)
\]

M: # of projections taken from \( 0 \rightarrow \pi \)

T: Physical spacing between detectors

c(\cdot) is the discretized version of the filter function \( h(x') \), whose FT is \( |\omega|W(\omega) \)

NPRE 435, Principles of Imaging with Ionizing Radiation, Fall 2022
Image Variance with Filtered Backprojection

We have

\[
\text{Var}[\hat{\mu}(x, y)] = \frac{\pi T}{M} \frac{1}{N} \int_0^\pi \int_{-\infty}^{\infty} \left[ c(x \cos \phi_j + y \sin \phi_j - x') \right]^2 dx'
\]

\[
= \frac{\pi T}{M} \frac{1}{N} \pi \int_{-\infty}^{\infty} \left[ c(x \cos \phi_j + y \sin \phi_j - x') \right]^2 dx'
\]

\[
= \frac{\pi T}{M} \frac{1}{N} \pi \int_{-\infty}^{\infty} [C(\omega)]^2 d\omega
\]

Remember that

\[
\hat{f}(x, y) = \frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\infty} \omega |W(\omega)P_\phi(\omega)e^{j2\pi wx'} d\omega
\]

and

\[
C(\omega) = |\omega| W(\omega)
\]

We can now derive the variance in image (attenuation function) for given filter functions...
Image Variance with Filtered Backprojection

For a Ram-Lak filter function

\[ W_{RL} = \begin{cases} 
|\omega|, & |\omega| < \omega_c \\
0, & \text{otherwise}
\end{cases} \]

Figure 3-4  (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.
If we assume that a Ram-Lak filter is used, we have

\[
Var[\hat{\mu}(x, y)] = \frac{\pi T}{M} \frac{1}{N} \pi \int_{-\infty}^{\infty} [C(\omega)]^2 d\omega
\]

\[
= \frac{\pi T}{M} \frac{1}{N} \frac{\omega_c}{\pi} \int_{-\omega_c}^{\omega_c} |\omega|^2 d\omega
\]

\[
= \frac{1}{M} \frac{1}{N/T} \frac{2\pi^2}{3} \omega_c^3
\]

Let’s see how things folded together ...
1. $1/M$: increasing the number of angular sampling always reduce noise.

2. $\bar{N}/T$ term ($\propto$ incident X-ray intensity): Number of photons detected on each detector element of width $T$.

3. $\omega_c^3$ term: accepting higher frequencies imaging features always increase noise.

$$Var[\hat{\mu}(x, y)] = \frac{1}{M} \frac{1}{\bar{N}/T} \frac{2\pi^2}{3} \omega_c^3$$

- Number of angular samplings
- Detector width
- Cutoff frequency
- No. of counts per detector of width $T$
SNR with Filtered Backprojection

\[ \bar{\mu}(x, y) \equiv \text{Mean}[\hat{\mu}(x, y)] = \frac{\pi T}{M} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} \tilde{g}_{ij} c(x \cos \varphi_j + y \sin \varphi_j - iT) \]

\[ \text{Var}[\hat{\mu}(x, y)] = \left( \frac{\pi T}{M} \right)^2 \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} \frac{1}{\tilde{N}_{ij}} \cdot [c(x \cos \varphi_j + y \sin \varphi_j - iT)]^2 \]

\[ \approx \frac{\pi^2 T}{M} \frac{1}{\tilde{N}} \int_{-\infty}^{\infty} [C(\omega)]^2 d\omega \]

\[ = \frac{1}{M \tilde{N}/T} \frac{2\pi^2}{3} \omega_c^3 \] (for Ram-Lak filter)

So the SNR for CT image reconstructed using Ram-Lak filter is

\[ \text{SNR}[\hat{\mu}(x, y)] = C \frac{\bar{\mu}(x, y)}{\sigma(\mu)} = C \frac{\bar{\mu}(x, y)}{\pi} \omega_c^{-3/2} \sqrt{\frac{3}{2} \left( \frac{\tilde{N}}{T} \right) \frac{M}{3}} \]

\[ \text{Width of detector element} \]

\[ \text{Number of angular samplings} \]
So the SNR for CT image reconstructed using Ram-Lak filter is

\[ SNR[\hat{\mu}(x, y)] = C \frac{\bar{\mu}(x, y)}{\sigma(\mu)} = \frac{C \bar{\mu}(x, y)}{\pi} \omega_C^{-3/2} \sqrt{\frac{3}{2} \left( \bar{N} / T \right) M} \]

So, want a better image SNR,

- Higher dose (if possible)
- More angular sampling
SNR with Filtered Backprojection

In reality, there is no point of setting the cutoff frequency higher than the highest frequency contained in the data.

Suppose, the detector width is d, the cutoff frequency is normally $\omega_c \approx k/d$, and $k \approx 1$. In this case the SNR is

$$SNR[\hat{\mu}(x, y)] \approx \frac{C \bar{\mu}(x, y)}{\pi} \frac{\omega_c^{-3/2}}{\omega_c^3} \sqrt{\frac{3}{2} (\bar{N}/T)M} \approx 0.4C \bar{\mu}(x, y)d \sqrt{(\bar{N}/T)M}$$

where

$$\bar{\mu}(x, y) = \frac{\pi T}{M} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} g_{ij} c(x \cos \phi_j + y \sin \phi_j - iT)$$
SNR and Radiation Dose

\[ SNR[\hat{\mu}(x, y)] \approx \frac{C \bar{\mu}}{\pi} \omega_c^{-3/2} \sqrt{\frac{3}{2}} \left( \frac{\bar{N}}{T} \right) M \]

This term here is proportional to the total radiation dose delivered to the center of the object. Consider the finite thickness (\( \Delta Z \)) of the slice along the axis,

\[ Dose(0, 0) \propto \left( \frac{SNR[\hat{\mu}(x, y)]}{\hat{\mu}(x, y)} \right)^2 \frac{1}{\Delta Z} \omega_c^3 \]
SNR and Radiation Dose

\[ Dose(0,0) \propto \left( \frac{SNR[\hat{\mu}(x, y)]}{\bar{\mu}(x, y)} \right)^2 \frac{1}{\Delta Z} \omega_c^3 \]

Normally, the cutoff frequency \( (\omega_c) \) is

\[ \omega_c \approx \frac{1}{2r} \]

\( \Delta Z \) : Thickness of the slice

The “spatial resolution” required in the image

To maintain an isotropic resolution, one would also reconstruct the image with a thickness comparable to the resolution achieved in-plane, therefore,

\[ Dose(0,0) \propto \left( \frac{SNR[\hat{\mu}(x, y)]}{\bar{\mu}} \right)^2 \frac{1}{r^4} \]

To improve the spatial resolution by a factor of 2 while maintaining the same SNR, one would need to increase the dose by a factor of 16!!
Discrete Filtered Backprojection

Note that this equation is a linear function of the projection data $g_{ij}$.

Therefore, the mean of the estimated attenuation coefficient at a given location $(x,y)$ is simply

$$\text{mean}[\hat{\mu}(x, y)] = \frac{\pi T}{M} \sum_{j=1}^{M} \sum_{i=-N/2}^{N/2} \overline{g}_{ij} c(x \cos \phi_j + y \sin \phi_j - iT)$$

Substituting $g_{ij}$ with its mean