Chapter 2: Mathematical Preliminaries
Contents

- Signals and Systems.
- Fourier Transform Basics
- Analytical Image Reconstructing Techniques
- Iterative Reconstruction Methods
- Image Quality Assessment and System Optimization.

Signals and Systems

Reading Material:
Chapter 2 in
Medical Imaging Signal and Systems, 2’nd Ed.
J. L. Prince et. al,
Key Questions and Concepts related to Signal:

• **Impulse signal** (δ-signal) and its properties.

• **Sampling function** and its application on the continuous-to-discrete operation.

• What are **line-impulse function, square function, sinc function**?

• **Separable signals and periodic signals**.
The Basic Problems in Imaging

The forward problem: Given an input signal and the known response of a imaging system, what is the output is going to be?

The inverse problem: Given a output signal and the known system response, what should be the input signal that gave rise to the output data?
How to Improve the Tradeoff between Spatial Resolution and Sensitivity?

The idea of multiplexing –

- Each detected photon no longer corresponds to a unique emission location in the 2-D source plane.
- Information content per detected photon is decreased.
- No of detected photons is increased.
Introduction to Signals

- Continuous signal:

  A continuous 2-D signal

  \[ f(x, y), \quad -\infty \leq x, y \leq \infty \]

- Discrete signal: Pixel and voxel representations of a continuous signal
Basic Idea of Fourier Transform and Its Implications on Signal Processing
Continuous Fourier Transform

• For any square-integrable function \( f(x,y) \), a continuous Fourier transform is defined as

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} \, dx \, dy
\]

where \( j = \sqrt{-1} \)

• We can also define an inverse Fourier transform as

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} \, du \, dv
\]

• Both \( f(x,y) \) and \( F(u,v) \) have infinite support.

• Both \( f(x,y) \) and \( F(u,v) \) are defined on a continuum of values.

• \( f(x,y) \) and \( F(u,v) \) must contain the same information.

\[
e^{-j \cdot 2\pi(ux + vy)} = \cos[2\pi(ux + vy)] - j \cdot \sin[2\pi(ux + vy)]
\]
Continuous Fourier Transform

www.revisemri.com
Discrete Fourier Transform in 1-D

The discrete Fourier transform (DFT) is defined as

$$F_n = \sum_{k=0}^{N-1} f_k e^{-j \frac{2\pi nk}{N}}, \ n = 0,1,2,...N-1$$

$n = 0$ corresponding to the DC component (spatial frequency is zero)
$n = 1,..., N/2 - 1$ are corresponding to the positive frequencies $0 < u < u_c$
$n = N/2, ..., N - 1$ are corresponding to the negative frequencies $-u_c < u < 0$

The inverse DFT is defined as

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{j \frac{2\pi nk}{N}}, \ k = 0,1,2,...N-1$$

$$e^{-j \frac{2\pi nk}{N}} = \cos \left[ \frac{2\pi nk}{N} \right] - j \cdot \sin \left[ \frac{2\pi nk}{N} \right]$$
Point Impulse Signal

- A point source is mathematically represented by the delta function or Dirac function.

\[
\delta(x, y) = \begin{cases} 
0, & x = 0 \text{ and } y = 0 \\
\neq 0, & \text{otherwise}
\end{cases}
\]

and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) \, dx \, dy = 1
\]
Point Impulse Signal

\[ \delta(x) = \lim_{\alpha \to \infty} \alpha \ e^{-\pi \alpha^2 x^2} \]

\[ \delta(x) = \lim_{\alpha \to \infty} \frac{\alpha}{\pi \ \text{sinc}(\alpha x)}; \ \text{sinc}(x) = \frac{\sin(x)}{x} \]
Point Impulse Signal

• The sampling property

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x - \xi, y - \eta) \, dx \, dy = f(\xi, \eta) \]

• The scaling property

\[ \delta(ax, by) = \frac{1}{|ab|} \delta(x, y) \]

\[ \delta(x, y) \begin{cases} \neq 0, & x = 0 \text{ and } y = 0 \\ = 0, & \text{otherwise} \end{cases} \]

and

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) \, dx \, dy = 1 \]
Comb and Sampling Function

- The 2-D comb function

\[
comb(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n)
\]
Comb and Sampling Function

- The 2-D sampling function

\[ \delta_s(x, y, \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y) \]

where \( \Delta x \) and \( \Delta y \) are the sampling intervals

\[ \delta_s(x, y, \Delta x, \Delta y) = \frac{1}{\Delta x \Delta y} \text{comb} \left( \frac{x}{\Delta x}, \frac{y}{\Delta y} \right) \]

\[ \text{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) \]

\[ \delta(ax, by) = \frac{1}{|ab|} \delta(x, y) \]
Line Impulse Signal (1)

\[ \delta_L(x, y) = \delta(x \cos \theta + y \sin \theta - l) \]

where \( \delta(x) = \begin{cases} > 0, & x \cos \theta + y \sin \theta = l \\ 0, & \text{otherwise} \end{cases} \)
Rect Function

• Rect function:

\[
rect(x, y) = \begin{cases} 
1, & \text{for } |x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\
0, & \text{otherwise}
\end{cases}
\]

• It is normally used to pick up a particulate section of a given function:

\[
f(x, y) \cdot rect\left(\frac{x - \xi}{w_X}, \frac{y - \eta}{w_Y}\right)
\]
Sinc Function

• The sinc function is defined as

\[ \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \]

• The sinc function is normalized.

\[ \int_{-\infty}^{\infty} \text{sinc}(x) \, dx = 1 \]

Any arbitrary band-limited signal can be written as a weighted sum of multiple sinc functions ... (the Nyquist Sampling Theorem)
Triangular Signals and Gaussian Signals

- **Triangular function:**

\[
Tri\left(\frac{x}{2L}\right) = 1 - \frac{|x|}{L} \quad \text{for } |x| < L \\
= 0 \quad \text{for } |x| > L
\]

- **Normalized Gaussian function:**

\[
G_{1D}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} \\
G_{2D}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}
\]
Separable Signals and Periodic Signals

- The separable signals is a class of continuous signals that satisfy

\[ f(x, y) = f_1(x) \cdot f_2(y) \]

- A signal is periodic if

\[ f(x, y) = f(x + X, y) = f(x, y + Y) \]

where \( X \) and \( Y \) are the signal periods
Key Questions and Concepts related to System:

- What is a **linear system** and what is a **shift-invariance system**?
- What is the **impulse response function**?
- For a linear and shift-invariant system, how to connect the input to the output signal?
General Concept of a System

- A continuous-to-continuous system is defined as
  
  \[ g(x, y) = \mathcal{S}[f(x, y)] \]

- A system is a mapping process from an input signal to the output signal
Linear Systems

A system is linear if it satisfies the superposition principle

\[
S \left[ \sum_{i=1}^{I} w_i \cdot f_i(x, y) \right] = \sum_{i=1}^{I} w_i \cdot S[f_i(x, y)]
\]

where

\( f(x, y) \) is the input signal,
\( S[\cdot] \) is an operator that represents the system,
\( f(x, y) \) is the total input signal and
\( w_i \)s are weighting factors.
Linear Systems – An Example

For example, consider an amplifier with gain $A$:

\[ S[w_1f_1 + w_2f_2] = A(w_1f_1 + w_2f_2) \]
\[ = Aw_1f_1 + Aw_2f_2 = w_1S[f_1] + w_2S[f_2] \]

It satisfies the Superposition Principle

\[ S \left[ f(x, y) = \sum_{i=1}^{L} w_i \cdot f_i(x, y) \right] = \sum_{i=1}^{L} w_i \cdot S[f_i(x, y)] \]
Linear Systems – Why Important?

- Linear systems is mathematically more “tractable”.

\[
    f(x, y) \leftarrow S \left[ f(x, y) = \sum_{i=1}^{I} w_i \cdot f_i(x, y) \right] \Rightarrow g(x, y)
\]

- Many imaging systems used in medical and other applications can be described as linear systems.
Linear Systems – Why Important?

• Linear systems satisfy the Superposition Principle.

\[
g(x,y) = S[f(x,y)] = S\left[ \sum_{i=1}^{I} w_i \cdot f_i(x,y) \right] = \sum_{i=1}^{I} w_i \cdot S[f_i(x,y)]
\]

• It would be good if we can decompose an arbitrary signal into a linear combination of a series of basis functions – such as the δ-function.

• If one can derive the response of the system to this basis function,

• then the response of a system to the arbitrary input signal should easily follow …
Continuous Fourier Transform
Impulse Response Function

One of the most common shape for impulse responses used in imaging application
Impulse Response Function

For a linear system, knowing the IRF, one could compute the output from any arbitrary input function as

\[
g(x, y) = S[f(x, y)] = S \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta(x - \xi, y - \eta) d\xi d\eta \right)
\]

The Sampling property of the Delta function

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x - \xi, y - \eta) dx dy = f(\xi, \eta)
\]

or

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta(\xi - x, \eta - y) dx dy = f(x, y)
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta(\xi - x, \eta - y) dx dy = f(x, y)
\]

Assuming the system is linear
Impulse Response Function (IRF)

For a linear system, knowing the IRF enables one to compute the output from any arbitrary input function.

\[ g(x, y) = \mathcal{S}[f(x, y)] \]

\[ = \mathcal{S}\left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta(x - \xi, y - \eta) d\xi d\eta \right] \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{S}[f(\xi, \eta) \delta(x - \xi, y - \eta)] d\xi d\eta \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \mathcal{S}[\delta(x - \xi, y - \eta)] d\xi d\eta \]

The impulse response function is defined as

\[ h(x, y, \xi, \eta) \equiv \mathcal{S}[\delta(x - \xi, y - \eta)] \]
Impulse Response Function

• For a 2-D problem, the impulse response is a 4-D function.

\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta)h(x, y, \xi, \eta) \, d\xi \, d\eta \]

• The computation can be greatly reduced with further simplifications ...

What exactly is this function again? What does it tell us about the system?
Shift Invariant Systems

Scanner

object (e.g. an organ or tumor)

shift

Final Image from Scanner

shift invariant (no change)

shift variant (changes with position)
Shift Invariant Systems

Shift-Invariance Rule

Original input
Sound Pressure Level

Output
Electrical Activity

time

time

Original input, later in time
Sound Pressure Level

Output, later in time
Electrical Activity

time

time
Shift Invariant Systems (II)

- A system is called shift-invariant if

\[ g(x - \Delta x, y - \Delta y) = S[f(x - \Delta x, y - \Delta y)] \]

- Shift-invariance does not require or imply linearity

- The impulse response function of a shift-invariant system is

\[ h(x, y, \xi, \eta) = S[\delta_{\xi,\eta}(x, y)] = h(x - \xi, y - \eta) \]

where

\[ h(x, y) \equiv S[\delta(x, y)] \]

4-D \( \rightarrow \) 2-D
Shift Invariant Systems (III)

• The impulse response function of a shift invariant system is

\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta)h(x, y, \xi, \eta)d\xi d\eta \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta)h(x - \xi, y - \eta)d\xi d\eta \]

\[ = f(x, y) * h(x, y) \]

• The output of a linear and shift-invariant system is the input convolved with the impulse response function.
Connection of LSI Systems

LSI systems may be decomposed into the combination of multiple sub-systems. This may lead to a simplified mathematical representation of the complete system...

\[
g(x, y) = [h_1(x, y) + h_2(x, y)] * f(x, y) = [h_2(x, y) + h_1(x, y)] * f(x, y)
\]

\[
g(x, y) = h_2(x, y) * [h_1(x, y) * f(x, y)] = h_1(x, y) * [h_2(x, y) * f(x, y)]
\]
Separable Systems

A system is called separable if

\[ h(x, y) = h_1(x)h_2(y) \]

in which case, the convolution between the input and the impulse response function is
Fourier Transform and Sampling

Reading Material:
Chapter 2, Medical Imaging Signals and Systems, 2’nd Edition,
Key Questions and Concepts related to Fourier Transform:

• What is and how to perform Fourier transform on a signal?

• Properties of Fourier transform, such as the linearity property, shifting property, and scaling property.

• The concept of spatial frequency.

• The Convolution theorem, and its implication on (a) modeling of a system, (b) filtering of signals ...
Continuous Fourier Transform

A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

So what does this mean exactly?

Let’s start with an example...in 1-D

Notice that it is symmetric around the central point and that the amount of points radiating outward correspond to the distinct frequencies used in creating the image.

Since this object can be made up of 3 fundamental frequencies an ideal Fourier Transform would look something like this:

- - - - - - - -

Increasing Frequency  Increasing Frequency

Notice that it is symmetric around the central point and that the amount of points radiating outward correspond to the distinct frequencies used in creating the image.

Can be represented by:

When you let these three waves interfere with each other you get your original wave function!
Fourier Transform – What and Why?

What is Fourier Transform?
• A function can be described by a summation of waves with different frequency, amplitudes and phases.

The importance of Fourier Transform in Imaging?
• Signal representations in the frequency domain provide unique information.
• Certain computations can be performed more efficiently in frequency domain.
• Certain hardware naturally measures signals in the frequency domain.
Continuous Fourier Transform

• For any square-integrable function \( f(x, y) \), a continuous Fourier transform is defined as

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)} \, dx \, dy
\]

where \( j = \sqrt{-1} \)

• We can also define an inverse Fourier transform as

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{j2\pi(ux+vy)} \, du \, dv
\]

• Both \( f(x, y) \) and \( F(u, v) \) have infinite support.

• Both \( f(x, y) \) and \( F(u, v) \) are defined on a continuum of values.

• \( f(x, y) \) and \( F(u, v) \) must contain the same information.
Fourier Transform and Spatial Frequency

- Fourier transform can be viewed as a decomposition of the function $f(x,y)$ into a linear combination of complex exponentials with strength $F(u,v)$.

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{j2\pi(ux+vy)} dudv$$

- Fourier transform provides information on the sinusoidal composition of a signal at different spatial frequencies.

$$e^{j2\pi(ux+vy)} = \cos[2\pi(ux + vy)] + j \sin[2\pi(ux + vy)]$$

- $F(u,v)$ is normally referred to as the spectrum of the function $f(x,y)$. 
Fourier transform provides information on the sinusoidal composition of a signal at different spatial frequencies.
Continuous Fourier Transform
An Example

Signal

Magnitude spectrum

Decreasing high-frequency content
# Basic Fourier Transform Pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta(u, v)$</td>
</tr>
<tr>
<td>$\delta(x, y)$</td>
<td>1</td>
</tr>
<tr>
<td>$\delta(x - x_0, y - y_0)$</td>
<td>$e^{-j2\pi(ux_0 + vy_0)}$</td>
</tr>
<tr>
<td>$\delta_s(x, y; \Delta x, \Delta y)$</td>
<td>$\text{comb}(u\Delta x, v\Delta y)$</td>
</tr>
<tr>
<td>$e^{j2\pi(u_0x + v_0y)}$</td>
<td>$\delta(u - u_0, v - v_0)$</td>
</tr>
<tr>
<td>$\sin[2\pi(u_0x + v_0y)]$</td>
<td>$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$</td>
</tr>
<tr>
<td>$\cos[2\pi(u_0x + v_0y)]$</td>
<td>$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$</td>
</tr>
<tr>
<td>$\text{rect}(x, y)$</td>
<td>$\text{sinc}(u, v)$</td>
</tr>
<tr>
<td>$\text{sinc}(x, y)$</td>
<td>$\text{rect}(u, v)$</td>
</tr>
<tr>
<td>$\text{comb}(x, y)$</td>
<td>$\text{comb}(u, v)$</td>
</tr>
<tr>
<td>$e^{-\pi(x^2 + y^2)}$</td>
<td>$e^{-\pi(u^2 + v^2)}$</td>
</tr>
</tbody>
</table>
Properties of Fourier Transform (1)

- **Linearity**

\[
F[a_1 f(x, y) + a_2 g(x, y)] = a_1 F[f(x, y)] + a_2 F[g(x, y)]
\]

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} \, dx \, dy
\]

where \( j = \sqrt{-1} \)

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} \, du \, dv
\]
Properties of Fourier Transform (2)

Shifting Property – Shift in spatial domain is equivalent to phase change in spatial frequency domain.

\[ \mathcal{F} \left[ f(x-x_0, y-y_0) \right] = \mathcal{F} \left[ f(u,v) \right] e^{-j2\pi(ux_0 + vy_0)} \]

An example

\[ \Lambda(x/16)\Lambda(y/16) \]

real and even

\[ F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)} \, dx \, dy \]

where \( j = \sqrt{-1} \)
Properties of Fourier Transform

Real\{F(u,v)\} = 256 \text{sinc}^2(16u)\text{sinc}^2(16v) \quad \text{Imag}\{F(u,v)\} = 0

\begin{align*}
|F(u,v)| &= \sqrt{\left\{\text{Real}[F(u,v)]\right\}^2 + \left\{\text{Imag}[F(u,v)]\right\}^2} \\
&= 256 \text{sinc}^2(16u)\text{sinc}^2(16v)
\end{align*}

\begin{align*}
\angle F(u,v) &= \tan^{-1} \frac{\text{Imag}[F(u,v)]}{\text{Real}[F(u,v)]} = 0
\end{align*}
Properties of Fourier Transform

\[ |F(u,v)| = 256 \text{sinc}^2(16u)\text{sinc}^2(16v) \]

and

\[ \angle F(u,v) = -2\pi u \]

\[ \Lambda[(x-1)/16] \Lambda[y/16] \]
shifted by 1
Properties of Fourier Transform

1-D Gaussian function: \( f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \)

Spatial Domain

A Gaussian transforms to a Gaussian

Spatial Frequency Domain

\[ \mathcal{F}[f(x-x_0, y-y_0)] = \mathcal{F}[f(u, v)] e^{-j2\pi(ux_0 + vy_0)} \]

Spectral phase is zero

Magnitude is a Gaussian
Properties of Fourier Transform

\[ F[f(x - a)] = F[f(x)]e^{j2\pi ua} \]

Spatial Domain

\[
F(u, v) = F_R(u, v) + j \cdot F_I(u, v)
\]
\[
|F(u, v)| = \sqrt{F_R^2(u, v) + F_I^2(u, v)}
\]
\[
\angle F(u, v) = \tan^{-1} \frac{F_I(u, v)}{F_R(u, v)}
\]
\[
F(u, v) = |F(u, v)|e^{j\angle F(u, v)}
\]

Spatial Frequency Domain

Linear shifting in spatial domain simply adds some linear phase to the pulse

Magnitude is unchanged
Properties of Fourier Transform

- Scaling

\[ F[f(ax, by)] = \frac{1}{ab} F\left(\frac{u}{a}, \frac{v}{b}\right) \]

An 1-D example

The shorter the pulse, the broader the spectrum!

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} \, dx \, dy \]

where \( j = \sqrt{-1} \)

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} \, du \, dv \]

Spatial domain, \( f(x) \)  
Fourier Transform, \( F(u) \)
Properties of Fourier Transform

- Scaling

\[ F[f(ax, by)] = \frac{1}{ab} F\left(\frac{u}{a}, \frac{v}{b}\right) \]

An 2-D example

![2-D Rect, 64 width](image1)
![2-D Rect, 8 width](image2)
![2-D Rect, 64 width](image3)
![2-D Rect, 8 width](image4)

FT

Linear scale

Log scale
Linear and Shift Invariant Systems Revisited

For a shift-invariant system, the output is the input convolved with the impulse response function.

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x, y, \xi, \eta) d\xi d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta
\]

\[
= f(x, y) * h(x, y)
\]

where \( h(x, y, \xi, \eta) \) is the response of the system to an delta impulse signal.
Convolution Theorem

• The convolution of two 2-D functions is defined as

\[ f(x, y) * h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot h(x - \xi, y - \eta) d\xi d\eta \]

• The Fourier transform of the convolution is equal to the product of the individual Fourier transforms:

\[ \mathcal{F}[f(x, y) * h(x, y)] = \mathcal{F}[f(x, y)] \cdot \mathcal{F}[h(x, y)] \]

where \( \mathcal{F}[\cdot] \) is the Fourier transform operator.

The output from a **linear and shift-invariant** system is therefore

\[ g(x, y) = f(x, y) * h(x, y) = \mathcal{F}^{-1} \left\{ \mathcal{F}[f(x, y)] \cdot \mathcal{F}[h(x, y)] \right\} \]
Convolution Theorem

Proof:

\[
\mathbf{F} \left[ f(x, y) * g(x, y) \right] \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot g(x - \xi, y - \eta) \, d\xi \, d\eta \right\} e^{-j2\pi(ux + vy)} \, dx \, dy \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot g(x', y') \cdot e^{-j2\pi[u(x' + \xi) + v(y' + \eta)]} \, dx' \, dy' \right\} \, d\xi \, d\eta \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot e^{-j2\pi(ux + vy)} \cdot g(x', y') \cdot e^{-j2\pi(ux' + vy')} \, dx' \, dy' \right\} \, d\xi \, d\eta \\
= \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot e^{-j2\pi(ux + vy)} \, d\xi \, d\eta \right] \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x', y') \cdot e^{-j2\pi(ux' + vy')} \, dx' \, dy' \right] \\
= \mathbf{F} \left[ f(x, y) \right] \cdot \mathbf{F} \left[ g(x, y) \right]
\]
Properties of Fourier Transform

• **Product**

\[
\mathbf{F}\left[ f(x,y) \cdot g(x,y) \right] = \mathbf{F}\left[ f(x,y) \right] * \mathbf{F}\left[ g(x,y) \right]
\]

Fourier transform of the product of two functions equals to the convolution of the Fourier transforms of individual function.

• **The Convolution Theorem**

\[
\mathbf{F}\left[ f(x,y) * g(x,y) \right] = \mathbf{F}\left[ f(x,y) \right] \cdot \mathbf{F}\left[ g(x,y) \right]
\]
\[ \mathbf{F}[f(x,y) \cdot g(x,y)] = \mathbf{F}[f(x,y)] \ast \mathbf{F}[g(x,y)] \]

Proof:

The convolution theorem states:
\[ \mathbf{F}[f(x,y) \ast g(x,y)] = \mathbf{F}[f(x,y)] \cdot \mathbf{F}[g(x,y)] \]

Similarly, we can prove that:
\[ \mathbf{F}^{-1}[f(x,y) \ast g(x,y)] = \mathbf{F}^{-1}[f(x,y)] \cdot \mathbf{F}^{-1}[g(x,y)] \]

If we define:
\[ F = \mathbf{F}^{-1}[f(x,y)], G = \mathbf{F}^{-1}[g(x,y)], \text{ then } f(x,y) = \mathbf{F}[F], g(x,y) = \mathbf{F}[G] \]

We can see that
\[ \mathbf{F}^{-1}\{\mathbf{F}[F] \ast \mathbf{F}[G]\} = F \cdot G \]

Therefore
\[ \mathbf{F}[F \cdot G] = \mathbf{F}[F] \ast \mathbf{F}[G] \]

Note that \( F \) and \( G \) are arbitrary functions, so that we can rewrite the above equation as
\[ \mathbf{F}[f(x,y) \cdot g(x,y)] = \mathbf{F}[f(x,y)] \ast \mathbf{F}[g(x,y)] \]
The convolution theorem enables one to perform convolution operation as multiplication process in spatial frequency domain.

\[ f(x, y) \ast h(x, y) = F^{-1} \{ F[f(x, y)] \cdot F[h(x, y)] \} \]

By using the Fast Fourier Transform (FFT) algorithms, the convolution operation can be performed very efficiently!! This provide a practical way for modeling linear shift-invariant systems ...

\[ g(x, y) = f(x, y) \ast h(x, y) \]
The Fourier transform of the impulse response function $h$ is called the system transfer function.

$$H(u, v) = \mathcal{F}[h(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) e^{-2\pi j (ux + vy)} \, dx \, dy$$

$$G(u, v) = F(u, v)H(u, v)$$
System Transfer Function

\[ G(u, v) = F(u, v)H(u, v) \]

An ideal low-pass filter is defined as

\[
H(u, v) = \begin{cases} 
1 & \text{for } \sqrt{u^2 + v^2} \leq c \\
0 & \text{for } \sqrt{u^2 + v^2} > c 
\end{cases}
\]

c is called the cut-off frequency.
Figure 2.12
The response of an ideal low-pass filter for two values of the cutoff frequency $c$ ($c_1 > c_2$).
Spatial Frequency Revisited

Image

Fourier Space (log magnitude)

Detail

Contrast
Image

Fourier Space (log magnitude)

Detail

Contrast

5%
10%
20%
50%

NPRE 435, Principles of Imaging with Ionizing Radiation, Fall 2022 Fourier Transform
Sampling
Key Questions and Concepts related to **Sampling**:

- How to do *continuous-to-discrete sampling*?
- Understanding the implication of the C-to-D sampling, and from which to answer the important question of *how best to recover from a sampled/discrete signal to its continuous signal*?
- The *Nyquist Sampling theorem*. 
Transformation from continuous signals into discrete signals is called **sampling**. The sampled data can then be processed by digital hardware.

\[
f(x, y) \Rightarrow f(m\Delta x, n\Delta y), \text{ for } m, n = 0, 1, \ldots
\]

where \(\Delta x, \Delta y\) are called sampling intervals or sampling periods.
Sampling in 1-D

Model

\[ f(x) \quad \ast \quad f_s(x) \]
\[ \delta_s(x, \Delta x) \]

The 1-D sampling function

\[ \delta_s(x, \Delta x) = \sum \delta(x - n \cdot \Delta x) \]

The sampled function

\[ f_s(x) = f(x) \cdot \delta_s(x, \Delta x) = \sum f(n \Delta x) \delta(x - n \Delta x) \]
Fourier Transform of Sampled Function

\[ F_{f_s}(u) = \mathcal{F}[f_s(x)] = \mathcal{F}[\delta_s(x, \Delta x) \cdot f(x)] \]

\[ = \text{comb}(u \cdot \Delta x) \ast \mathcal{F}[f(x)] \]

\[ = \left\{ \sum_{n=-\infty}^{n=\infty} \delta \left( \Delta x \left( u - \frac{n}{\Delta x} \right) \right) \right\} \ast \mathcal{F}(u) \]

\[ = \frac{1}{\Delta x} \sum_{n=-\infty}^{n=\infty} \left\{ \delta \left( u - \frac{n}{\Delta x} \right) \ast \mathcal{F}(u) \right\} \]

\[ = \frac{1}{\Delta x} \sum_{n=-\infty}^{n=\infty} \mathcal{F}(u - \frac{n}{\Delta x}) \]

Combination of functions:

\[ \text{comb}(x) = \sum_{m=-\infty}^{\infty} \delta(x - m) \]

\[ \delta_s(x, \Delta x) = \sum_{m=-\infty}^{\infty} \delta(x - m \Delta x) \]

\[ \mathcal{F}[\delta_s(x, \Delta x)] = \text{comb}(u \cdot \Delta x) \]

Multiplication in one domain becomes convolution in the other,
Point Impulse Signal

• The sampling property

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x - \xi, y - \eta) \, dx \, dy = f(\xi, \eta) \]

• The scaling property

\[ \delta(ax, by) = \frac{1}{|ab|} \delta(x, y) \]

\[ \delta(x, y) \begin{cases} \neq 0, & x = 0 \text{ and } y = 0 \\ = 0, & \text{otherwise} \end{cases} \]

and

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) \, dx \, dy = 1 \]
Fourier Transform of Sampled Function

\[ F[f_s(x)] = \frac{1}{\Delta x} \sum_{n=-\infty}^{n=\infty} F(u - \frac{n}{\Delta x}) \]

Prior to sampling,

\[ F(u) = F[f(x)] \]

After sampling,

\[ F_s(u) = F[f_s(x)] \]

center-to-center: \( \frac{1}{\Delta x} \)

\( \frac{1}{\Delta x} > 2u_c \) ?
Fourier Transform of Sampled Function

If $F(u)$ is band limited to $u_c$, (cutoff frequency)

$$F(u) = 0 \text{ for } |u| > u_c.$$

To avoid overlap (aliasing),

$$F_s(u)$$

We would need

$$\frac{1}{\Delta x} - u_c > u_c \text{ and therefore } \frac{1}{\Delta x} > 2u_c$$
Rect Function

- Rect function:

\[ rect(x, y) = \begin{cases} 
1, & \text{for } |x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\
0, & \text{otherwise} 
\end{cases} \]

- It is normally used to pick up a particulate section of a given function:

\[ f(x, y) \cdot rect\left(\frac{x - \xi}{w_X}, \frac{y - \eta}{w_Y}\right) \]
Fourier Transform of Sampled Function

To avoid overlap (aliasing),

\[ F_s(u) \]

We would need \( \frac{1}{\Delta x} > u_c \) and therefore \( \frac{1}{\Delta x} > 2u_c \)

\[ f(x) = \mathcal{F}^{-1} \left[ F_{f_s}(u) \cdot \Pi \left( \frac{u}{2u_c} \right) \right] \]
Can we restore \( g(x) \) from the sampled frequency-domain signal? Yes, using the Interpolation Filter

\[
H(u) = \prod \left( \frac{u}{2u_c} \right)
\]

\[\downarrow \text{ Fourier transform}\]

\[
h(x) = 2u_c \cdot \text{sinc}(2u_c x)
\]
The original function $f$ can be recovered as

$$f(x) = f_s(x) * h(x)$$

$$f_s(x) = \sum f(n\Delta x)\delta(x - n\Delta x)$$

$$H(u) = \prod \left(\frac{u}{2u_c}\right)$$

$$h(x) = 2u_c \cdot \text{sinc}(2u_c x)$$

$$\mathcal{F} \updownarrow$$

$$\int_{-\infty}^{\infty} \delta(u - \xi) \cdot f(u) du = f(\xi)$$

$f(x)$ is restored from a combination of sinc functions, each weighted and shifted according to its corresponding sampling point.
An Example

\[ f(x) = \sum_{n=-\infty}^{\infty} 2 \cdot u_c \cdot f(n \cdot \Delta x) \cdot \text{sinc}(2u_c(x - n\Delta x)) \]

- Original Signal
- Sampled Signal
- Interpolated Signal
- Intermediate Step

Adding all vertical values gives back the original function.
Nyquist/Shannon Theory:
We must sample at twice the highest frequency in x and in y (U and V) to reconstruct the original signal.
Nyquist Condition

Nyquist Theorem:
In order to restore the original function, the sampling rate must be greater than twice the highest frequency component of the function.

Nyquist Sampling Interval:
The maximum sampling interval allowed without introduce aliasing is

$$\Delta x \leq \frac{1}{2u_c}$$
Two Dimensional Sampling

\[ f_s(x, y) = f(x, y) \cdot \delta_s(x, y, \Delta x, \Delta y) \]

\[ = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(x, y) \cdot \delta(x - n\Delta x, y - m\Delta y) \]

\[ = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y) \cdot \delta(x - n\Delta x, y - m\Delta y) \]
Fourier Transform of Sampled Image

\[ F_{fs}(u) = \mathcal{F}[f_s(x, y)] \]
\[ = \mathcal{F}[\delta_s(x, y, \Delta x, \Delta y) \cdot f(x, y)] \]
\[ = \text{comb}(u \cdot \Delta x, v \cdot \Delta y) \ast \mathcal{F}[f(x, y)] \]
\[ = \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} \delta \left[ \Delta x \left( u - \frac{n}{\Delta x} \right), \Delta y \left( v - \frac{m}{\Delta y} \right) \right] \ast F(u, v) \]
\[ = \frac{1}{\Delta x \Delta y} \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} \delta(u - \frac{n}{\Delta x}, v - \frac{m}{\Delta y}) \ast F(u, v) \]
\[ = \frac{1}{\Delta x \Delta y} \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} F(u - \frac{n}{\Delta x}, v - \frac{m}{\Delta y}) \]

The result: Replicated \( F(u,v) \), or “islands” every \( 1/ \Delta x \) in \( u \), and \( 1/\Delta y \) in \( v \).
Fourier Transform of a Sampled 2-D Function

The diagram illustrates the Fourier transform of a sampled 2-D function. The original function $F(u,v)$ is shown on the left, and its sampled version $F_s(u,v)$ is depicted on the right, highlighting the effects of aliasing.
Consequence of Under-Sampling

$F_s(u, v)$

$\frac{1}{\Delta x}$

$\frac{1}{\Delta y}$

Signal

Aliasing

Fourier Transform
Restoration of the Original 2-D Function

Interpolation Filter in 2-D

\[ H(u, v) = \prod (u\Delta x) \prod (v\Delta y) \]

\[ h(x, y) = \left[ \frac{1}{\Delta x} \cdot \text{sinc}(\frac{x}{\Delta x}) \right] \left[ \frac{1}{\Delta y} \cdot \text{sinc}(\frac{y}{\Delta y}) \right] \]
Restoration of the Original 2-D Function

Given that the Nyquist sampling condition is met, the original function can be recovered exactly.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(u - \xi, v - \eta) \cdot f(u, v) \, du \, dv = f(\xi, \eta)
\]

\[
f(x, y) = f_s(x, y) * h(x, y)
\]

\[
f_s(x, y) * \left[ \frac{1}{\Delta x} \cdot \text{sinc} \left( \frac{x}{\Delta x} \right) \right] \left[ \frac{1}{\Delta y} \cdot \text{sinc} \left( \frac{y}{\Delta y} \right) \right]
\]

\[
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y) \cdot \delta(x - n \cdot \Delta x, y - m \cdot \Delta y) \ast \left[ \frac{1}{\Delta x} \cdot \text{sinc} \left( \frac{x}{\Delta x} \right) \right] \left[ \frac{1}{\Delta y} \cdot \text{sinc} \left( \frac{y}{\Delta y} \right) \right]
\]

\[
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} f(n\Delta x, m\Delta y) \cdot \text{sinc} \left( \frac{x - n \cdot \Delta x}{\Delta x} \right) \cdot \text{sinc} \left( \frac{y - m \cdot \Delta y}{\Delta y} \right)
\]
Chapter 2: Mathematical Preliminaries
Mathematical Preliminaries for 2-D Image Reconstructions
Key Questions and Concepts related to Radon Transform and Inverse Radon Transform:

• What is Radon transform?
• What is a sinogram?
• What does the Central Slice theorem tell us?
• How to design an inverse Radon transform?
• What is the point behind Filtered Backprojection as a way to reconstruct the image from projection data?
The integral of a line impulse function and a given 2-D signal gives the projection data from a given view ...

\[ p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') \, dx \, dy \]
The Radon transform of a 2-D function is defined as

\[ p(\phi, x') \equiv \mathcal{R}[f(x, y)] \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} f(x' \cos \phi - y' \sin \phi, x' \sin \phi + y' \cos \phi) \, dy' \]

where

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  \cos \phi & \sin \phi \\
  -\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]
Radon Transform and Sinogram

- Radon transform maps a 2-D function $f(x,y)$ into a sinogram.

$$ P(\phi, x') = \mathcal{R}[f(x, y)] $$

Any point represented by polar coordinates $(r, \theta)$ will simply follow the equation $x' = r \cos(\phi - \theta)$ in the sinogram space.
Radon Transform and Sinogram

Sinogram is a 2-D function representing the original function $f(x,y)$ into the projection data space.

Sinogram is the basic data format for reconstruction.

**Figure 3-3** (a) Phantom with several distinct objects at various positions; (b) corresponding sinogram.
Radon Transform and Sinogram

http://tech.snmjournals.org/cgi/content-nw/full/29/1/4/F3
Radon Transform and Sinogram

http://tech.snmjournals.org/cgi/content-nw/full/29/1/4/F3
What Exactly Does Projection Data Tell Us?
Central Slice Theorem

\[ F\{p(\phi, x')\} = F(r, \phi) \]
Central Slice Theorem

Integrate intensities along x-direction

Projection profiles

The more angles used, the better the Fourier space image is filled

Create lines in central slice of entire DFT image

1-D DFTs

http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf
Central Slice Theorem

DFT image represents integration of original projections DFT transformed and summed together.

1-D DFTs at each projection

This is the fast way to create the DFT image from projection data. The more projections taken, the more complete the sampling.

http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf
Central Slice Theorem

Let’s consider the 2D FFT of an arbitrarily given function

\[ F(u,v) = \int \int f(x,y) \cdot e^{-i \frac{2\pi}{(ux + vy)}} \, dx \, dy \]

In polar coordinates within the spatial-frequency domain, let

\[ u = \rho \cos \beta \quad \text{and} \quad v = \rho \sin \beta, \]

Then

\[ F(\rho,\beta) = \int \int f(x,y) \cdot e^{-i \frac{2\pi}{\rho (x \cos \beta + y \sin \beta)}} \, dx \, dy \]

If we treat \((x \cos \beta + y \sin \beta)\) as a constant, then \(\exp[-i \frac{2\pi}{\rho (x \cos \beta + y \sin \beta)}]\) could be written as a linear phase shift, which is the Fourier transform of a shifted delta function. Let’s write the complex exponential as the FT of a \(\delta\) function,

\[
F(\rho,\beta) = \int_y \int_x f(x,y) \cdot \delta[x' - (x \cos \beta + y \sin \beta)] \, dx \cdot dy ,
\]

and

\[
F(\rho,\beta) = \int_y \int_x f(x,y) \cdot \{\int_x \delta[x' - (x \cos \beta + y \sin \beta)] \cdot e^{-i \frac{2\pi}{\rho x'}} \, dx'\} \cdot dx \, dy ,
\]
Central Slice Theorem

\[
F(\rho, \phi) = \int_x \int_y f(x, y) \{\int_x \left[ \delta(x \cos \phi + y \sin \phi - x') \right] e^{-i2\pi \rho x'} dx' \} dx dy.
\]

At this point, we can change the order of the integration operators as

\[
F\{p(\phi, x')\} = \int_{x'} \left[ \int_y \int_x f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy \right] e^{-i2\pi \rho x'} dx'.
\]

Recall how we wrote the projection as a double integral of \(f(x,y)\), where a delta function performs the line integral,

\[
p(\phi, x') = \int_x \int_y [(x,y) \cdot \delta(x \cos \phi + y \sin \phi - x')] dx dy
\]

We take the Fourier Transform of \(p(\phi, x')\):

\[
F\{p(\phi, x')\} = \int_{x'} \left[ \int_y \int_x f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy \right] e^{-i2\pi \rho x'} dx'
\]

which is exactly what we wrote for \(F(\rho, \beta)\) above! Therefore, we have

\[
F\{p_\phi(x')\} = F(\rho, \phi)
\]
Projection Theorem or Central Slice Theorem

An alternative proof:

See Page 77, Foundations of Medical Imaging, Z. H. Cho.
Review of 2-D Analytical Reconstruction Methods

Projection Data

Projection data \( p(\phi, x') \)

Incident X-rays

Detect \( p(\phi, x') \)

\[
p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy
\]
Review of 2-D Analytical Reconstruction Methods

Back Projection Operation

Incident X-rays

\( f(x,y) \)

\( R \)

\( \phi \)

\( x \)

\( y \)

\( x' \)

\( r \)

Detected \( p(\phi, x') \)
Simple Backprojection

inverse = 1 back projection

Source

Detector

Source

Detector
Simple Backprojection

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

\[ b_{\phi}(x,y) = \int p_{\phi}(x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

Adding up all the back projections from all the angles gives,

\[ f_{\text{simple back-projection}}(x,y) = \int b_{\phi}(x,y) \, d\phi \]

\[ f_{\text{simple back-projection}}(x,y) = \int_{0}^{\pi} \int_{-\infty}^{\infty} p_{\phi}(x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \, d\phi \]
Simple Backprojection
Simple Backprojection

3 projections

4 projections

many projections

Original object
Impulse Response Function of Simple Backprojection Operator

\[ h_b(r) = \frac{1}{r} \]

\[ f_b(x, y) = f(x, y) \times \frac{1}{r} \]

\[ F_b(\rho, \phi) = \frac{F(\rho, \phi)}{\rho}, \quad \text{since } F\{1/r\} = \frac{1}{\rho} \]

Back projected image is blurred by convolution with \( 1/r \)
Central Slice Theorem

\[ F\{p (\phi, x')\} = F(r, \phi) \]
Simple Back-projection and the 1/r Blurring

The nature of the 1/r blurring:
Radon transform produced equally spaced radial sampling in Fourier domain.

FIGURE 18 The discrete sampling pattern of $F(u_x, u_y)$ contained in $B(u_z, u_y)$, resulting from the use of discretely sampled projections.
Review of 2-D Analytical Reconstruction Methods

Projection Data

Projection data \( p(\phi, x') \)

Incident X-rays

Detected \( p(\phi, x') \)

\[
p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy
\]
Simple Backprojection and Inverse Radon Transform

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

\[ b_\phi (x,y) = \int p_\phi (x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

Adding up all the back projections from all the angles gives,

\[ f_{\text{back-projection}} (x,y) = \int b_\phi (x,y) \, d\phi \]

\[ \hat{f}_{\text{simple backprojection}} (x,y) = \int_0^\pi \int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} p_\phi (x') \cdot \delta(x\cos\phi + y\sin\phi - x') \, dx' \]

\[ \hat{f}_{\text{Inverse Radon transform}} (x,y) = \int_0^\pi \int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} F^{-1}[F[p_\phi (x')] \cdot |w|] \cdot \delta(x\cos\phi + y\sin\phi - x') \, dx' \]
Filtered Back-projection

Ideal filter

The Ram-Lak filter

\[ H_{RL}(\omega) = \begin{cases} 
|\omega|, & (|\omega| \leq 2\pi B) \\
0, & \text{(otherwise)}
\end{cases} \]

Ram-Lak filter
Filtered Back-projection

Figure 3-4  (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.
Filtered Back-Projection

\[ \Delta x = 1/(2B), \]

**Sampled version**

\[ h_{RL}(0) = B^2 = \frac{1}{4\Delta x^2} \quad \text{(if } k = 0\text{)} \]

\[ h_{RL}(k) = 0 \quad \text{(if } k \text{ even)} \]

\[ h_{RL}(k) = \frac{-4B^2}{\pi^2k^2} = \frac{-1}{\pi^2k^2 \Delta x^2} \quad \text{(if } k \text{ odd)} \]

\[ h_{SL}(k) = \frac{-2}{\pi^2 \Delta x^2(4k^2 - 1)} \]

\[ = \frac{-8B^2}{\pi^2(4k^2 - 1)} \]

*Figure 3-4* (b) Spatial domain filter kernels corresponding to the filter functions shown in the Ram-Lak filter is a high-pass filter with a sharp response but results in some noise enhancement, while the Shepp-Logan and the Hamming window filters are noise-smoothed filters and therefore have better SNR.
Filtered Back-projection

The Ram-Lak filter in spatial domain

\[ h_{RL}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{RL}(\omega) \exp(ix\omega) \, d\omega \]

\[ = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} |\omega| \exp(ix\omega) \, d\omega \]

\[ = 2B^2 \text{sinc}(2\pi Bx) - B^2 \text{sinc}^2(\pi Bx) \]

Applying the Ram-Lak filter in filtered backprojection

\[ \hat{f}(x, y) = \frac{1}{\pi} \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} dx' \, p_\phi(x') h(x \cos \phi + y \sin \phi - x') \]

\[ p_\phi^*(x') \]
Simple and Filtered Back-projection

\[ p_\phi(x') = \int_{-\infty}^{\infty} dx' \; p_\phi(x') h(x \cos \phi + y \sin \phi - x') \]

From Computed Tomography, Kalender, 2000.
Simple and Filtered Back-projection

**FIGURE 13-28.** Simple backprojection is shown on the left; only three views are illustrated, but many views are actually used in computed tomography. A profile through the circular object, derived from simple backprojection, shows a characteristic $1/r$ blurring. With filtered backprojection, the raw projection data are convolved with a convolution kernel and the resulting projection data are used in the backprojection process. When this approach is used, the profile through the circular object demonstrates the crisp edges of the cylinder, which accurately reflects the object being scanned.

Chapters 12 & 13, *The Essential Physics of Medical Imaging*, Bushberg
Radon Transform and Sinogram

http://tech.snmjournals.org/cgi/content-nw/full/29/1/4/F3
Filtered Back-projection

https://www.youtube.com/watch?v=ddZeLNh9aac
Statistical Methods

Advantages

- Can accurately model the image formation process (use with non-standard geometries, e.g. not all angles measured, gaps)
- Allow use of constraints and \textit{a priori} information (non-negativity, boundaries)
- Corrections can be included in the reconstruction process (attenuation, scatter, etc)

Disadvantages

- Slow
- Less predictive behaviour (noise? convergence?)
Key Questions and Concepts related to **Iterative Reconstruction Techniques**:

- What is the **maximum likelihood estimation** (MLE) approach?
- **How to formulate** MLE for tomographic image reconstruction?
- **How to numerically implement** MLE?
- How to further improve MLE with **prior information**?
Review of 2-D Analytical Reconstruction Methods
Filtered Back Projection

The solution:

Put what ever we know about the system into the model.

Try to model the projection data with better accuracy.

Using iterative optimization methods to get around the numerical stability problem in large scale inverse (reconstruction) problem.
Image Reconstruction and General Linear Inverse Problem

A basic problem for image reconstruction in emission tomography

Find the object distribution $f$, given (1) a set of projection measurements $g$, (2) information (in the form of a matrix $H$) about the imaging system that produced the measurements, and, possibly, (3) a statistical description of the data and (4) a statistical description of the object (Fig. 1).
Image Reconstruction and General Linear Inverse Problem

An Emission Tomography System

FIGURE 1 A general model of tomographic projection in which the measurements are given by weighted integrals of the emitting object distribution.
Image Reconstruction and General Linear Inverse Problem

Linear imaging systems – An alternative representation

\[ \bar{g} = Hf \]

**FIGURE 2** A discrete model of the projection process.
The Basic Idea of Maximum Likelihood Estimation (MLE)

We can estimate the underlying parameter $N$ by the following maximum-likelihood estimation (MLE) strategy:

$$\hat{N}_{\text{Maximum likelihood}} = \arg \max_N \{P[k|N]\}$$

$P[k(N)]$ is call the Likelihood function: the probability of detecting $k$ counts given there were a total of $N$ gamma rays emitted by the source.
Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- It consists of N trails.
- Each trail has a binary outcome: success or failure (decay or not).
- The probability of success (decay) is a constant from trail to trail – all atoms have equal probability to decay.
- The trails are independent.

In statistics, these four conditions characterize a Bernoulli process.
Source of Noise

Source of noise:

Measurement error, instrument malfunction ...

Random fluctuation in the number of counts detected by detectors

Other additive noise

In emission tomography systems, detector readout noise is generally much smaller than the statistical fluctuation associated with the counting process.

Noise introduced by other physical effects. The general frame work of iterative (or some-times called statistical) reconstruction methods allows one to incorporate many other effects ...
Binomial Distribution

The binomial distribution gives the discrete probability distribution of obtaining exactly k successes out of N Bernoulli trials (where the result of each Bernoulli trial is true with probability p and false with probability 1-p). The binomial distribution is therefore given by

\[
p(k|N, p) = \binom{N}{k} p^k (1-p)^{N-k}
\]

where

\[
\binom{N}{k} = \frac{N!}{k!(N-k)!}
\]
Binomial Distribution and Imaging Applications

Suppose we have a point source generated N gamma rays and we have a pixilated detector for detecting these photons.

Each gamma ray photon has a fixed probability $p$ of reaching a given detector element. $p$ is defined by the relative distance between the source and the pixel and the collimation configuration used.

So the probability of detecting $k$ gamma rays on the pixel is given as

$$p(k|N, p) = \binom{N}{k} p^k (1 - p)^{N-k}$$

Furthermore, the number of counts detected on different pixels are independent ...
Binomial Distribution

For a binomial distribution, the mean or the expectation of the number of disintegration in time \( t \) is given by

\[
\mu = \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np
\]

and the fluctuation on the number of disintegrations is given by the variance or the standard deviation of the

\[
\sigma^2 = \sum_{n=0}^{N} (n - \mu)^2 \cdot P_n = Npq
\]

and

\[
\sigma = \sqrt{\sum_{n=0}^{N} (n - \mu)^2 \cdot P_n} = \sqrt{Npq}
\]
What are the mean and standard deviation of a Binomial distribution?
Poisson Distribution

The counting statistics related to nuclear decay processes is often more conveniently described by the Poisson distribution, is related to situations that involves a collection of multiple trails that satisfy the following conditions:

1. The number of trails, $N$, is very large, e.g. $N \gg 1$.
2. Each trail is independent.
3. The probability that each single trail is successful is a constant and approaching zero, $p \ll 1$. So the number of successful trails is fluctuating around a finite number.
Poisson Distribution

The probability of having $n$ successful trails can be approximated with the Poisson distribution.

$$P(n \mid \mu) = \frac{\mu^n e^{-\mu}}{n!}$$

and the mean and the variance of number of successful trail are given by

$$Mean(n) = \mu = N \cdot p$$

$$Std(n) = \sigma = \sqrt{\mu} = \sqrt{Np}$$
Binomial and Poisson Random Variable

When $N \to \infty$ and $p \to 0$, $k$ follows the Poisson distribution, whose probability function is

$$p(k|N, p) = \binom{N}{k} p^k (1-p)^{N-k} \Rightarrow p(k) = \frac{\mu^k}{k!} e^{-\mu}$$

where $\mu$ is the expected value of the Poisson variable defined as

$$\mu = \lim_{N \to \infty} N p$$
The Basic Idea of Statistical Reconstruction

We can estimate the underlying parameter $N$ by the following maximum-likelihood estimation (MLE) strategy:

$$\hat{N}_{\text{Maximum likelihood}} = \arg \max_N \{P[k|N]\}$$

$P[k(N)]$ is call the **Likelihood function**: the probability of detecting $k$ counts given there were a total of $N$ gamma rays emitted by the source.
The Basic Idea of Statistical Reconstruction

Since
\[ P[k(N)] = \frac{(N \cdot p)^k}{k!} e^{-N \cdot p} \]

Then
\[ \hat{N}_{\text{Maximum likelihood}} = \arg \max_{N} \{P[k|N]\} = \arg \max_{N} \left\{ \frac{(N \cdot p)^k}{k!} e^{-N \cdot p} \right\}, \]
alternatively
\[ \hat{N}_{\text{ML}} = \arg \max_{N} \{ \log[P[k|N]] \} \]
\[ = \arg \max_{N} \{ k \cdot \log(N \cdot p) - N \cdot p - \log k! \} \]
\[ = \arg \max_{N} \{ k \cdot \log(N \cdot p) - N \cdot p \} \]
So the ML estimate of the number of gamma rays emitted by the source is
\[ \hat{N}_{ML} = \arg \max_{N} \{ \log[P[k|N]] \} = \arg \max_{N} \{ k \cdot \log(N \cdot p) - N \cdot p \} \]
Image Reconstruction and General Linear Inverse Problem

An Emission Tomography System

Given a measured data set $g$, how to formulate the ML estimator of $f$?

$$\hat{f}_{ML} = \arg \max_g P(g|f)$$

Then how to evaluate $P(g|f)$?
An Typical Imaging System Described in Matrix Form

\[
\begin{pmatrix}
\bar{g}_1 \\
\bar{g}_2 \\
\bar{g}_3 \\
\vdots \\
\bar{g}_M
\end{pmatrix}
= 
\begin{pmatrix}
p_{11} & p_{12} & p_{13} & \cdots & p_{1N} \\
p_{21} & p_{22} & p_{23} & \cdots & p_{2N} \\
p_{31} & p_{32} & p_{33} & \cdots & p_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{M1} & p_{M2} & p_{M3} & \cdots & p_{MN}
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
f_3 \\
\vdots \\
f_N
\end{pmatrix}
\]

- \(\bar{g}_m\): the expected number of photons detected on each detector element.
- \(p_{nm}\): the probability for a gamma ray emitted by the \(n^{th}\) source voxel to be detected by the \(m^{th}\) detector pixel.
- \(f_m\): the number of photons being emitted from each source voxel.
- Everything we know about the imaging system - The system response function.
Poisson Statistics of the Projection Data

The probability of the measured projection data \( g=(g_1, g_2, g_3, \ldots, g_M) \) may be written as

\[
p(g) = \prod_{m=1}^{M} p(g_m) = \prod_{m=1}^{M} \frac{g_m^{g_m} e^{-g_m}}{g_m!}
\]

where \( g_m \) is the expected value for the number of counts on detector pixel \( \#m \)

Remember that

\[
\begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
\vdots \\
g_M \\
\end{pmatrix} = \begin{pmatrix}
p_{11} & p_{12} & p_{13} & \cdots & p_{1N} \\
p_{21} & p_{22} & p_{23} & \cdots & p_{2N} \\
p_{31} & p_{32} & p_{33} & \cdots & p_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{M1} & p_{M2} & p_{M3} & \cdots & p_{MN} \\
\end{pmatrix} \begin{pmatrix}
f_1 \\
f_2 \\
f_3 \\
\vdots \\
f_N \\
\end{pmatrix}
\]

In the context of emission tomography, \( p_{nm} \) is the probability of a gamma ray generated at a source pixel \( n \) is detected by detector element \( m \).
The Maximum Likelihood Reconstruction

Recall that given a measured dataset \( g \), the \textit{likelihood function}, \( L(f,g) \), of a possible source function \( f \) is

\[
L(f, g) \equiv p(g \mid f)
\]

So that the maximum likelihood solution (the image that maximizing the likelihood function) can be found as

\[
\hat{f}_{ML} = \arg\max_f L(f, g)
\]

or equivalently

\[
\hat{f}_{ML} = \arg\max_f \log[L(f, g)] = \arg\max_f l(f, g)
\]

where \( l(f, g) \) is the log-likelihood function

\[
l(f, g) = \log[L(f, g)]
\]
The Maximum Likelihood Reconstruction

\[ \hat{f}_{ML} = \arg\max_f \ L(f, g) = \arg\max_f \ \log[L(f, g)] \]

\[ = \arg\max_f \ \{ \log[\prod_{m=1}^{M} \frac{\tilde{g}^m g_m}{g_m!} e^{-\tilde{g}_m}] \} \]

So the ML reconstruction for Poisson distributed data is

\[ \hat{f}_{ML} = \arg\max_f \ l(f, g) = \arg\max_f \ \left[ \sum_{m=1}^{M} (g_m \log \tilde{g}_m - \tilde{g}_m - \log g_m !) \right] \]

since \( g_m \) is not function of \( f \), we get

\[ \hat{f}_{ML} = \arg\max_f \ l(f, g) = \arg\max_f \ \left[ \sum_{m=1}^{M} (g_m \log \tilde{g}_m - \tilde{g}_m) \right] \]

\[ \tilde{g}_m = \sum_{n=1}^{N} f_n p_{nm} \]
The Maximum Likelihood Reconstruction

ML estimate

\[ \hat{f}_{ML} = \arg \max_f \ l(f, g) = \arg \max_f \ \left[ \sum_{m=1}^{M} g_m \cdot \log \left( \sum_{n=1}^{N} f_n p_{nm} \right) - \left( \sum_{n=1}^{N} f_n p_{nm} \right) \right] \]

Underlying/unknown image to be estimated

Measured data
The Maximum Likelihood Expectation Maximization (MLEM) Algorithm

The source (image) function \( f \) that has the maximum likelihood of giving rise to the observed data \( g \) can be found by the following iterative updating scheme

\[
f_n^{(new)} = \frac{f_n^{(old)}}{P_m \sum_{n=1}^{N} g_m} \frac{P_{nm}}{P_{n,m} \sum_{n'=1}^{N} P_{n',m} f_{n'}^{(old)}}, \quad n = 1, 2, \ldots, N
\]

where

- \( m = 1, 2, \ldots, M \) is the index of detector elements in the imaging system
- \( n = 1, 2, \ldots, N \) is the index of source object pixels
- \( f_n^{(old)} \): the current estimate of the source function in pixel \( n \)
- \( f_n^{(new)} \): the updated estimate of the source pixel \( n \)
- \( P_{nm} \): the probability of a gamma ray generated in source pixel \( n \) and detected by the detector element \( m \)
- \( g_m \): the observed number of counts in detector pixel \( m \)
The Maximum Likelihood Expectation Maximization (MLEM) Algorithm

\[
\begin{align*}
&f_n^{(new)} = \frac{f_n^{(old)}}{\sum_{m=1}^{M} \sum_{n=1}^{N} p_{nm}} \sum_{m=1}^{M} g_m \sum_{n'=1}^{N} p_{n'm} f_{n'}^{(old)}, \quad n = 1, 2, \ldots, N \\
&\sum_{m=1}^{M} p_{nm} \sum_{n'=1}^{N} p_{n'm} f_{n'}^{(old)}
\end{align*}
\]

Each iteration updates all elements in the source function \(f\) sequentially.

Each iteration is guaranteed to provide a new image function that has an *increased likelihood* compared to the previous image function (unless the maximum likelihood solution has been achieved).
Structure of MLEM Algorithm

\[
f_n^{(new)} = \frac{f_n^{(old)}}{\sum_{m=1}^{M} P_{nm}} \sum_{m=1}^{M} g_m p_{nm} \sum_{n'=1}^{N} P_{n'm} f_n^{(old)}
\]

**FIGURE 6** Flowchart of a generic iterative reconstruction algorithm.
Maximum Likelihood Estimation Applied to Tomographic Image Reconstruction – An Example

Consider a simple pinhole imaging problem as illustrated below

Q1. Can you write down the equation for the likelihood function? Please be as detailed as possible.

\[ L(g; f) = P(g|f) \]

Q2. Please write down the equation for the Maximum-Likelihood (ML) estimate of the number of gamma rays emitted by the four source voxels during the measurement.

Note: The probability distribution function of a Poisson random number \( k \), with a mean of \( \mu \), is given by

\[ p(k|\mu) = \frac{\mu^k}{k!} e^{-\mu} \]
A Typical Imaging System Described in Matrix Form

\[
\begin{pmatrix}
\overline{g}_1 \\
\overline{g}_2 \\
\overline{g}_3 \\
\vdots \\
\overline{g}_M
\end{pmatrix} =
\begin{pmatrix}
p_{11} & p_{12} & p_{13} & \cdots & p_{1N} \\
p_{21} & p_{22} & p_{23} & \cdots & p_{2N} \\
p_{31} & p_{32} & p_{33} & \cdots & p_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{M1} & p_{M2} & p_{M3} & \cdots & p_{MN}
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
f_3 \\
\vdots \\
f_N
\end{pmatrix}
\]

- Expected number of photons detected on each detector element
- Everything we know about the imaging system – The system response function.
- The source object.
- The number of photons being emitted from each source voxel

\( p_{nm} \): the probability for a gamma ray emitted by the \( n \)'th source voxel to be detected by the \( m \)'th detector pixel.
The Maximum Likelihood Reconstruction

\[ \hat{f}_{ML} = \underset{f}{\text{argmax}} \ L(f, g) = \underset{f}{\text{argmax}} \ \log[L(f, g)] \]

\[ = \underset{f}{\text{argmax}} \ \{ \log[\prod_{m=1}^{M} \left( \frac{\hat{g}_m^g m}{g_m!} e^{-\hat{g}_m} \right) ] \} \]

So the ML reconstruction for Poisson distributed data is

\[ \hat{f}_{ML} = \underset{f}{\text{argmax}} \ l(f, g) = \underset{f}{\text{argmax}} \ \left[ \sum_{m=1}^{M} (g_m \log \hat{g}_m - \hat{g}_m - \log g_m !) \right] \]

since \( g_m \) is not function of \( f \), we get

\[ \hat{f}_{ML} = \underset{f}{\text{argmax}} \ l(f, g) = \underset{f}{\text{argmax}} \ \left[ \sum_{m=1}^{M} (g_m \log \hat{g}_m - \hat{g}_m) \right] \]

\[ \hat{f}_{ML} = \underset{f}{\text{argmax}} \ l(f, g) = \underset{f}{\text{argmax}} \ \left[ \sum_{m=1}^{M} \left( g_m \cdot \log \left( \sum_{n=1}^{N} f_n p_{nm} \right) - \left( \sum_{n=1}^{N} f_n p_{nm} \right) \right) \right] \]
Properties of MLEM Algorithm

Advantages

• Relatively simple but highly useful.

• Provides room for incorporating a wide range of physical effects and detector system characteristics and therefore provide an improved image quality for low statistics data.

• Consistent and predictable behavior.

• Automatically incorporates non-negativity constraint. So the image values are always non-negative.

• Provides theoretically optimum solution (Please be careful when saying this!!).
Properties of MLEM Algorithm

Shortcomings

- *Relatively slow* convergence. For typical 3-D reconstruction, it takes several tens of iterations to converge. In principle, each iteration takes similar amount of computation to that for a complete FBP reconstruction!

- Tends to yield *noisy reconstructions* when projection data has low statistics.

What exactly is the problem?

The algorithm tends to fit to the noise in the (observed) projection data!
Properties of MLEM Algorithm

With noise free data

With noisy data
Properties of MLEM Algorithm

How to improve?

Adding extra (or sometime called \textit{a priori}) information in the reconstruction process.

This can be done using two numerically equivalent approaches

- This can be done by trying to discourage possible image solutions that has large fluctuations (noisy images). – Penalized Maximum Likelihood (PML) methods.
- Using the Bayes’ law to incorporate the \textit{a priori} information – Maximum a posteriori (MAP) methods.
Further Improvements – Penalized ML Algorithms

The basic idea:

Instead of finding the original image by maximizing the likelihood function:

\[
\hat{f} = \arg\max_f p(g \mid f) = \arg\max_f l(f, g)
\]

Probability of a measurement \( g \) given the underlying image \( f \)

The so-called log-likelihood function

We can use a different selection criterion

\[
\hat{f} = \arg\max_f [l(f, g) - \beta U(f)]
\]

The so-called penalty function that has increased value when an undesired imaging feature presents in the solution.

\( \beta \) is a scaling constant that controls “how strong the penalty is for the presence of a given feature”

So by minimizing, we are selecting an image by dis-encouraging the undesired features to be presented in the image.

Penalized Maximum Likelihood (PML) reconstruction ...
Further Improvements –
Maximum A Posteriori (MAP) Algorithms

We have to change our mind a little bit ... by considering that the image to be estimated is itself a random variable that follows some statistical law $p(f)$.

The Bayes’s law:

If the underlying image is $f$, the conditional probability of observing the measurement $g$ is

$$p(f \mid g) = \frac{p(g \mid f)p(f)}{p(g)}$$

*a posteriori probability.*

Suppose we know something about the underlying image, and we can describe this knowledge with a statistical law.

So the underlying image can be best estimated by maximizing this probability

This leads to the maximum a posteriori (MAP) approach...
Further Improvements – Bayesian Estimation and Maximum A Posteriori (MAP) Algorithms

The maximum a posteriori (MAP) approach...

$$\hat{f}_{\text{MAP}} = \arg \max_f \log p(f \mid g) = \arg \max_f \log \frac{p(g \mid f)p(f)}{p(g)}$$

$$= \arg \max_f \left[ \log p(g \mid f) + \log p(f) \right]$$

This is numerically equivalent to PML approach is we recognize that

$$\text{if } p(f) = -\beta U(f)$$

$$\hat{f}_{\text{PML}} = \arg \max_f \left[ \log p(g \mid f) - \beta U(f) \right] \iff \hat{f}_{\text{MAP}} = \arg \max_f \left[ \log p(g \mid f) + \log p(f) \right]$$

The a priori probability of the data. This is where you can fold in your prior knowledge about the underlying image... for example, the image “should” be relatively smooth...
Chapter 2: Mathematical Preliminaries

Image Quality
Key Questions and Concepts related to Image Quality:

• What is the **contrast** in an image?

• The use of **modulation** as a measure of the contrast?

• What is the **modulation transfer function**? What does it tell us about the performance of an imaging system?
Contrast
Contrast

- What is contrast?

- The difference in the image gray scale between closely adjacent regions of the image.

- Medical image contrast is the result of many steps during image acquisition, processing and display.
Subject (or Object) Contrast

- Difference in some aspects of the signal prior to it being recorded
- Consequence of fundamental differences in the object, e.g., in x-ray intensity based on attenuation
- \( C = \frac{(A-B)}{A} \)
Detector Contrast

- Detector Contrast ($C_d$)
- A detector's characteristics play an important role in producing contrast in the final image
- $C_d$ determined principally by how the detector 'maps' detected energy into the output signal
- Characteristic curve: input radiation exposure to output value (analog or digital)
A Revisit to Key Image Quality Measures

Modulation

- What is modulation?
- Definition of modulation transfer function
- How to experimentally measure MTF
Modulation

- The modulation $m_f$ is an effective way to quantify the contrast of a periodic signal

$$m_f = \frac{f_{\text{max}} - f_{\text{min}}}{f_{\text{max}} + f_{\text{min}}}.$$

- In general, $m_f$ is referred to as the contrast of a periodic signal $f(x,y)$ relative to its average value.

- So within two signals, $f(x,y)$ and $g(x,y)$, with the same average value, $f(x,y)$ is said to have more contrast if $m_f > m_g$. 

Continuous Fourier Transform

www.revisemri.com
Modulation

- Suppose an input signal function

\[ f(x, y) = A + B \sin(2\pi u_0 x), \]

where \( A > B \) and both are non-negative constants.

\[ m_f = \frac{f_{\text{max}} - f_{\text{min}}}{f_{\text{max}} + f_{\text{min}}} \quad \Rightarrow \quad m_f = \frac{B}{A}. \]

Greater \( m_f \), more contrast
Modulation

- Now let this signal to pass through an LSI imaging system. Suppose an input signal function. Since

$$f(x, y) = A + B \sin(2\pi u_0 x) = A + \frac{B}{2j} \left[ e^{j2\pi u_0 x} - e^{-j2\pi u_0 x} \right],$$

- Suppose the system impulse response function $h(x,y)$ is circularly symmetric, then

$$g(x, y) = AH(0, 0) + B |H(u_0, 0)| \sin(2\pi u_0 x).$$

- Then

$$g_{\min} = AH(0,0) - B |H(u_0, 0)| \quad \text{and} \quad g_{\max} = AH(0,0) + B |H(u_0, 0)|,$$

- The modulation of the input and output signals are

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}} = \frac{B}{A}, \quad \text{and} \quad m_g = \frac{g_{\max} - g_{\min}}{g_{\max} + g_{\min}} = \frac{B |H(u_0,0)|}{AH(0,0)} = m_f \frac{|H(u_0,0)|}{H(0,0)}$$

$$\frac{m_g}{m_f} = \frac{H(u_0,0)}{H(0,0)} \leq 1$$
Modulation

- The effect of an LSI having circular symmetric impulse response function on an input sinusoidal signal is to scale the input signal by a factor equal to the magnitude spectrum at the same frequency $u_0$.

\[
m_f = \frac{B}{A} \quad \text{and} \quad m_g = \frac{B|H(u_0, 0)|}{A H(0, 0)}
\]

- It is often true that $H(0,0) \approx 1$, and $H(U_0,0) < 1$. So that the output signal has less contrast, $m_g < m_f$. 
Modulation

- Now let this signal to pass through an LSI imaging system. Suppose an input signal function. Since

\[ f(x, y) = A + B \sin(2\pi u_0 x) = A + \frac{B}{2j} \left[ e^{j2\pi u_0 x} - e^{-j2\pi u_0 x} \right], \]

- Suppose the system impulse response function \( h(x,y) \) is circularly symmetric,

\[ h(u, v) = \mathcal{F}[-h(x, y)] \]

\[ g(x, y) = AH(0, 0) + B |H(u_0, 0)| \sin(2\pi u_0 x). \]
Modulation

- Now let this signal to pass through an LSI imaging system. Suppose an input signal function. Since
  \[ f(x, y) = A + B \sin(2\pi u_0 x) = A + \frac{B}{2j} \left[ e^{j2\pi u_0 x} - e^{-j2\pi u_0 x} \right], \]

- Suppose the system impulse response function \( h(x,y) \) is circularly symmetric,

  \[ g(x, y) = AH(0, 0) + B |H(u_0, 0)| \sin(2\pi u_0 x). \]

- So the modulation of the output signal is

  \[ m_g = \frac{B|H(u_0, 0)|}{AH(0, 0)} = m_f \frac{|H(u, 0)|}{H(0, 0)}. \]

- Modulation Transfer Function (MTF).

  \[ \text{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u, 0)|}{H(0, 0)}. \]
Modulation Transfer Function (MTF)

- In order to fully quantify the response of an LSI for an arbitrary signal, \( f(x,y) \), we would need to know the response of the system to sinusoidal signals at different frequencies.

- Modulation Transfer Function (MTF).

\[
MTF(u) = \frac{m_g}{m_f} = \frac{|H(u, 0)|}{H(0, 0)}.
\]

- MFT is, in effect, the “frequency response function” of a given imaging system. It is normally evaluated for positive frequencies only.

- Most imaging systems lead to decreased contrast, so that

\[
0 \leq MTF(u) \leq MTF(0) = 1, \text{ for every } u,
\]
Modulation Transfer Function (MTF)

- In case of non-isotropic impulse response function \( h(x,y) \) is not circularly symmetric, the MTF can be defined as

\[
MTF(u, v) = \frac{m_g}{m_f} = \frac{|H(u, v)|}{H(0, 0)},
\]

- A typical MTF of an imaging system

\[
|H(u, v)| = \sqrt{H_R^2(u, v) + H_I^2(u, v)}
\]

*Figure 3.3*
A typical MTF of a medical imaging system.
Modulation Transfer Function (MTF)

- In case of non-isotropic impulse response function ($h(x,y)$ is not circularly symmetric), the MTF can be defined as

$$MTF(u, v) = \frac{m_g}{m_f} = \frac{|H(u, v)|}{H(0, 0)},$$

$$|H(u, v)| = \sqrt{H_R^2(u, v) + H_I^2(u, v)}$$
Modulation Transfer Function (MTF)

\[
MTF(u) = \frac{m_g}{m_f} = \frac{|H(u, 0)|}{H(0,0)}
\]

If an imaging system is designed to faithfully reproduce a DC signal, then \((0,0)=1\). So the MTF becomes

\[
MTF(u) = \frac{m_g}{m_f} = |H(u, 0)| .
\]
Modulation Transfer Function (MTF)

$$MTF(u) = \frac{m_g}{m_f} = \frac{|H(u, 0)|}{H(0, 0)}.$$
System Cascade

- Modulation Transfer Function (MTF).

\[ g(x, y) = h_K(x, y) \ast \cdots \ast (h_2(x, y) \ast (h_1(x, y) \ast f(x, y))) \]

- The overall MTF is the product of the MTF for sub-systems:

\[ MTF_{total}(u) = \prod_{i=1}^{i=L} MTF_i(u) \]

- MTF of an imaging system that can be modeled as a chain if systems is often determined by the MTF of the worst system in the cascade.
Modulation transfer Function -- Revisited

- Modulation Transfer Function (MTF).
Modulation Transfer Function -- Revisited

Standard Test Chart for Determining the Modulation Transfer Function (MTF) of an Imaging System.
Modulation Transfer Function (MTF)

- An example of the same signal passing through three imaging systems with decreasingly poor MTF, which leads to decreasing contrast in images.
Spatial Resolution

• Resolution: the ability of an given imaging system to accurately depict two distinct events in space, time or frequency respectively.
• Resolution can also be thought as the degree of blurring, smearing.

• Spatial resolution is fully described by the point-spread function or impulse response function, $h(x,y)$. 
The Point Spread Function (PSF)

- One method is to stimulate the imaging system with a point impulse and observe the resulting point-response function.
Full-width at Half Maximum (FWHM) of The Point Spread Function (PSF)

Figure 3.6
An example of the effect of system resolution on the ability to differentiate two points. The FWHM equals the minimum distance that the two points must be separated in order to be distinguishable.
Stationary and Non-stationary PSF

- Spatial variation of the PSF is another important aspect of a given imaging system.
Other Ways to Measure the Spatial Resolution
Line Response Function

- The resolution of an imaging system can also be estimated with the line-spread function.
- Given a line impulse:
  \[ f(x, y) = \delta_\ell(x, y) = \delta(x \cos \theta + y \sin \theta - \ell) \]
- Assuming the impulse response function of the imaging system \( h(x, y) \) is isotropic, it is sufficient to consider the response of the system to a vertical line through the origin. For this case, line response function is:
  \[
  g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) f(x - \xi, y - \eta) \, d\xi \, d\eta,
  \]
  \[
  = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\xi, \eta) \delta(x - \xi) \, d\xi \right] \, d\eta,
  \]
  \[
  = \int_{-\infty}^{\infty} h(x, \eta) \, d\eta,
  \]

Line Response Function: \( l(x) \equiv \int_{-\infty}^{\infty} h(x, \eta) \, d\eta \)
Line Response Function

Line Response Function: \( l(x) \equiv \int_{-\infty}^{\infty} h(x, \eta) d\eta \)

- The 1-D Fourier transform of the line spread function is

\[
L(u) = \mathcal{F}_{1D}[l](u),
\]

\[
= \int_{-\infty}^{\infty} l(x)e^{-j2\pi ux} \, dx,
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, \eta)e^{-j2\pi ux} \, dx \, d\eta,
\]

\[
= H(u, 0).
\]

Remember that

\[
MTF(u) = \frac{mg}{mf} = |H(u, 0)|
\]

- So the values of the Fourier transform of the LSF crossing a horizontal line passing through the origin is numerically equivalent to the MTF,

\[
L(u) = F_1[l(x)] = |H(u, 0)| = \frac{mg}{mf} = MTF(u)
\]
LSF and MTF

- Modulation Transfer Function (MTF).

\[
\text{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u, 0)|}{H(0, 0)} = \frac{|L(u)|}{L(0)},
\]

for every \(u\).

- For a “reasonable” imaging system, the \(L(0)=1\), so that

\[
\text{MTF}(u) = L(u)
\]

- MTF is an effective way to compare two imaging systems in terms of spatial resolution and contrast.

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta)f(x - \xi, y - \eta) \, d\xi \, d\eta,
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\xi, \eta) \delta(x - \xi) \, d\xi \right] d\eta,
\]

\[
= \int_{-\infty}^{\infty} h(x, \eta) \, d\eta, \quad \equiv \ast_{\infty}, \quad L\omega = \mathcal{F}_{\omega}[l\ast_{\infty}]
\]
• System 1 has better low frequency contrast and it is better for imaging coarse features.
• System has a better high frequency contrast. It is better for resolving fine details.
• The resolution of a system is related to the higher frequency components and the cut-off frequency of the MFT.
Image Noise

- The random fluctuation is referred to as the image noise. That has dramatic effects on the subsequent analysis, for example, for signal detection and quantification tasks ...

![Images showing varying levels of noise from low to high](image.png)
Image Noise Reduces Contrast
Where is the noise coming from?
Image Noise

- Due to the random nature of the data acquired by any imaging system, the output images are normally multivariate random variables.
Central Slice Theorem

DFT image represents integration of original projections DFT transformed and summed together.

1-D DFTs at each projection

This is the fast way to create the DFT image from projection data. The more projections taken, the more complete the sampling.

http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf
The nature of the 1/r blurring:
Radon transform produced equally spaced radial sampling in Fourier domain.
Simple Backprojection and Inverse Radon Transform

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

\[ b_\phi(x,y) = \int p_\phi(x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

Adding up all the back projections from all the angles gives,

\[ f_{\text{back-projection}}(x,y) = \int b_\phi(x,y) \, d\phi \]

\[ f_{\text{simple backprojection}}(x,y) = \int_0^\pi \int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} p_\phi(x') \cdot \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

\[ f_{\text{Inverse Radon transform}}(x,y) = \int_0^\pi \int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x')] \cdot |w|\} \cdot \delta(x \cos \phi + y \sin \phi - x') \, dx' \]
An Example – Noise on Images Acquired with FBP

The Ram-Lak filter in spatial domain

\[
\begin{align*}
    h_{RL}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{RL}(\omega) \exp(i x \omega) \, d\omega \\
    &= \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} |\omega| \exp(i x \omega) \, d\omega \\
    &= 2B^2 \text{sinc}(2\pi Bx) - B^2 \text{sinc}^2(\pi Bx)
\end{align*}
\]

\[
\hat{f}(x, y) = \frac{1}{\pi} \int_{0}^{\pi} \int_{-\infty}^{\infty} \text{d}\phi \, \text{d}x' \, p_{\phi}(x') h(x \cos \phi + y \sin \phi - x')
\]

Or written in discrete form: \( \hat{f}_i = \sum_{j=1}^{m} S_{ij} \cdot g_j \)
Filtered Back-projection

Figure 3-4  (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.
Covariance of Random Variables

Covariance provides a measure of the strength of the correlation between two or more random variables. The covariance for two random variables and, each with sample size, is defined by the expectation value

\[
\text{Covariance}(N_1, N_2) = \iint_{N_1 N_2} p(N_1) \cdot p(N_2) \cdot (N - \tilde{N}_1) \cdot (N_2 - \tilde{N}_2) \cdot dN_2 dN_2
\]

\[
\text{Variance}(N_1) \equiv \sigma^2 = \int_{N_1} p(N_1) \cdot (N - \tilde{N}_1)^2 \cdot dN_1
\]
Why Gaussian Random Variable is Important?

- When a quantity is derived as the result of a large number of accumulative effects, and each effect has a small contribution to the final outcome, then the distribution of the quantity tends to follow Gaussian distribution.

- The measured value on a detector is the result of accumulated interactions during a measurements session...

- The reconstructed image at a given pixel is the sum of the contributions from the data acquired with a large number of detector elements in the system...

\[ f_i = \sum_{j=1}^{m} s_{ij} \cdot g_j \] 

or 

\[ f = S \cdot g \]

- It should follow Gaussian distribution

Mean of the reconstructed image: 

\[ \hat{f} = S \cdot \bar{g} \]

Covariance: 

\[ \text{Cov}(\hat{f}) = S \cdot \text{Cov}(g) \cdot S^T \]

In this simple case, we know exactly what is the probability of a given reconstructed image!!

NPRE 435, Principles of Imaging with Ionizing Radiation, Fall 2022
The Maximum Likelihood Reconstruction

Recall that the *likelihood function*, $L(f, g)$, of a possible source function $f$ is

$$L(f, g) = p(g \mid f)$$

So that the maximum likelihood solution (the image that maximizing the likelihood function) can be found as

$$\hat{f}_{ML} = \arg\max_f L(f, g)$$

or equivalently

$$\hat{f}_{ML} = \arg\max_f \log[L(f, g)] = \arg\max_f l(f, g)$$

where $l(f, g)$ is the log-likelihood function

$$l(f, g) = \log[L(f, g)]$$
Poisson Statistics of the Projection Data

The probability of a given projection data \( g=(g_1, g_2, g_3, \ldots, g_M) \) is

\[
p(g) = \prod_{m=1}^{M} p(g_m) = \prod_{m=1}^{M} \frac{g_m}{g_m!} e^{-g_m}
\]

where \( g_m \) is the expected value for the number of counts on detector pixel \( #m \).

Remember that

\[
\bar{g}_m = \sum_{n=1}^{N} f_n p_{nm}
\]

In the context of emission tomography, \( p_{nm} \) is the probability of a gamma ray generated at a source pixel \( n \) is detected by detector element \( m \).
The Maximum Likelihood Reconstruction

For data that follows Poisson distribution, the likelihood function, \( L(f,g) \), of a possible source function \( f \) is

\[
L(f, g) = p(g \mid f) = \prod_{m=1}^{M} p(g_m) = \prod_{m=1}^{M} \frac{\bar{g}_m^{g_m}}{g_m!} e^{-\bar{g}_m}
\]

\( \bar{g}_m = \sum_{n=1}^{N} f_n p_{nm} \), is the expected number of counts on detector pixel \( m \).

\( g_m \) is the measured number of counts on detector pixel \( m \).

The log-likelihood function is

\[
l(f, g) = \log[L(f, g)] = \log[p(g \mid f)] = \prod_{m=1}^{M} p(g_m) = \log \left[ \prod_{m=1}^{M} \frac{\bar{g}_m^{g_m}}{g_m!} e^{-\bar{g}_m} \right] = \sum_{m=1}^{M} \left( g_m \log \bar{g}_m - \bar{g}_m - \log g_m! \right)
\]
The Maximum Likelihood Reconstruction

So the ML reconstruction for Poisson distributed data is

\[
\hat{f}_{ML} = \arg \max_f l(f, g) = \arg \max_f \left[ \sum_{m=1}^{M} \left( g_m \log \bar{g}_m - \bar{g}_m - \log g_m! \right) \right]
\]

since \( g_m \) is not function of \( f \), we get

\[
\hat{f}_{ML} = \arg \max_f l(f, g) = \arg \max_f \left[ \sum_{m=1}^{M} \left( g_m \log \bar{g}_m - \bar{g}_m \right) \right]
\]
The Maximum Likelihood Expectation Maximization (MLEM) Algorithm

The source (image) function $f$ that has the maximum likelihood of giving rise to the observed data $g$ can be found by the following iterative updating scheme

$$
\begin{align*}
    f_n^{(new)} &= \frac{f_n^{(old)}}{\sum_{m=1}^{M} p_{nm}^{old}} \sum_{m=1}^{M} \frac{g_m}{\sum_{n'=1}^{N} p_{n'm} f_{n'}^{(old)}} p_{nm}, & n = 1, 2, \ldots, N
\end{align*}
$$

where

$m = 1, 2, \ldots, M$ is the index of detector elements in the imaging system

$n = 1, 2, \ldots, N$ is the index of source object pixels

$f_n^{(old)}$: the current estimate of the source function in pixel $n$

$f_n^{(new)}$: the updated estimate of the source pixel $n$

$p_{nm}$: the probability of a gamma ray generated in source pixel $n$ and detected by the detector element $m$

$g_m$: the observed number of counts in detector pixel $m$