Fourier Transform and Sampling

Reading Material:

Chapter 2, Medical Imaging Signals and Systems, 2’nd Edition,

by Prince and Links, Prentice Hall, 2006.

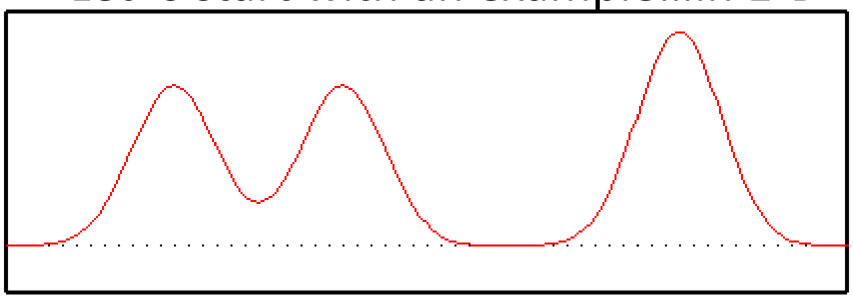
Fourier Transform Basics

Continuous Fourier Transform

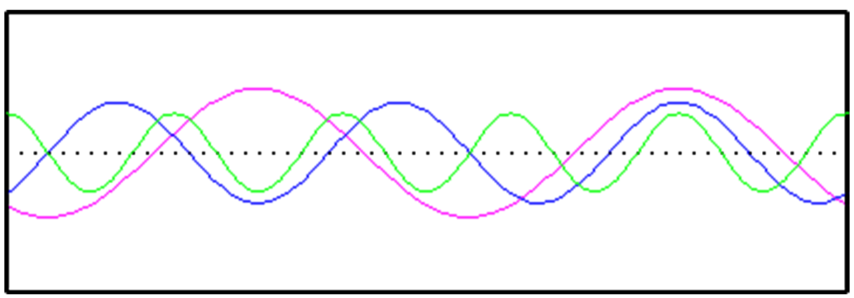
A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

So what does this mean exactly?

Let's start with an example…in 1-D

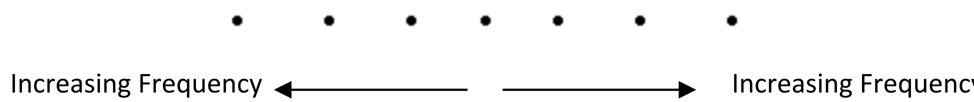


Can be represented by:



When you let these three waves interfere with each other you get your original wave function!

Since this object can be made up of 3 fundamental frequencies an ideal Fourier Transform would look something like this:



Notice that it is symmetric around the central point and that the amount of points radiating outward correspond to the distinct frequencies used in creating the image.

Fourier Transform – What and Why?

What is Fourier Transform?

- A function can be described by a summation of waves with different frequency, amplitudes and phases.

The importance of Fourier Transform in Imaging?

- Signal representations in the frequency domain provide **unique information.**
- Certain **computations** can be performed more efficiently in frequency domain.
- Certain **hardware** naturally measures signals in the frequency domain.

Continuous Fourier Transform

- For any square-integrable function f(x,y), a continuous Fourier transform is defined as

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$where\ j = \sqrt{-1}$$

- We can also define an inverse Fourier transform as

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

- Both f(x,y) and F(u,v) have infinite support.
- Both f(x,y) and F(u,v) are defined on a continuum of values.
- f(x,y) and F(u,v) must contain the same information.

Fourier Transform and Spatial Frequency

- Fourier transform can be viewed as a decomposition of the function f(x,y) into a linear combination of complex exponentials with strength F(u,v).

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

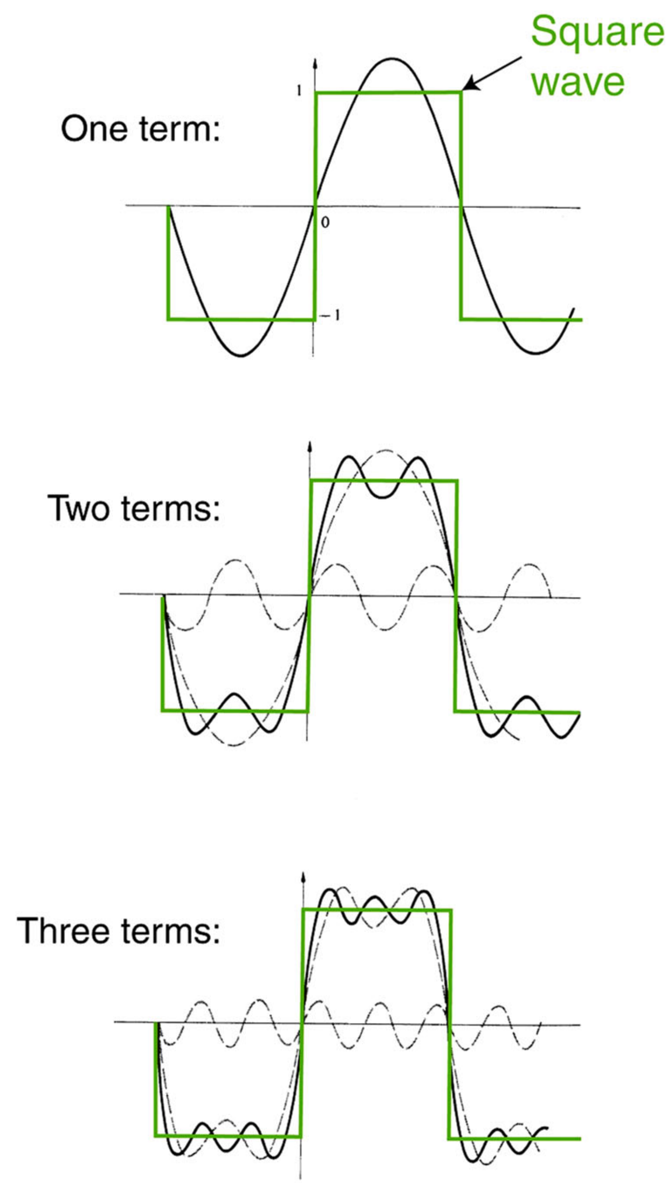
- Fourier transform provides information on the sinusoidal composition of a signal at different spatial frequencies.

$$e^{j2\pi(ux+vy)} = \cos[2\pi(ux+vy)] + j\sin[2\pi(ux+vy)]$$

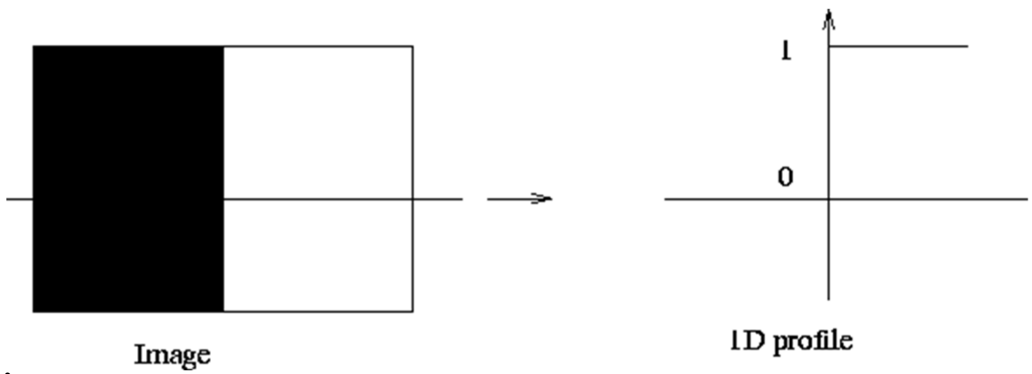
- F(u,v) is normally referred to as the spectrum of the function f(x,y).

Fourier Transform and Spatial Frequency

Fourier transform provides information on the sinusoidal composition of a signal at different spatial frequencies.



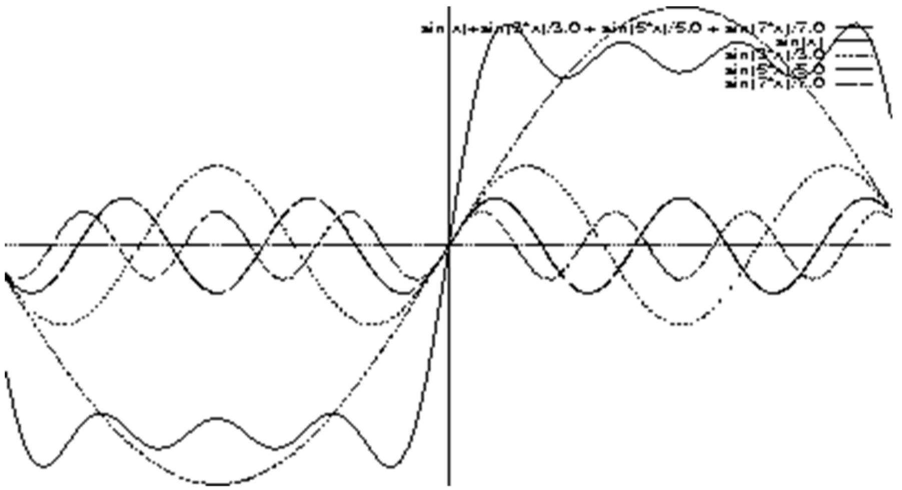
What is Spatial Frequency?



$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{-j2\pi(ux+vy)}dudv$$

$$e^{-j2\pi(ux+vy)}$$

$$= \cos[2\pi(ux+vy)] + j\sin[2\pi(ux+vy)]$$



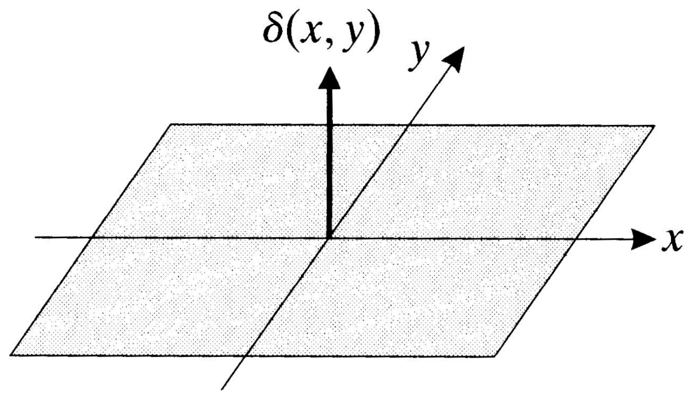
Point Impulse Signal

- A point source is mathematically represented by the delta function or Dirac function.

$$\delta(x,y)\begin{cases} \neq 0, & x=0 \; and \; y=0 \\ =0, & otherwise \end{cases}$$

and

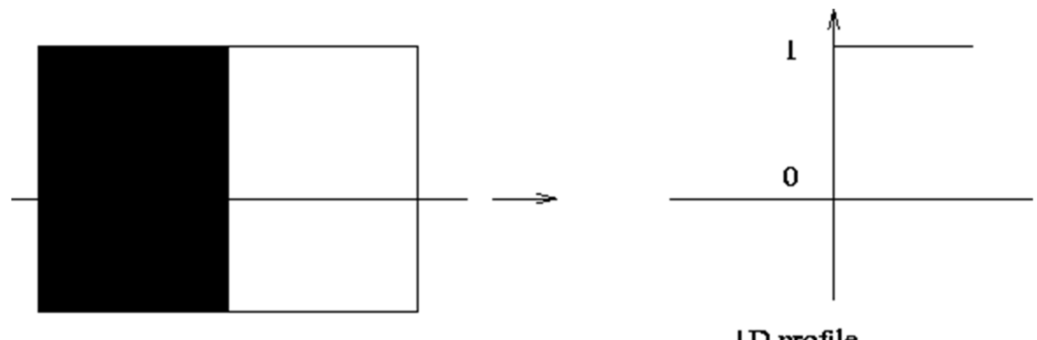
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy=1$$



- The sampling property

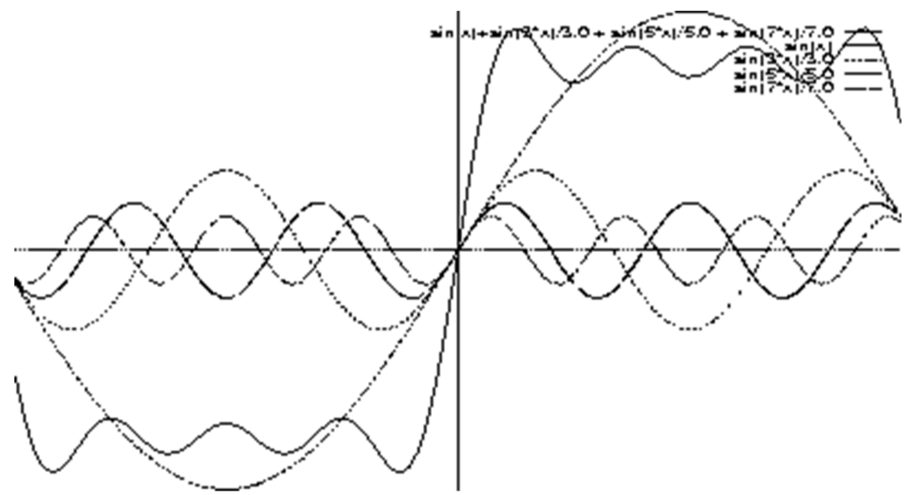
$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\cdot\delta(x-\xi,y-\eta)d\xi d\eta$$

What is Spatial Frequency?

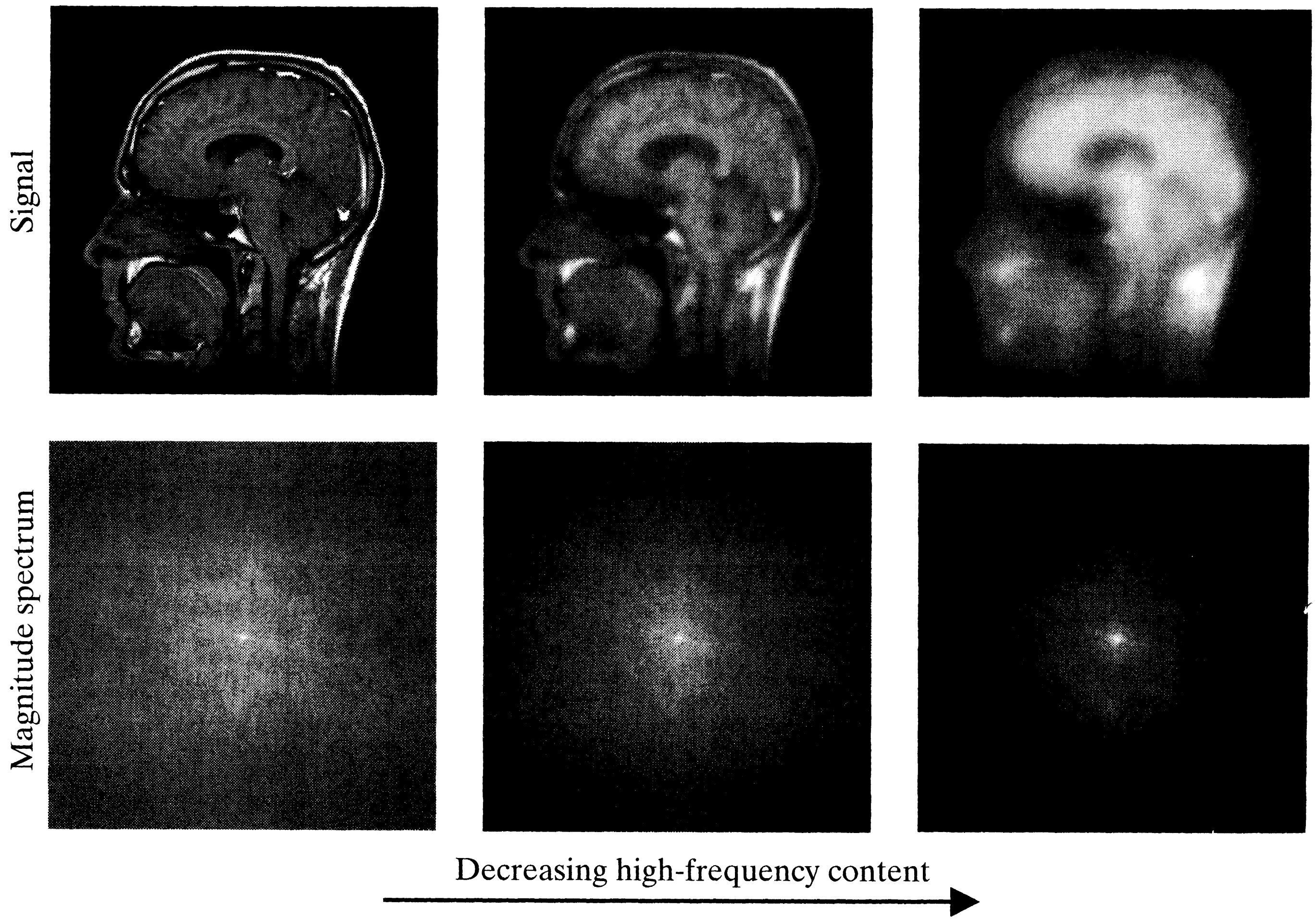


$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{-j2\pi(ux+vy)}dudv$$

$$e^{-j2\pi(ux+vy)}$$
$$=\cos[2\pi(ux+vy)]+j\sin[2\pi(ux+vy)]$$

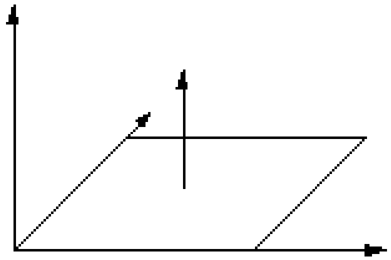


An Example

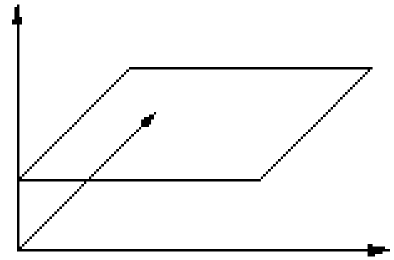

Signal
Magnitude spectrum
Decreasing high-frequency content

Examples

Delta function

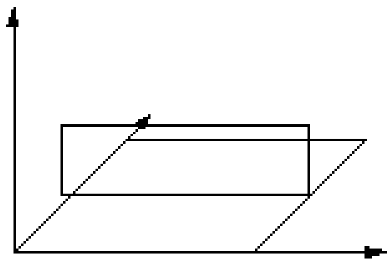


FFT



2-D DC plane

2-D line impulse



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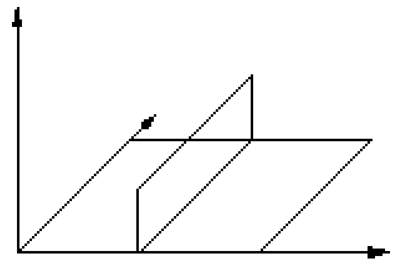
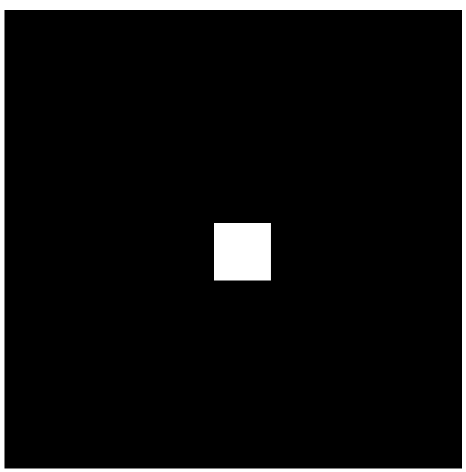
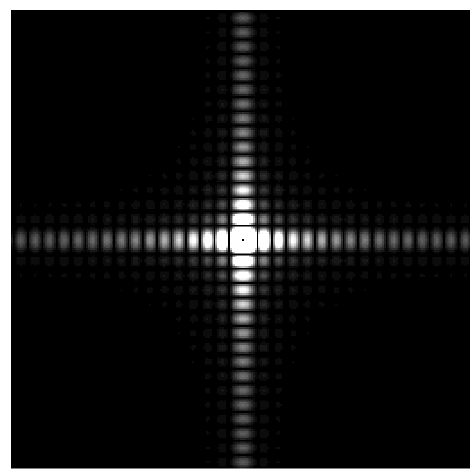


2-D line impulse

Square signal





2-D sinc function

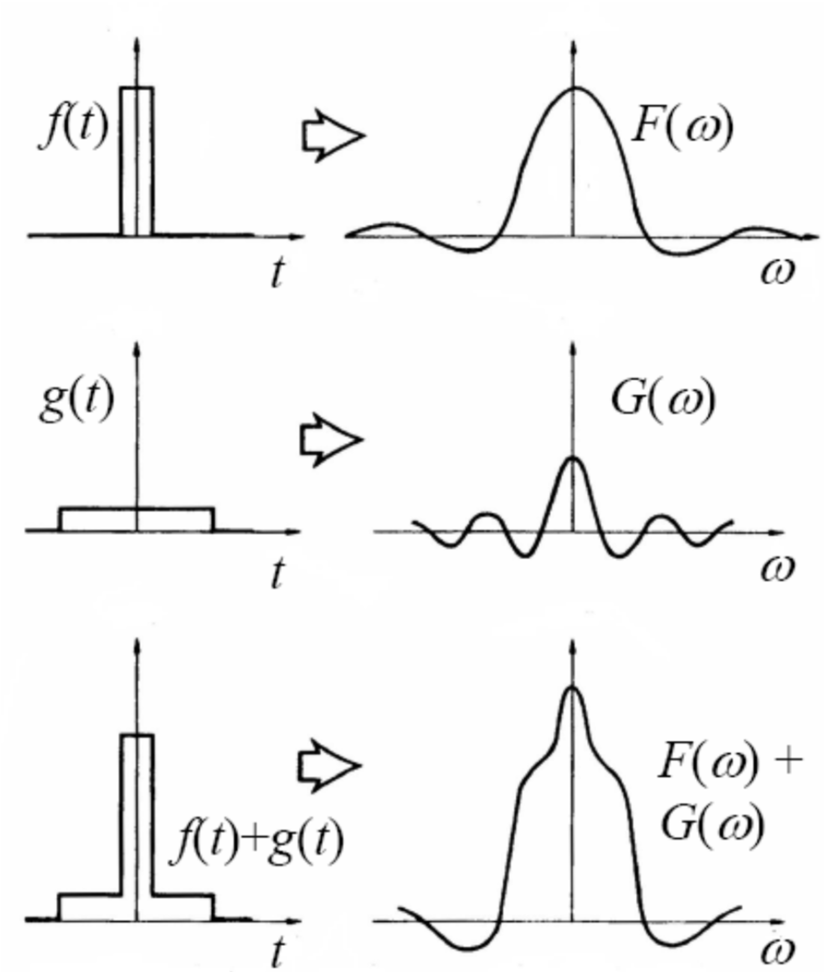
Basic Fourier Transform Pairs

Basic Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0 + vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\text{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0 x + v_0 y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0 x + v_0 y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0 x + v_0 y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\text{rect}(x, y)$	$\text{sinc}(u, v)$
$\text{sinc}(x, y)$	$\text{rect}(u, v)$
$\text{comb}(x, y)$	$\text{comb}(u, v)$
$e^{-\pi(x^2 + y^2)}$	$e^{-\pi(u^2 + v^2)}$

Properties of Fourier Transform (1)

- Linearity $F[a_1 f(x,y) + a_2 g(x,y)] = a_1 F[f(x,y)] + a_2 F[g(x,y)]$



$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$where\ j = \sqrt{-1}$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

Magnitude and Phase

- In general, Fourier transform is a complex valued signal, even if f(x,y) is real valued.
- It is sometimes useful to consider the magnitude and phase of the Fourier transform separately.

Fourier coefficients are complex: $F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$

Magnitude: $|F(u,v)| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$

Phase: $\angle F(u,v) = \tan^{-1} \frac{F_I(u,v)}{F_R(u,v)}$

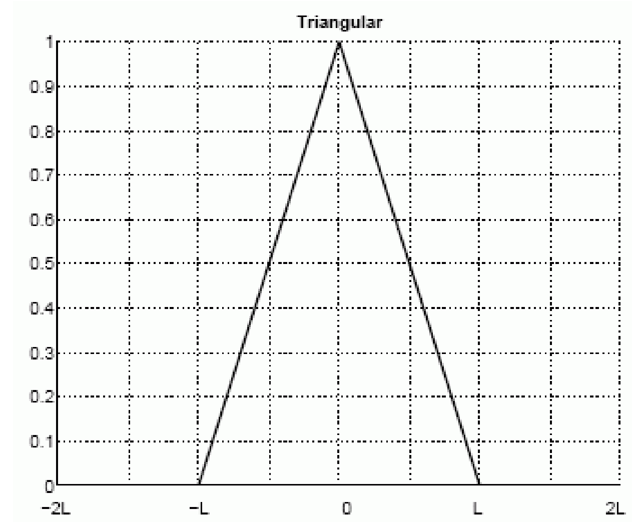
An alternative representation: $F(u,v) = |F(u,v)| e^{j\angle F(u,v)}$

- The square of the magnitude $|F(u,v)|^2$ is referred to as the power spectrum of the original function.

Triangular Signals and Gaussian Signals

- Triangular function:

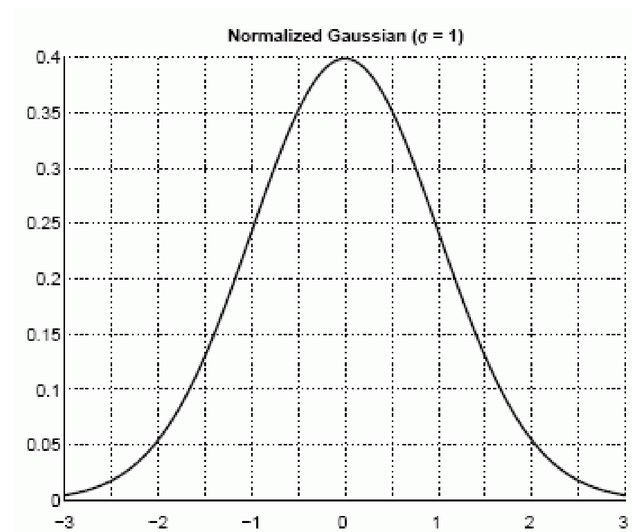
$$Tri\left(\frac{x}{2L}\right) = 1 - \frac{|x|}{L} \quad \text{for} \quad |x| < L$$
$$= 0 \quad \text{for} \quad |x| > L$$



- Normalized Gaussian function:

$$G_{1D}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}}$$

$$G_{2D}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

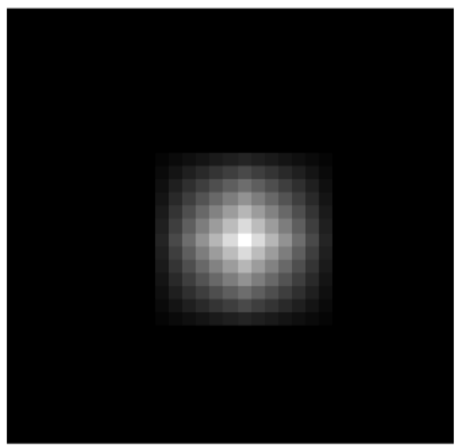


Properties of Fourier Transform (2)

Shifting Property – Shift in spatial domain is equivalent to phase change in spatial frequency domain.

$$\mathscr{F}\left[f(x-x_0, y-y_0)\right] = \mathscr{F}\left[f(u,v)\right]e^{-j2\pi(ux_0+vy_0)}$$

An example



$\Lambda(x/16)\Lambda(y/16)$

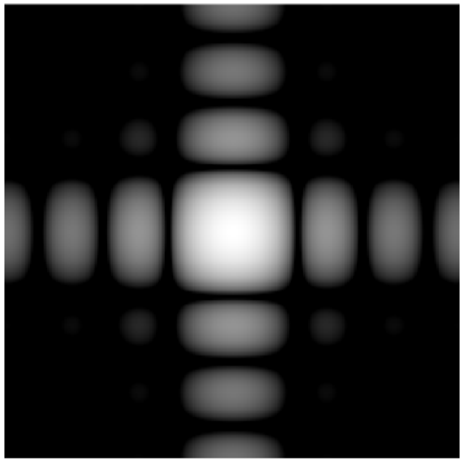
real and even

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$where\ j = \sqrt{-1}$$

Properties of Fourier Transform

Shown in Log scale



$\text{Real}\{F(u,v)\} = 256\ \text{sinc}^2(16u)\text{sinc}^2(16v)$ $Imag\{F(u,v)\} = 0$

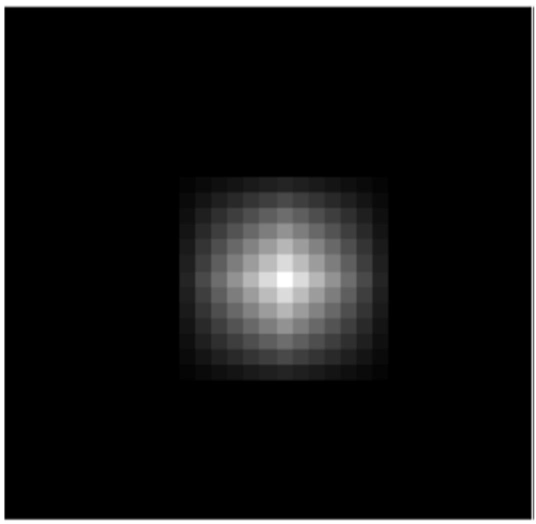
$$|F(u,v)| = \sqrt{\{Real[F(u,v)]\}^2 + \{Imag[F(u,v)]\}^2}$$

$$= 256\ \text{sinc}^2(16\text{u})\text{sinc}^2(16\text{v})$$

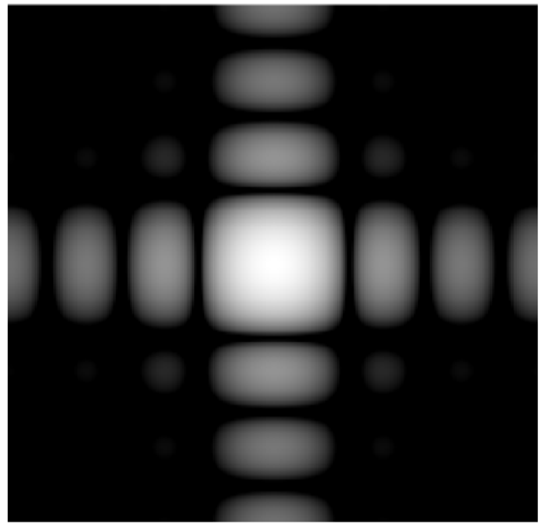
and

$$\angle F(u,v) = \tan^{-1}\frac{Imag[F(u,v)]}{Real[F(u,v)]} = 0$$

Properties of Fourier Transform



$\Lambda[(x-1)/16]\Lambda[y/16]$
shifted by 1



$|F(u,v)| =$

$256\,\text{sinc}^2(16\text{u})\text{sinc}^2(16\text{v})$

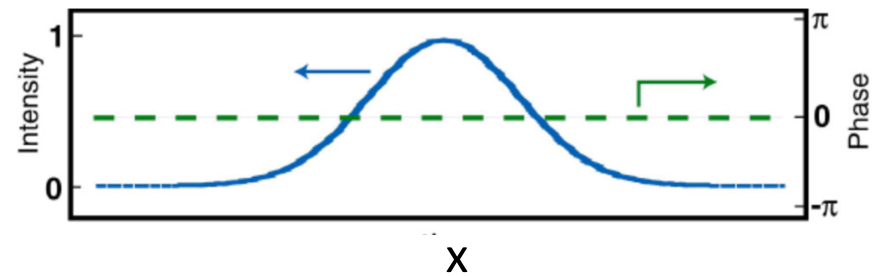
and

$\angle F(u,v) = -2\pi u$

Properties of Fourier Transform

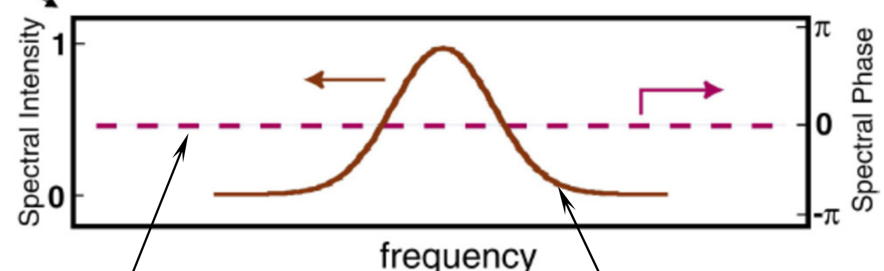
$$\text{1-D Gaussian function}: f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

Spatial Domain



A Gaussian transforms to a Gaussian

Spatial Frequency Domain



$$\mathscr{F}\left[f(x-x_0, y-y_0)\right] = \mathscr{F}\left[f(u,v)\right] e^{-j2\pi(ux_0+vy_0)}$$

Spectral phase is zero

Magnitude is a Gaussian

Properties of Fourier Transform

$$\boldsymbol{F}[\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{a})] = \boldsymbol{F}[\boldsymbol{f}(\boldsymbol{x})]\boldsymbol{exp}(-\boldsymbol{j}2\boldsymbol{\pi}\boldsymbol{u}\boldsymbol{a})$$

Spatial Domain

$$F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$$

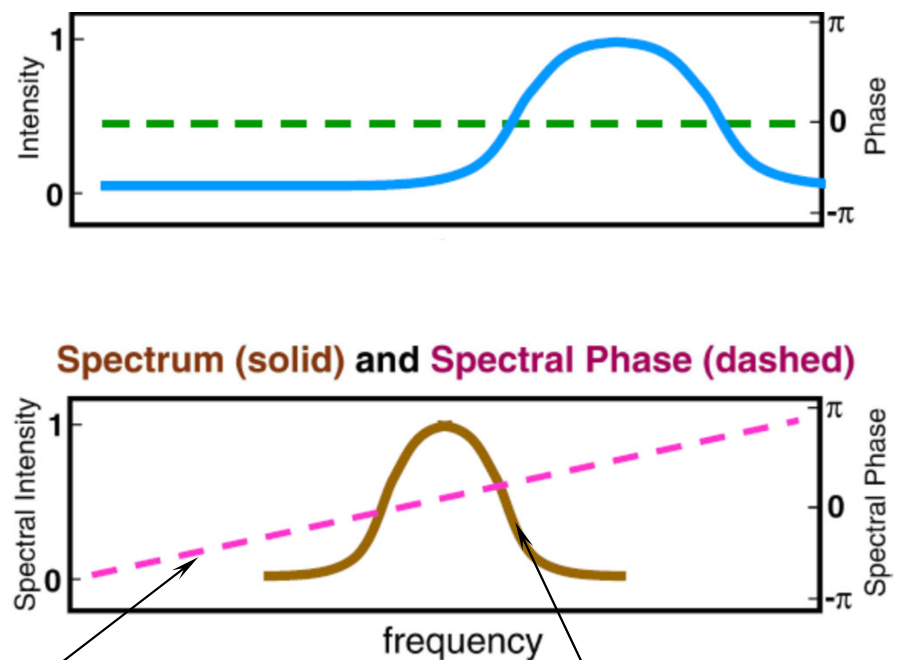
$$|F(u,v)| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$$

$$\angle F(u,v) = \tan^{-1}\frac{F_I(u,v)}{F_R(u,v)}$$

$$F(u,v) = |F(u,v)|e^{j\angle F(u,v)}$$

Spatial Frequency Domain

Properties



Linear shifting in spatial domain simply adds some linear phase to the pulse

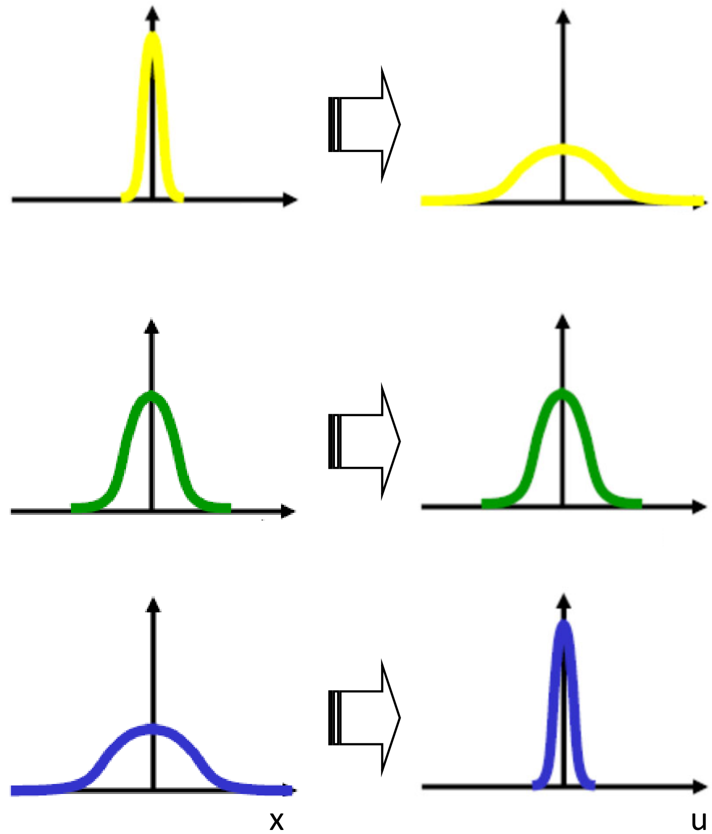
Magnitude is unchanged

Properties of Fourier Transform

- Scaling

$$F\left[f(ax,by)\right]=\frac{1}{ab}F\left(\frac{u}{a},\frac{v}{b}\right)$$

An 1-D example



Spatial domain, f(x)   Fourier Transform, F(u)

The shorter the pulse, the broader the spectrum !

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

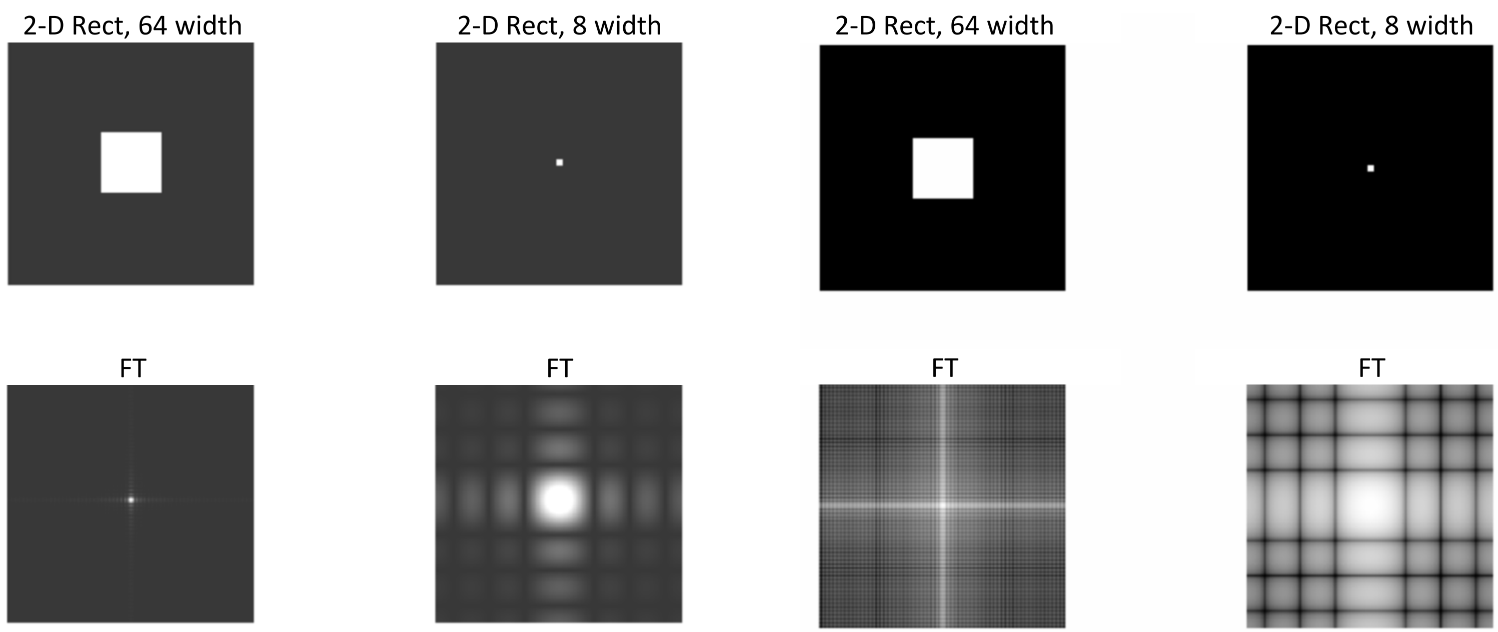
$$where j=\sqrt{-1}$$

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}dudv$$

Properties of Fourier Transform

- Scaling $$F\left[f(ax,by)\right]=\frac{1}{ab}F\left(\frac{u}{a},\frac{v}{b}\right)$$

An 2-D example



Linear scale

Log scale

Linear and Shift Invariant Systems Revisited

For a shift-invariant system, the output is the input convolved with the impulse response function.

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x,y,\xi,\eta)d\xi d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta$$

$$= f(x,y) * h(x,y)$$

where $h(x,y,\xi,\eta)$ is the response of the system to an delta impulse signal

Convolution Theorem

- The convolution of two 2-D functions is defined as

$$f(x,y)*h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\cdot h(x-\xi,y-\eta)d\xi d\eta$$

- The Fourier transform of the convolution is equal to the product of the individual Fourier transforms:

$$\mathscr{F}\left[f(x,y)*h(x,y)\right]=\mathscr{F}\left[f(x,y)\right]\cdot\mathscr{F}\left[h(x,y)\right]$$

where $\mathscr{F}\left[\cdot\right]$ is the Fourier transform operator.

The output from a shift invariant system is therefore

$$g(x,y)=f(x,y)*h(x,y)=\mathscr{F}^{-1}\left\{\mathscr{F}\left[f(x,y)\right]\cdot\mathscr{F}\left[h(x,y)\right]\right\}$$

Properties of Fourier Transform

- ***Product***

$$\boldsymbol{F}\left[f(x,y)\cdot g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]*\boldsymbol{F}\left[g(x,y)\right]$$

Fourier transform of the product of two functions equals to the convolution of the Fourier transforms of individual function.

- ***The Convolution Theorem***

$$\boldsymbol{F}\left[f(x,y)*g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]\cdot\boldsymbol{F}\left[g(x,y)\right]$$

Convolution Theorem

Proof:

$$\mathbf{F}\left[f(x,y)*g(x,y)\right]$$

Assuming $x' = x - \xi$ ***and*** $y' = y - \eta$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\cdot g(x-\xi,y-\eta)\,d\xi\,d\eta\right\}e^{-j2\pi(ux+vy)}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\cdot g(x',y')\cdot e^{-j2\pi[u(x'+\xi)+v(y'+\eta)]}\cdot dx'\,dy'\right\}d\xi\,d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\cdot e^{-j2\pi(u\xi+v\eta)}\cdot g(x',y')\cdot e^{-j2\pi(ux'+vy')}\,dx'\,dy'\right\}d\xi\,d\eta$$

$$= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\cdot e^{-j2\pi(u\xi+v\eta)}\,d\xi\,d\eta\right]\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x',y')\cdot e^{-j2\pi(ux'+vy')}\,dx'\,dy'\right]$$

$$= \mathbf{F}\left[f(x,y)\right]\cdot\mathbf{F}\left[g(x,y)\right]$$

$$\mathcal{F}\left[f(x,y)\cdot g(x,y)\right] = \mathcal{F}\left[f(x,y)\right] * \mathcal{F}\left[g(x,y)\right]$$

Proof:

Note that

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

The convolution theorem states:

$$\mathbf{F}\left[f(x,y) * g(x,y)\right] = \mathbf{F}\left[f(x,y)\right]\cdot \mathbf{F}\left[g(x,y)\right]$$

Similarly we can prove that:

$$\mathbf{F}^{-1}\left[f(x,y) * g(x,y)\right] = \mathbf{F}^{-1}\left[f(x,y)\right]\cdot \mathbf{F}^{-1}\left[g(x,y)\right]$$

If we define:

$$F = \mathbf{F}^{-1}\left[f(x,y)\right], G = \mathbf{F}^{-1}\left[g(x,y)\right], \quad then \quad f(x,y) = \mathbf{F}\left[F\right], g(x,y) = \mathbf{F}\left[G\right]$$

We can see that

$$\mathbf{F}^{-1}\left\{\mathbf{F}\left[F\right] * \mathbf{F}\left[G\right]\right\} = F\cdot G$$

Therefore

$$\mathbf{F}\left[F\cdot G\right] = \mathbf{F}\left[F\right] * \mathbf{F}\left[G\right]$$

Note that F and G are arbitrary function, so that

we can re - write the above equation as

$$\mathbf{F}\left[f(x,y)\cdot g(x,y)\right] = \mathbf{F}\left[f(x,y)\right] * \mathbf{F}\left[g(x,y)\right]$$

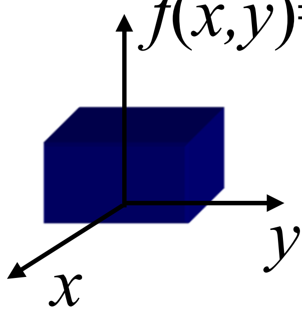
Properties of Fourier Transform

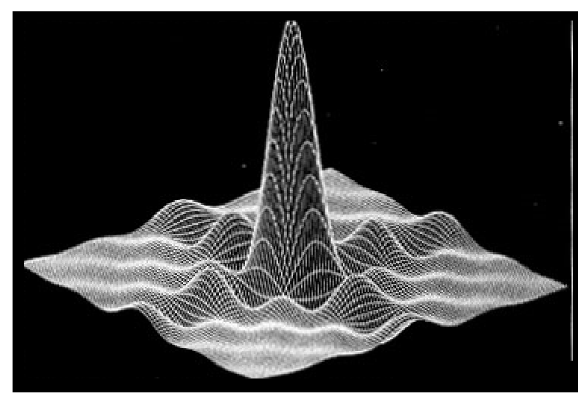
- Separable product

$$\textit{if}$$
$$f(x,y) = f_1(x)\cdot f_2(y)$$
$$\textit{then}$$
$$\boldsymbol{F}_{u,v}\left[f(x,y)\right] = \boldsymbol{F}_u\left[f_1(x)\right]\cdot \boldsymbol{F}_v\left[f_2(y)\right]$$

$$\mathscr{F}\{f(x,y)\}=\mathrm{sinc(u)sinc(v)}$$

$$f(x,y)=\mathrm{Rect(x)Rect(y)}$$





Symmetry Properties

Expanding the Fourier transform of a function, f(t):

$$F(\omega) = \int_{-\infty}^{\infty} [\mathrm{Re}\{f(t)\} + i\,\mathrm{Im}\{f(t)\}]\ [\cos(\omega t) - i\sin(\omega t)]\,dt$$

Expanding more, noting that: $\int_{-\infty}^{\infty} O(t)\,dt = 0$ if O(t) is an odd function

= 0 if Re{f(t)} is odd ↓ = 0 if Im{f(t)} is even ↓

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\cos(\omega t)\,dt + \int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\sin(\omega t)\,dt \quad \leftarrow \mathbf{Re}\{F(\omega)\}$$

= 0 if Im{f(t)} is odd ↓ = 0 if Re{f(t)} is even ↓

$$+\ i\int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\cos(\omega t)\,dt - i\int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\sin(\omega t)\,dt \quad \leftarrow \mathbf{Im}\{F(\omega)\}$$

Even functions of w Odd functions of w

Fourier transform of Circularly Symmetric Functions (1)

A function is circularly symmetric if

$$f_\theta(x,y) = f(x,y), \quad \text{for every } \theta,$$

where $f_\theta(x,y)$ is a rotated version of $f(x,y)$.

In this case, the function

$$f(x,y) = f(r), \ \text{where } r = \sqrt{x^2 + y^2}$$

The Fourier transform is also real and circularly symmetric.

$$\left|F(u,v)\right| = F(u,v) \text{ and } \angle F(u,v) = 0$$

and

$$F(u,v) = F(q), \text{ where } q = \sqrt{u^2 + v^2}$$

Fourier Transform of Circularly Symmetric Functions (2)

The Fourier transform of a circularly symmetric function is called Hankel transform

$$F(q) = 2\pi \int_{-\infty}^{\infty} f(r) J_0(2\pi q r) r dr$$

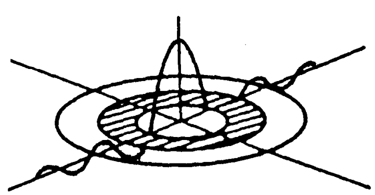
where $J_n(r)$ is the zero-order Bessel function,

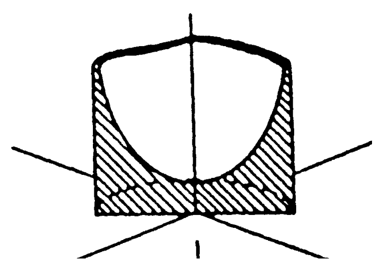
$$J_n(r) = \frac{1}{\pi} \int_0^{\pi} \cos(nr - r\sin\phi) d\phi, \quad n = 0,1,2,\ldots,$$

The inverse Hankel transform is given by

$$f(r) = 2\pi \int_{-\infty}^{\infty} F(q) J_0(2\pi q r) q dq$$

$$\frac{\sin(2\pi a r)}{r}$$





$$\frac{\Pi(q/2a)}{\left(a^2 - q^2\right)^{\frac{1}{2}}}$$

Properties of Fourier Transform (4)

- Parseval's Theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y)\right|^2 dudv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(u,v)\right|^2 dudv$$

Total energy of a signal of a signal f(x,y) in spatial domain equals its total energy in spatial frequency domain.

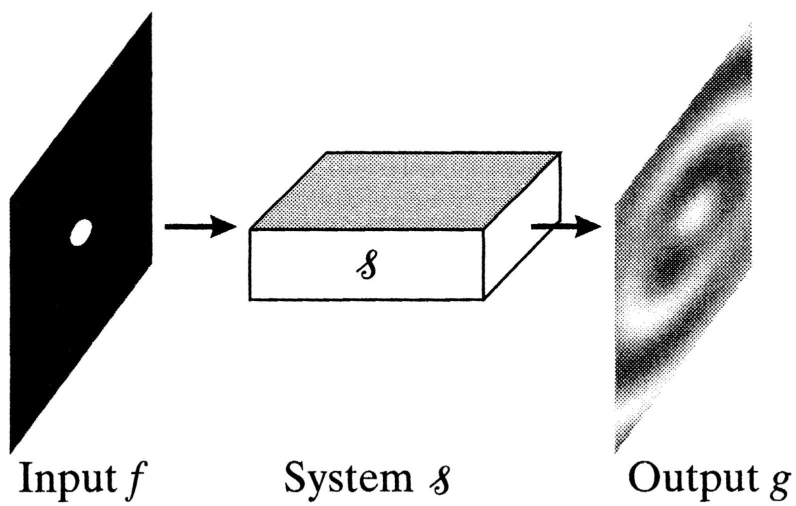
Fourier transform and its inverse is energy preserving!

Convolution Theorem Revisited

The convolution theorem enables one to perform convolution operation as multiplication process in spatial frequency domain.

$$f(x,y) * h(x,y) = \boldsymbol{F}^{-1}\left\{\boldsymbol{F}\left[f(x,y)\right]\cdot \boldsymbol{F}\left[h(x,y)\right]\right\}$$

By using the Fast Fourier Transform (FFT) algorithms, the convolution operation can be performed very efficiently !! This provide a practical way for modeling linear shift-invariant systems …



$$g(x,y) = f(x,y) * h(x,y)$$

System Transfer Function

The Fourier transform of the impulse response function h is called the system transfer function.

$$H(u,v) = \boldsymbol{F}\left[h(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)e^{-2\pi j(ux+vy)}\,dx\,dy$$

$$G(u,v) = F(u,v)H(u,v)$$

System Transfer Function

$$G(u,v) = F(u,v)H(u,v)$$

An ideal low-pass filter is defined as

$$H(u,v) = \begin{cases} 1 & \text{for } \sqrt{\mathrm{u}^2 + \mathrm{v}^2} \le c \\ 0 & \text{for } \sqrt{\mathrm{u}^2 + \mathrm{v}^2} > c \end{cases}$$

c is called the cut-off frequency.

System Transfer Function

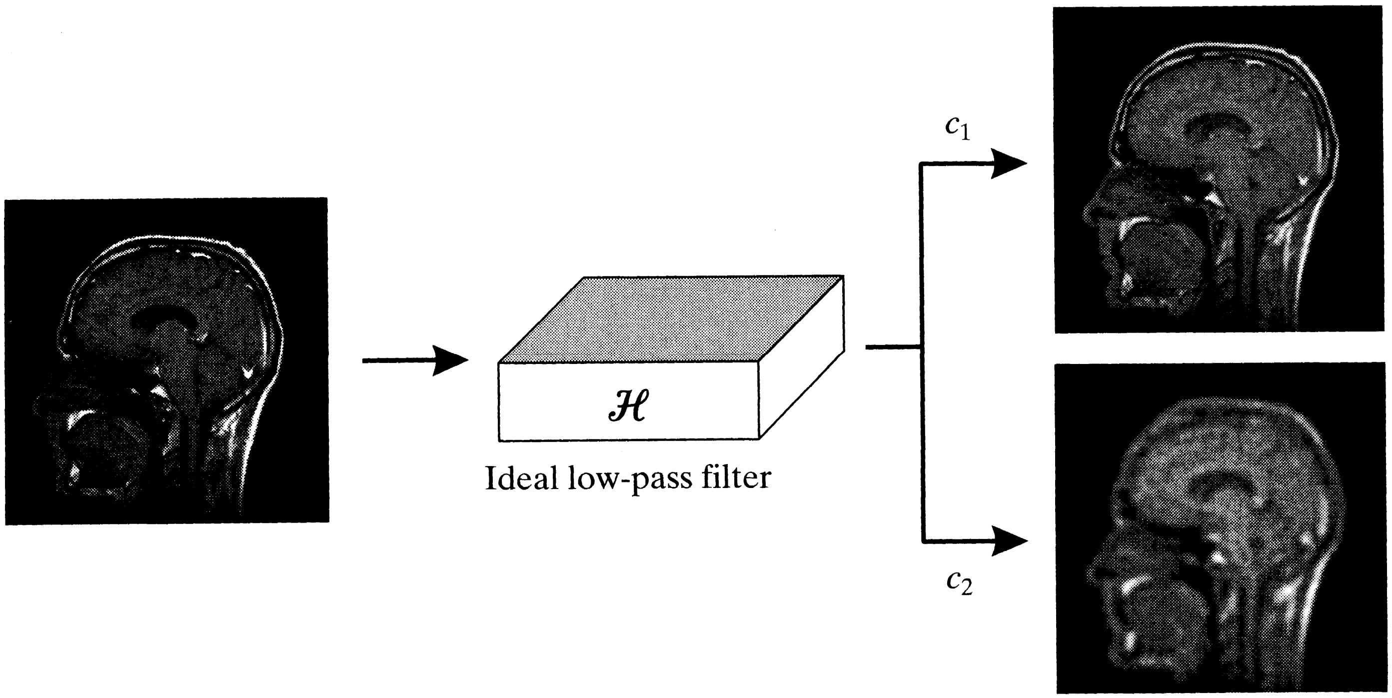
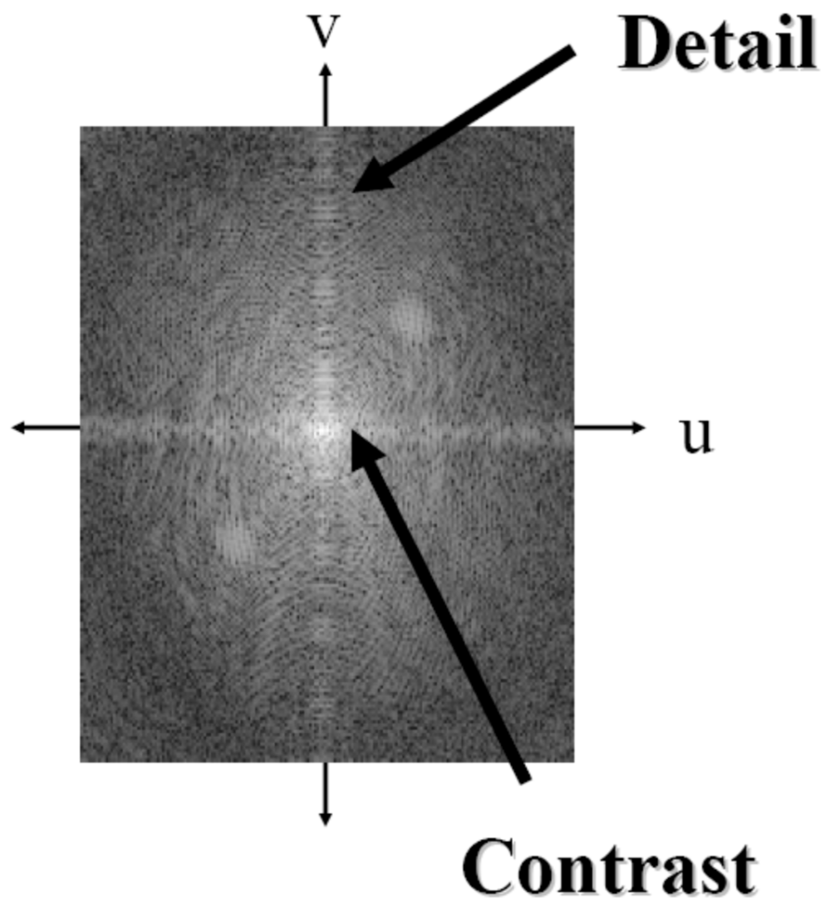


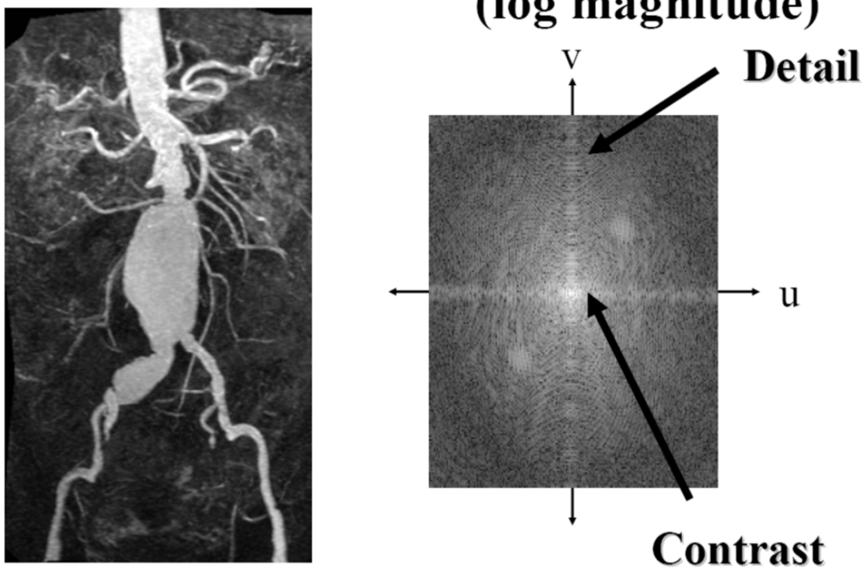
Figure 2.12
The response of an ideal low-pass filter for two values of the cutoff frequency c ($c_1 > c_2$).

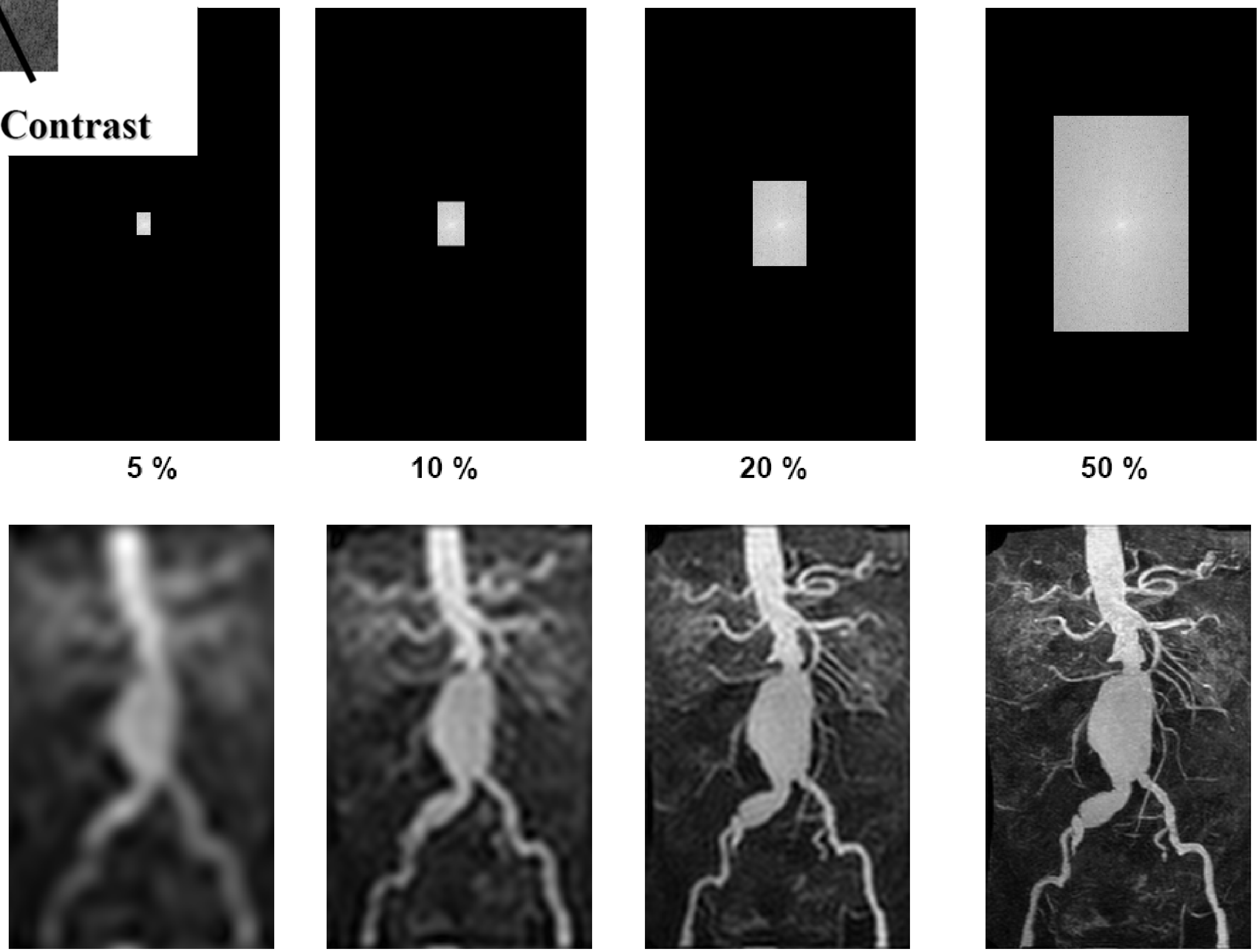
Spatial Frequency Revisited



Image

Fourier Space (log magnitude)





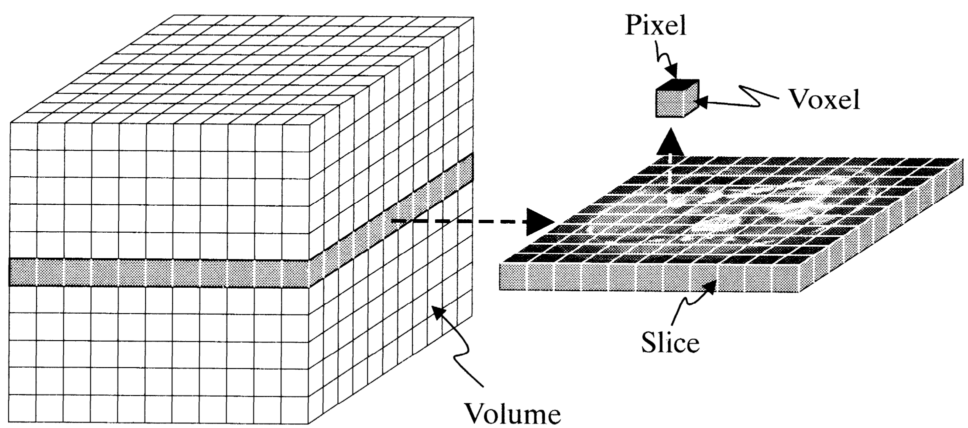
Sampling

What and Why?

Transformation from continuous signals into discrete signals is called ***sampling***. The sampled data can then be processed by digital hardware.

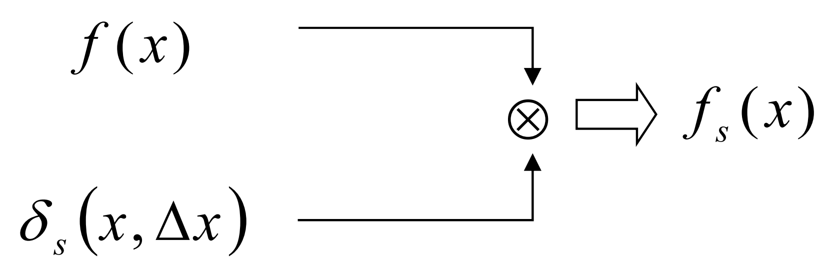
$$f(x,y) \Rightarrow f(m\Delta x, n\Delta y), \text{ for m, n} = 0,1,...$$

where $\Delta x, \Delta y$ are called sampling intervals or sampling periods.



Sampling in 1-D

Model

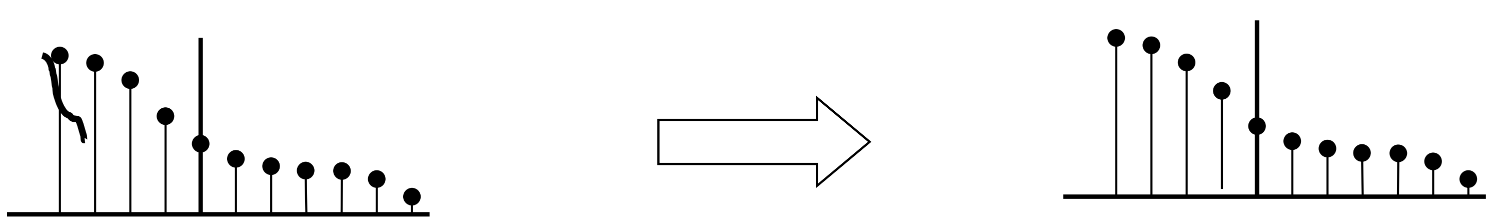


The 1-D sampling function

$$\delta_s(x, \Delta x) = \sum \delta(x - n \cdot \Delta x)$$

The sampled function

$$f_s(\mathrm{x}) = f(\mathrm{x}) \cdot \delta_s(x, \Delta x) = \sum f(n\Delta x)\delta(x - n\Delta x)$$



Fourier Transform of Sampled Function

$$F_{f_s}(u) = \mathcal{F}\left[f_s(x)\right] = \mathcal{F}\left[\delta_s(x,\Delta x)\cdot f(x)\right]$$

$$= comb(u\cdot \Delta x) * \mathcal{F}\left[f(x)\right]$$

$$= \left\{\sum_{n=-\infty}^{n=\infty} \delta\left[\Delta x\left(u - \frac{n}{\Delta x}\right)\right]\right\} * F(u)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{n=\infty}\left\{\delta\left(u - \frac{n}{\Delta x}\right) * F(u)\right\}$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{n=\infty} F\left(u - \frac{n}{\Delta x}\right)$$

$$comb(x) = \sum_{m=-\infty}^{\infty}\delta(x-m)$$

$$\delta_s(x,\Delta x) = \sum_{m=-\infty}^{\infty}\delta(x - m\Delta x)$$

$$\mathcal{F}[\delta_s(x,\Delta x)] = comb(u\cdot\Delta x)$$

Multiplication in one domain becomes convolution in the other,

Point Impulse Signal

- The sampling property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-\xi, y-\eta)dxdy = f(\xi,\eta)$$

- The scaling property

$$\delta(ax,by) = \frac{1}{|ab|}\delta(x,y)$$

$$\delta(x,y)\begin{cases} \neq 0, & x=0 \; and \;\; y=0 \\ =0, & otherwise \end{cases}$$

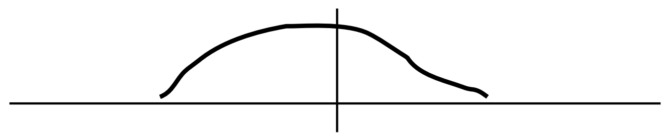
and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$$

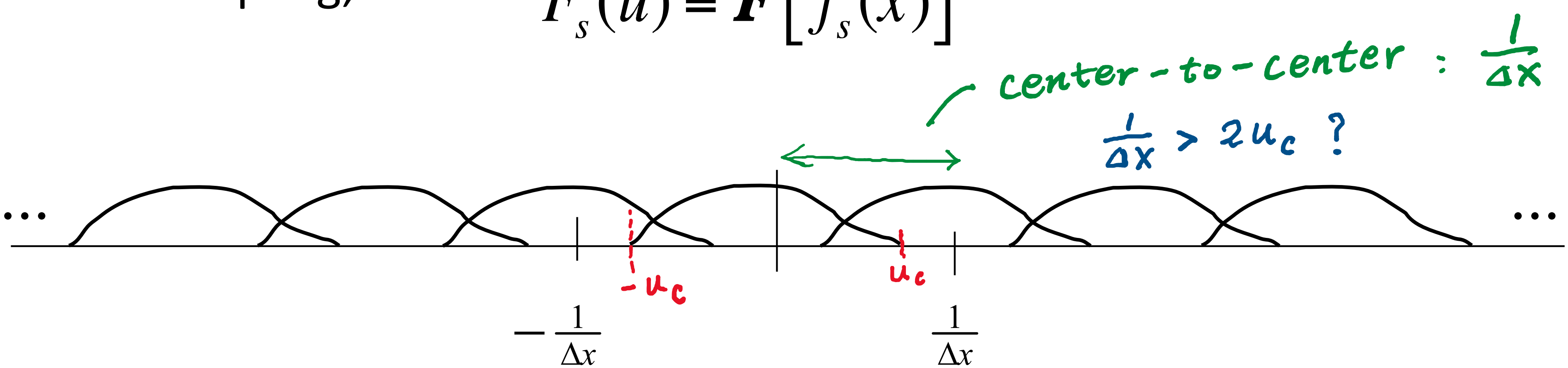
Fourier Transform of Sampled Function

$$\boldsymbol{F}\left[f_s(x)\right] = \frac{1}{\Delta x}\sum_{n=-\infty}^{n=\infty} \mathrm{F}\left(u-\frac{n}{\Delta x}\right)$$

Prior to sampling, $\mathrm{F}(u) = \boldsymbol{F}\left[f(x)\right]$



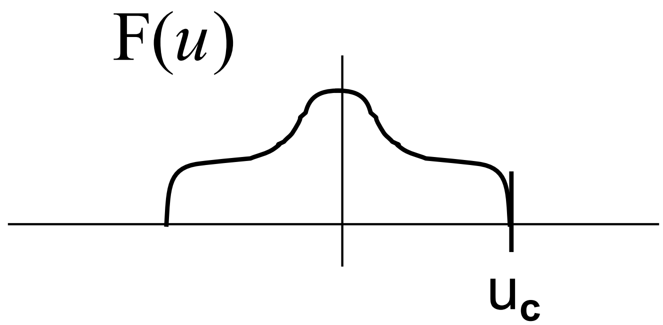
After sampling, $F_s(u) = \boldsymbol{F}\left[f_s(x)\right]$



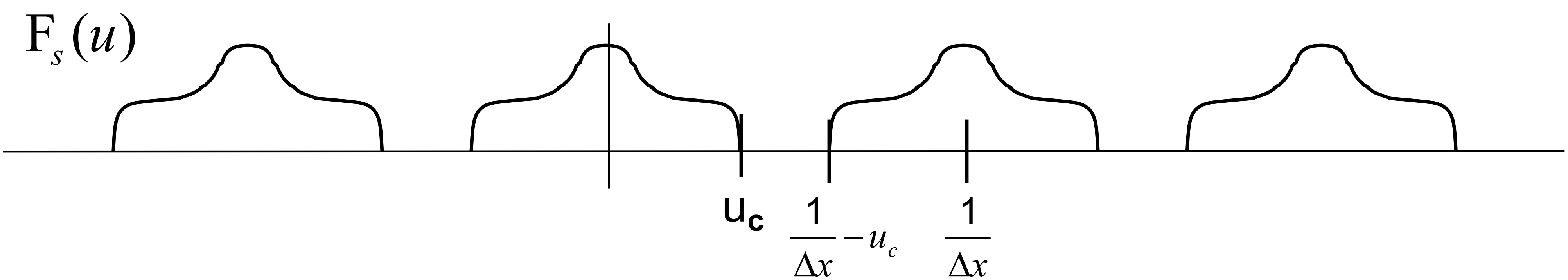
Fourier Transform of Sampled Function

If F(u) is band limited to u_c, (cutoff frequency)

F(u) = 0 for |u| > u_c.



To avoid overlap (aliasing),

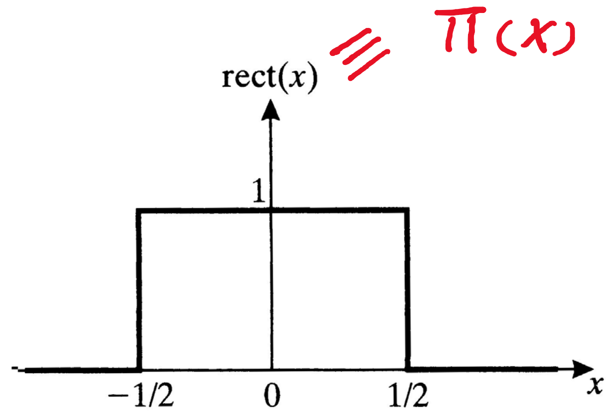


We would need $\frac{1}{\Delta x} - u_c > u_c$ **and therefore** $\frac{1}{\Delta x} > 2u_c$

Rect Function

- Rect function:

$$rect(x,y) = \begin{cases} 1, \text{ for } |x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\ 0, \text{ otherwise} \end{cases}$$

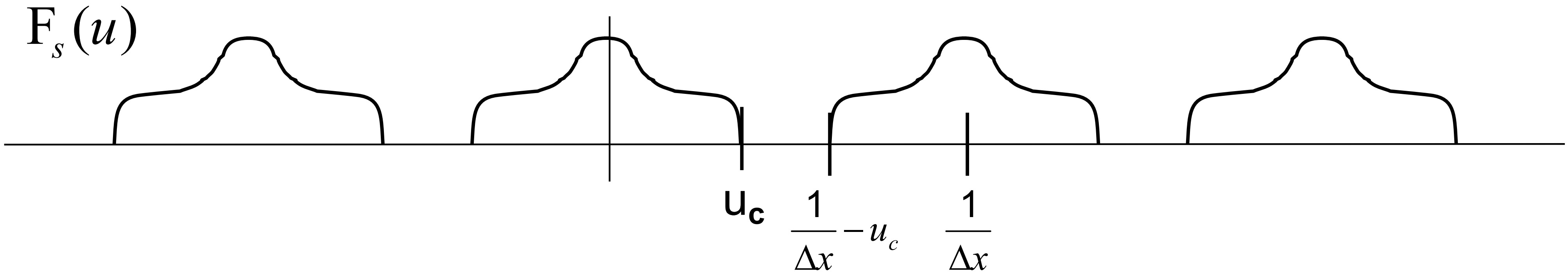


- It is normally used to pick up a particulate section of a given function:

$$f(x,y) \cdot rect\left(\frac{x-\xi}{w_X}, \frac{y-\eta}{w_Y}\right)$$

Fourier Transform of Sampled Function

To avoid overlap (aliasing),

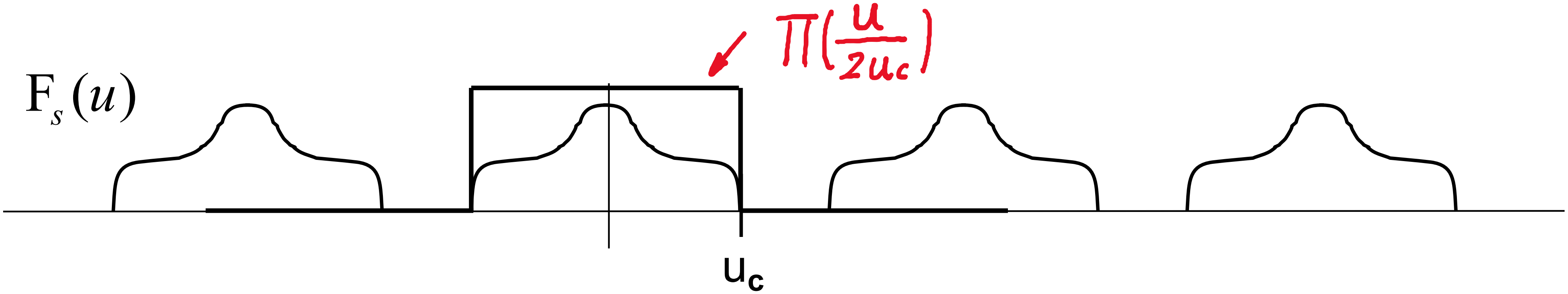


We would need $\frac{1}{\Delta x} - u_c > u_c$ **and therefore** $\frac{1}{\Delta x} > 2u_c$

$$f(x) = \mathcal{F}^{-1}\left[F_{f_s}(u) \cdot \Pi\left(\frac{u}{2u_c}\right)\right]$$

Covered in lecture

Restoration of Original Signal



Can we restore g(x) from the sampled frequency-domain signal? Yes, using the Interpolation Filter

$$H(u) = \Pi\left(\frac{u}{2u_c}\right)$$

$\Updownarrow$ Fourier transform

$$h(x) = 2u_c \cdot \mathrm{sinc}(2u_c x)$$

Restoration of Original Signal

The original function f can be recovered as

$$f(x) = \mathcal{F}^{-1}\left[F_{f_s}(u) \cdot \Pi\left(\frac{u}{2u_c}\right)\right]$$

$$f_s(x) = \sum f(n\Delta x)\delta(x - n\Delta x)$$

$$\begin{aligned} f(x) &= f_s(x) * h(x) \\ &= f_s(x) * [2u_c \cdot sinc(2u_c x)] \\ &= \left[\sum_{n=-\infty}^{\infty} f(n \cdot \Delta x) \cdot \delta(x - n \cdot \Delta x)\right] * [2u_c \cdot \mathrm{sinc}(2u_c x)] \\ &= \sum_{n=-\infty}^{\infty} 2 \cdot u_c \cdot f(n \cdot \Delta x) \cdot sinc(2u_c(x - n\Delta x)) \end{aligned}$$

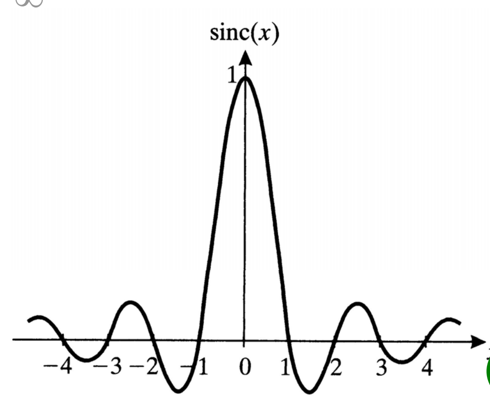
$$\mathrm{H}(u) = \Pi\left(\frac{u}{2u_c}\right)$$

$$\mathcal{F} \Updownarrow \quad \Updownarrow$$

$$h(x) = 2u_c \cdot \mathrm{sinc}(2u_c x)$$

$$\int_{-\infty}^{\infty} \delta(u - \xi) \cdot f(u) du = f(\xi)$$

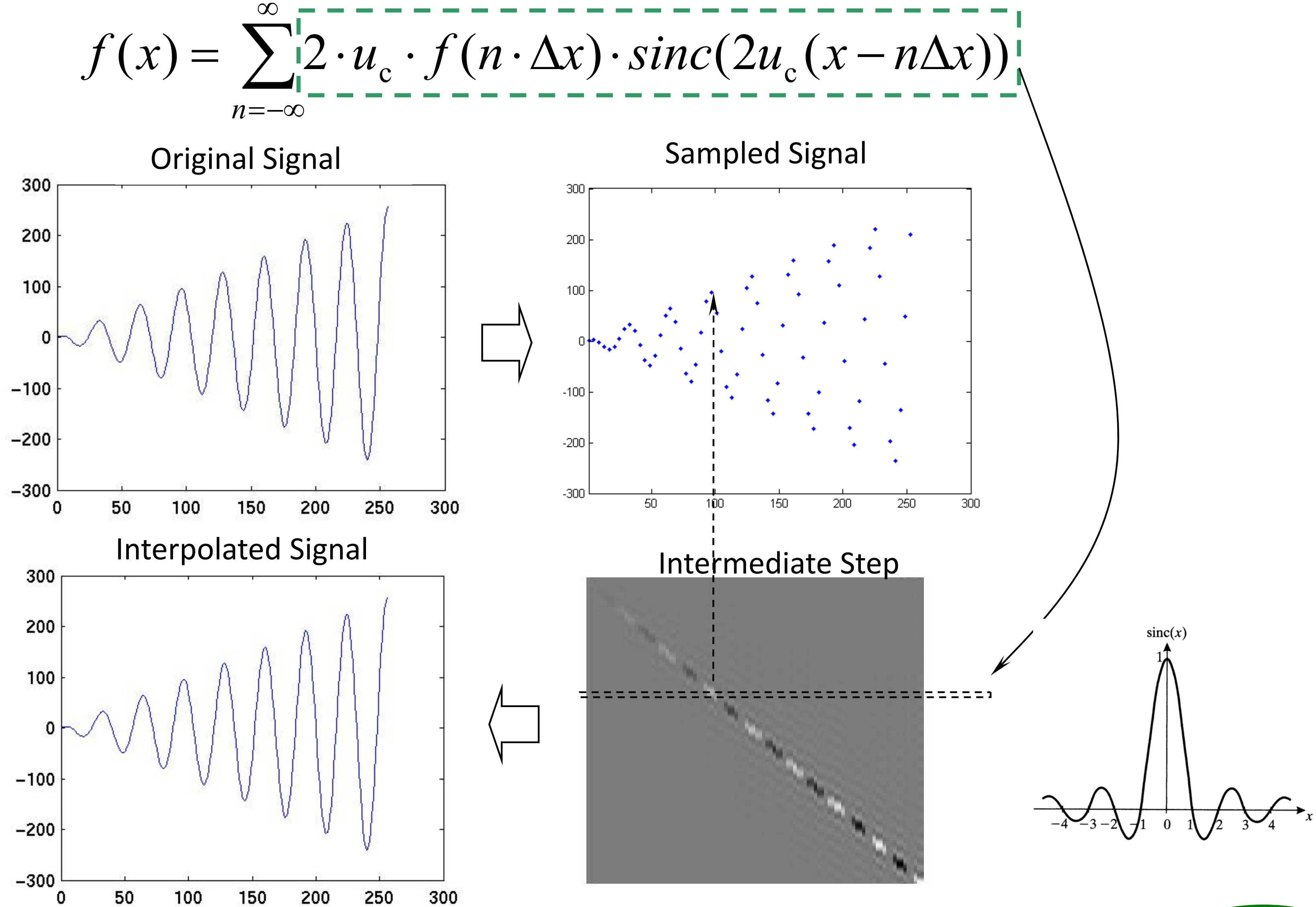
f(x) is restored from a combination of sinc functions, each weighted and shifted according to its corresponding sampling point.



Covered in lecture

An Example

$$f(x) = \sum_{n=-\infty}^{\infty} 2 \cdot u_c \cdot f(n \cdot \Delta x) \cdot sinc(2u_c(x - n\Delta x))$$



Adding all vertical values gives back the original function

Covered in lecture

Nyquist Condition

Nyquist Theorem:

In order to restore the original function, the sampling rate must be greater than twice the highest frequency component of the function.

Nyquist Sampling Interval:

The maximum sampling interval allowed without introduce aliasing is

$$\Delta x \leq \frac{1}{2u_c}$$

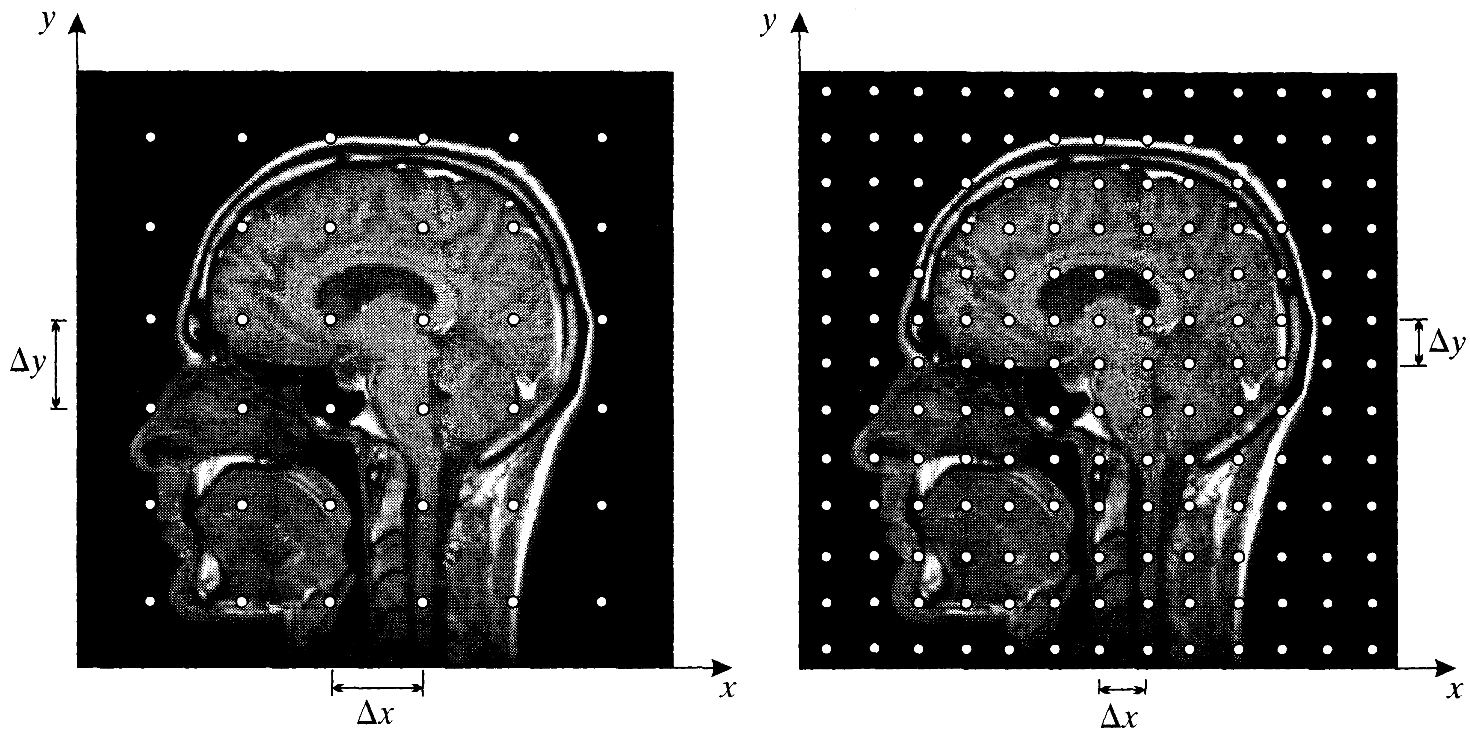
Covered in lecture

Two Dimensional Sampling

$$f_s(x,y) = f(x,y)\cdot\delta_s(x,y,\Delta x,\Delta y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(x,y)\cdot\delta(x-n\Delta x, y-m\Delta y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y)\cdot\delta(x-n\Delta x, y-m\Delta y)$$



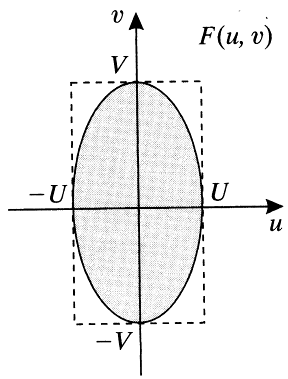
Fourier Transform of Sampled Image

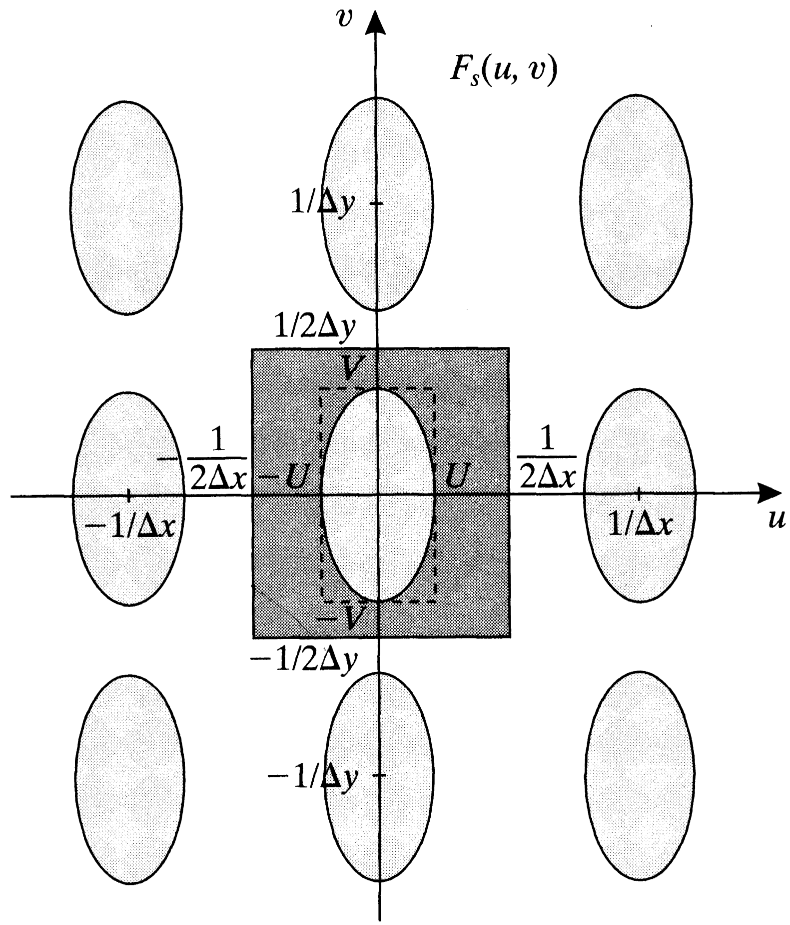
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(u-\xi,v-\eta)\cdot f(u,v)\,du\,dv = f(\xi,\eta)$$

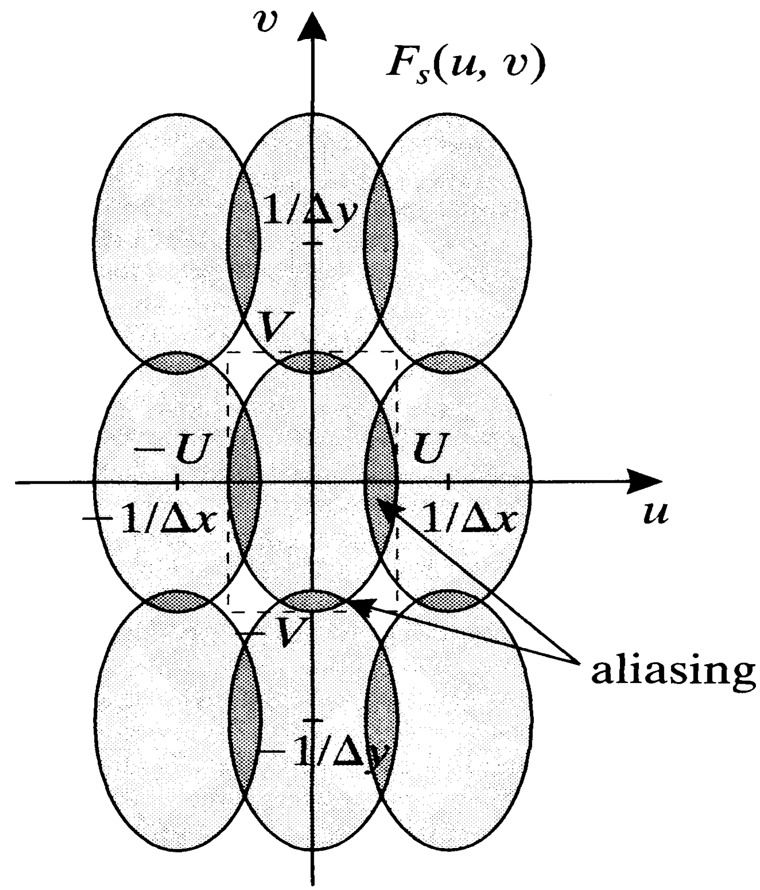
$$\mathrm{F}_{f_s}(u) = \mathfrak{F}\left[f_s(x,y)\right]$$
$$= \mathfrak{F}\left[\delta_s(x,y,\Delta x,\Delta y)\cdot f(x,y)\right]$$
$$= comb(u\cdot\Delta x, v\cdot\Delta y) * \mathfrak{F}\left[f(x,y)\right]$$
$$= \sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\delta\left[\Delta x\left(u-\frac{n}{\Delta x}\right),\Delta y\left(v-\frac{m}{\Delta y}\right)\right] * \mathrm{F}(u,v)$$
$$= \frac{1}{\Delta x\Delta y}\sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\delta\left(u-\frac{n}{\Delta x},v-\frac{m}{\Delta y}\right) * \mathrm{F}(u,v)$$
$$= \frac{1}{\Delta x\Delta y}\sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\mathrm{F}\left(u-\frac{n}{\Delta x},v-\frac{m}{\Delta y}\right)$$

The result: Replicated F(u,v), or "islands" every 1/ Δx in u, and 1/Δy in v.

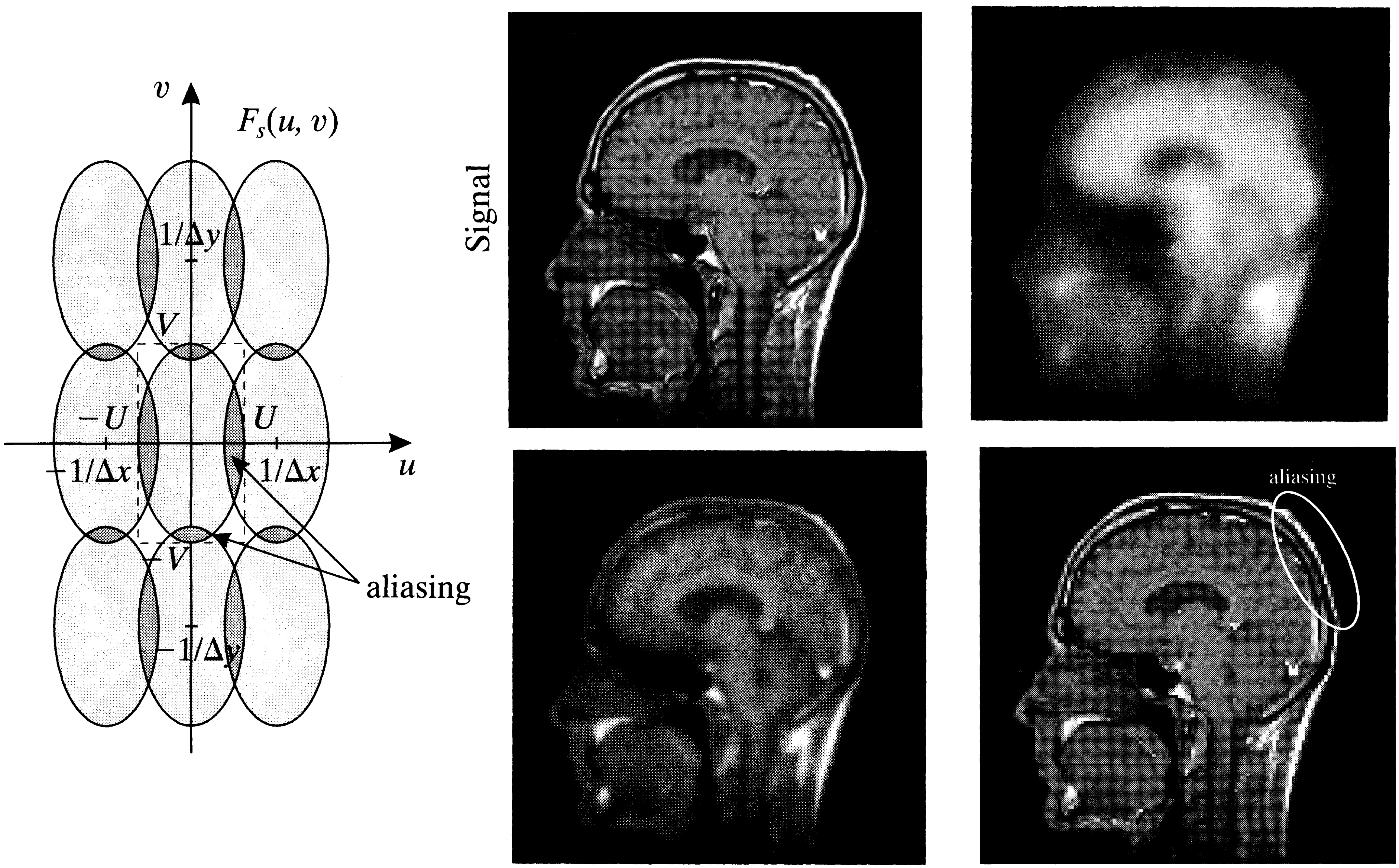
Fourier Transform of a Sampled 2-D Function







Consequence of Under-Sampling



Restoration of the Original 2-D Function

Interpolation Filter in 2-D

$$\mathrm{H}(u,v) = \prod(u\Delta x)\prod(v\Delta y)$$

$$\Updownarrow$$

$$h(x,y) = \left[\frac{1}{\Delta x}\cdot \mathrm{sinc}(\frac{x}{\Delta x})\right]\left[\frac{1}{\Delta y}\cdot \mathrm{sinc}(\frac{y}{\Delta y})\right]$$

Restoration of the Original 2-D Function

Given that the Nyquist sampling condition is met, the original function can be recovered exactly.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(u-\xi, v-\eta)\cdot f(u,v)\,du\,dv = f(\xi,\eta)$$

$$f(x,y) = f_s(x,y) * h(x,y)$$

$$= f_s(x,y) * \left[\frac{1}{\Delta x}\cdot \mathrm{sinc}(\frac{x}{\Delta x})\right]\left[\frac{1}{\Delta y}\cdot \mathrm{sinc}(\frac{y}{\Delta y})\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y)\cdot\delta(x-n\cdot\Delta x, y-m\cdot\Delta y) * \left\{\left[\frac{1}{\Delta x}\cdot \mathrm{sinc}(\frac{x}{\Delta x})\right]\left[\frac{1}{\Delta y}\cdot \mathrm{sinc}(\frac{y}{\Delta y})\right]\right\}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{\Delta x\Delta y} f(n\Delta x, m\Delta y)\cdot \mathrm{sinc}(\frac{x-n\cdot\Delta x}{\Delta x})\cdot \mathrm{sinc}(\frac{y-m\cdot\Delta y}{\Delta y})$$

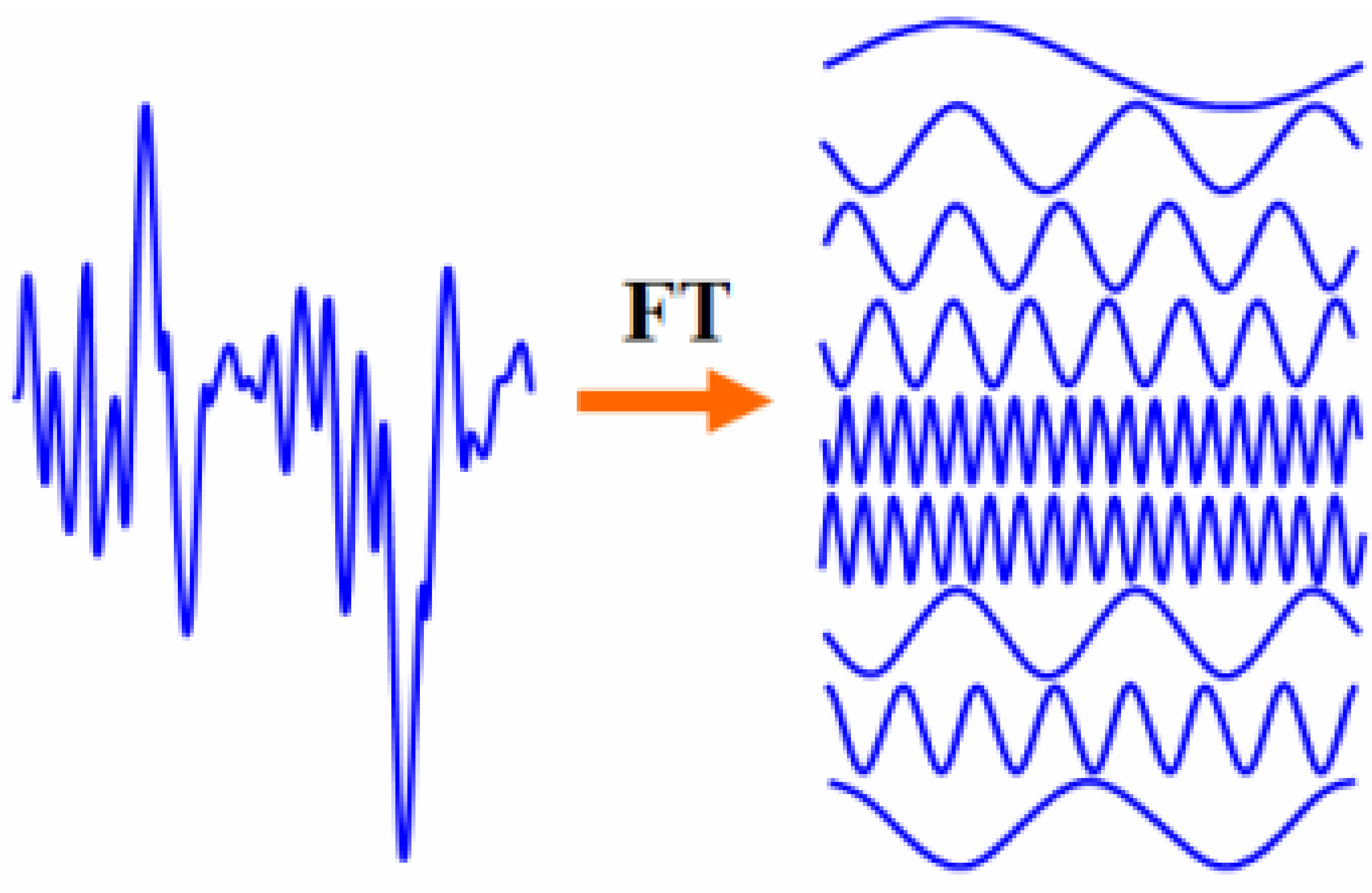
Fourier Transform of a Sampled 2-D Function

Harry Nyquist

Nyquist/Shannon Theory:
We must sample at twice the highest frequency in x and in y (U and V) to reconstruct the original signal.

From continuous to discrete Fourier transform

Fourier Transform and Spatial Frequency



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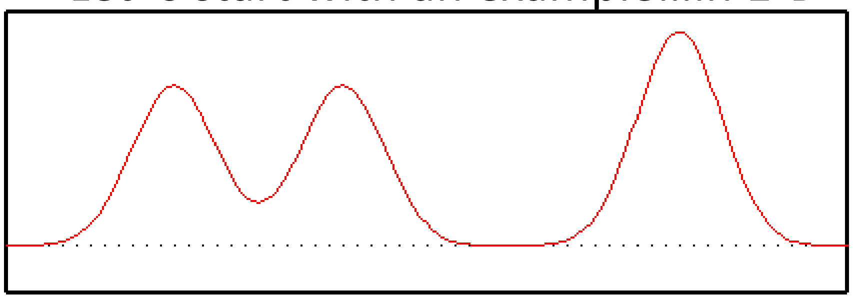
Fourier Transform and Spatial Frequency

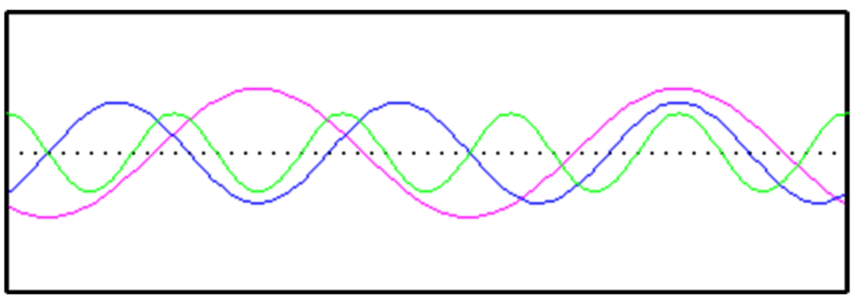
A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

So what does this mean exactly?

Let’s start with an example…in 1-D

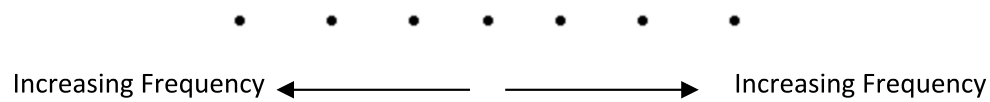


Can be represented by:



When you let these three waves interfere with each other you get your original wave function!

Since this object can be made up of 3 fundamental frequencies an ideal Fourier Transform would look something like this:



Notice that it is symmetric around the central point and that the amount of points radiating outward correspond to the distinct frequencies used in creating the image.

System Transfer Function

$$G(u,v) = F(u,v)H(u,v)$$

An ideal low-pass filter is defined as

$$H(u,v) = \begin{cases} 1 & \text{for} \sqrt{\mathrm{u}^2 + \mathrm{v}^2} \le c \\ 0 & \text{for} \sqrt{\mathrm{u}^2 + \mathrm{v}^2} > c \end{cases}$$

c is called the cut-off frequency.

Covered in lecture

System Transfer Function

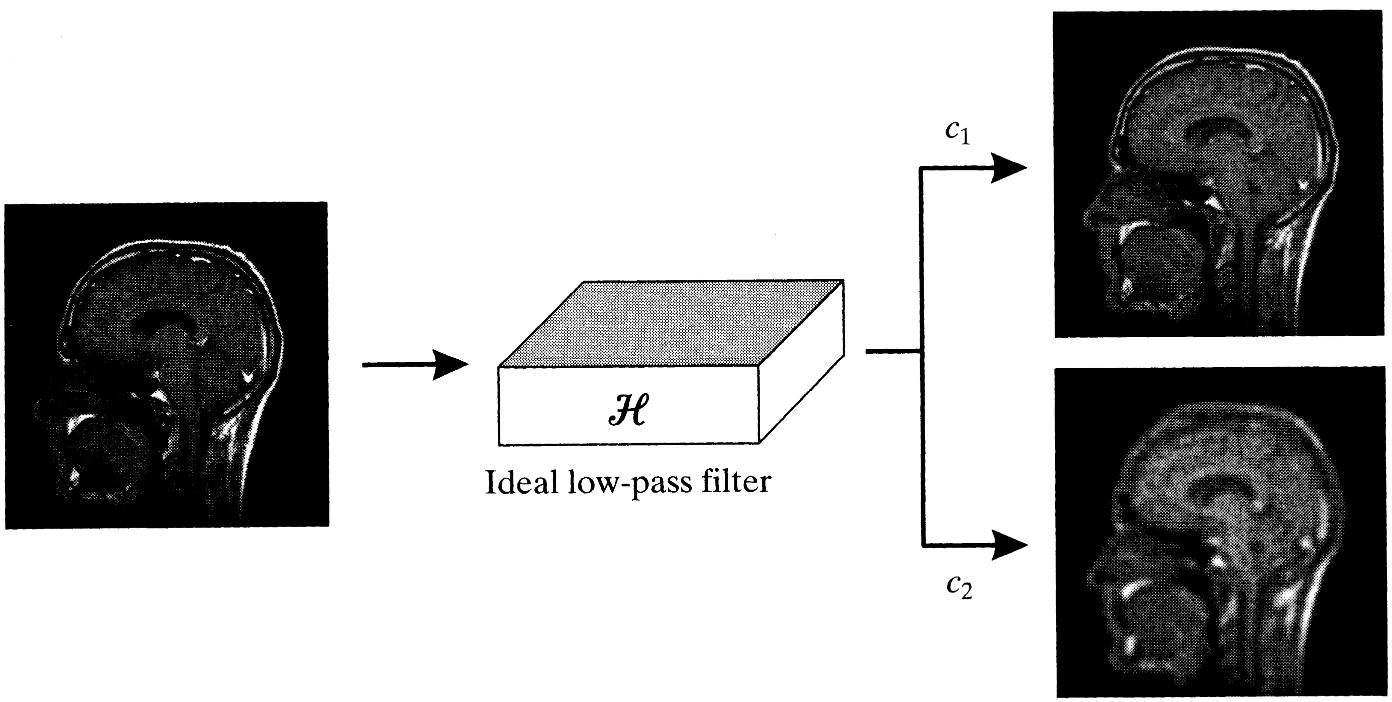


Figure 2.12
The response of an ideal low-pass filter for two values of the cutoff frequency c ($c_1 > c_2$).

Covered in lecture

Fourier Transform of Sampled Function

$$\mathrm{F}_{f_s}(u) = \mathrm{F}\left[f_s(x)\right] = \mathrm{F}\left[\delta_s(x,\Delta x)\cdot f(x)\right]$$

$$= comb(u\cdot \Delta x) * \mathrm{F}\left[f(x)\right]$$

$$= \left\{\sum_{n=-\infty}^{n=\infty} \delta\left[\Delta x\left(u - \frac{n}{\Delta x}\right)\right]\right\} * \mathrm{F}(u)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{n=\infty}\left\{\delta\left(u - \frac{n}{\Delta x}\right) * \mathrm{F}(u)\right\}$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{n=\infty} \mathrm{F}\left(u - \frac{n}{\Delta x}\right)$$

$$comb(x) = \sum_{m=-\infty}^{\infty} \delta(x-m)$$

$$\delta_S(x,\Delta x) = \sum_{m=-\infty}^{\infty} \delta(x - m\Delta x)$$

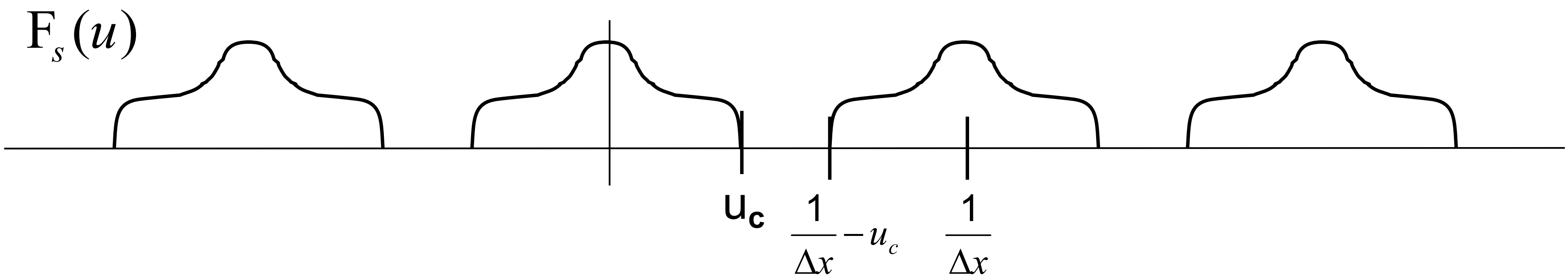
$$\mathfrak{F}\left[\delta_s(x,\Delta x)\right] = comb(u\cdot \Delta x)$$

Multiplication in one domain becomes convolution in the other,

Covered in lecture

Fourier Transform of Sampled Function

To avoid overlap (aliasing),



We would need $\frac{1}{\Delta x} - u_c > u_c$ **and therefore** $\frac{1}{\Delta x} > 2u_c$

$$f(x) = \mathcal{F}^{-1}\left[F_{f_s}(u) \cdot \Pi\left(\frac{u}{2u_c}\right)\right]$$

Covered in lecture

Discrete Fourier Transform

Nyquist Condition - Revisited

Nyquist Theorem:

In order to restore the original function, the sampling rate must be greater than twice the highest frequency component of the function.

For a continuous but band-width limited function, all “information content” it contains can be preserved by a finite number of samples …

Continuous Fourier Transform of Sampled Function

$$F_{f_s}(u) = \mathfrak{F}\left[f_s(x)\right] = \mathfrak{F}\left[\delta_s\left(x, \Delta x\right) \cdot f(x)\right]$$

$$= \frac{1}{\Delta x} \sum_{n=-\infty}^{n=\infty} F(u - \frac{n}{\Delta x})$$

Given that all "information content" of function f(x) is "carried" by a finite number (N) of samples in the spatial domain, can we equally use only N Fourier coefficients in spatial frequency domain to represent the sampled function?

Discrete Fourier Transform in 1-D

In reality, most experimental data comes in as sampled function defined on finite number of sampling points.

$$f_n = f(n\Delta x),\ \text{for n} = 0, 1, \ldots, \text{N} - 1$$
$$\text{where } \Delta x \text{ is the sampling interval}$$

The discrete Fourier transform (DFT) provides a de-composition of the sampled function with N spatial frequencies

$$U_n = \frac{n}{N\Delta x},\ \text{n} = -\frac{N}{2}, \ldots, \frac{N}{2}$$
$$\text{where } \Delta x \text{ is the sampling interval}$$

Note that there are really N+1 spatial frequencies here, but the two extreme values will be identical.

Discrete Fourier Transform

In reality, most experimental data comes in as sampled function defined on finite number of sampling points.

$$\boldsymbol{F}\left[f_n\right] \equiv U_n = \int_{-\infty}^{\infty} f(x) e^{-j2\pi u_n x}\, dx \approx \sum_{k=0}^{N-1} f(k\Delta x) e^{-j2\pi u_n x} \Delta x$$

$$= \Delta x \sum_{k=0}^{N-1} f(k\Delta x) e^{-\frac{j2\pi nk}{N}} = \Delta x \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}$$

where Δx is the sampling interval and $u_n = \dfrac{n}{N\Delta x}$, is a given spatial frenquency.

Note that U_n is periodic in n with a period N,

$$U_{-n} = U_{N-n}, \mathrm{n} = 1,2,\ldots$$

So we normally let n in U_n varying from 0 to N-1.

Discrete Fourier Transform in 1-D

The discrete Fourier transform (DFT) is defined as

$$F_n = \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}, \mathrm{n = 0,1,2,...N - 1}$$

$\mathrm{n} = 0$ corresponding to the DC component (spatial frenquency is zero)

$\mathrm{n} = 1,..., \mathrm{N}/2 - 1$ are corresponding to the positive frenquencies $0 < \mathrm{u} < \mathrm{u_c}$

$\mathrm{n} = \mathrm{N}/2, ..., \mathrm{N} - 1$ are corresponding to the negative frenquencies $-\mathrm{u_c} < \mathrm{u} < 0$

The inverse DFT is defined as

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{-\frac{j2\pi nk}{N}}, \mathrm{k = 0,1,2,...N - 1}$$

Discrete Fourier Transform

The 1-D Discrete Fourier Transform (DFT) is defined as

$$F_n = \sum_{k=0}^{N-1} f_k W^{nk}, \quad n=0,1,2,\ldots,N-1$$

$$\text{with} \quad W \equiv e^{\frac{j2\pi}{N}}$$

So calculating the DFT involves multiplication of a vector with a matrix

$$\mathbf{F} = \mathbf{Df} \quad \Rightarrow \quad \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

It requires $O(N^2)$ complex operations!

Inverse Discrete Fourier Transform

The original spatial domain function can be recovered

$$f_n = \sum_{k=0}^{N-1} F_k W^{-nk}, 0 \leq \mathrm{n} \leq \mathrm{N-1} \text{ and } W \equiv e^{\frac{j2\pi}{N}}$$

So calculating the DFT involves multiplication of a vector with a matrix

$$\mathbf{f} = \mathbf{D}^{-1}\mathbf{F} \quad \Rightarrow \quad \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W^{-1} & W^{-2} & \cdots & W^{-(N-1)} \\ 1 & W^{-2} & W^{-4} & \cdots & W^{-2(N-1)} \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{-(N-1)} & W^{-2(N-1)} & \cdots & W^{-(N-1)^2} \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{pmatrix}$$

Fast Fourier Transform

In fact, the DFT can be calculated using the Fast Fourier Transform (FFT) algorithm …

$$F_n = \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}, \mathrm{n} = 0,1,2,\ldots,\mathrm{N-1}$$

$$F_n = \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}$$

$$= \sum_{k=0}^{N/2-1} e^{-\frac{j2\pi n(2k)}{N}} f_{2k} + \sum_{k=0}^{N/2-1} e^{-\frac{j2\pi n(2k+1)}{N}} f_{2k+1}$$

$$= \sum_{k=0}^{N/2-1} e^{-\frac{j2\pi nk}{N/2}} f_{2k} + W^n \sum_{k=0}^{N/2-1} e^{-\frac{j2\pi nk}{N/2}} f_{2k+1} \qquad W \equiv e^{\frac{j2\pi}{N}}$$

$$= F_n^e + W^n F_n^o, \qquad \mathrm{n} = 0,1,2,\ldots,\mathrm{N-1}$$

Fast Fourier Transform

The wonderful thing about the previous results is that it can be used recursively … so that …

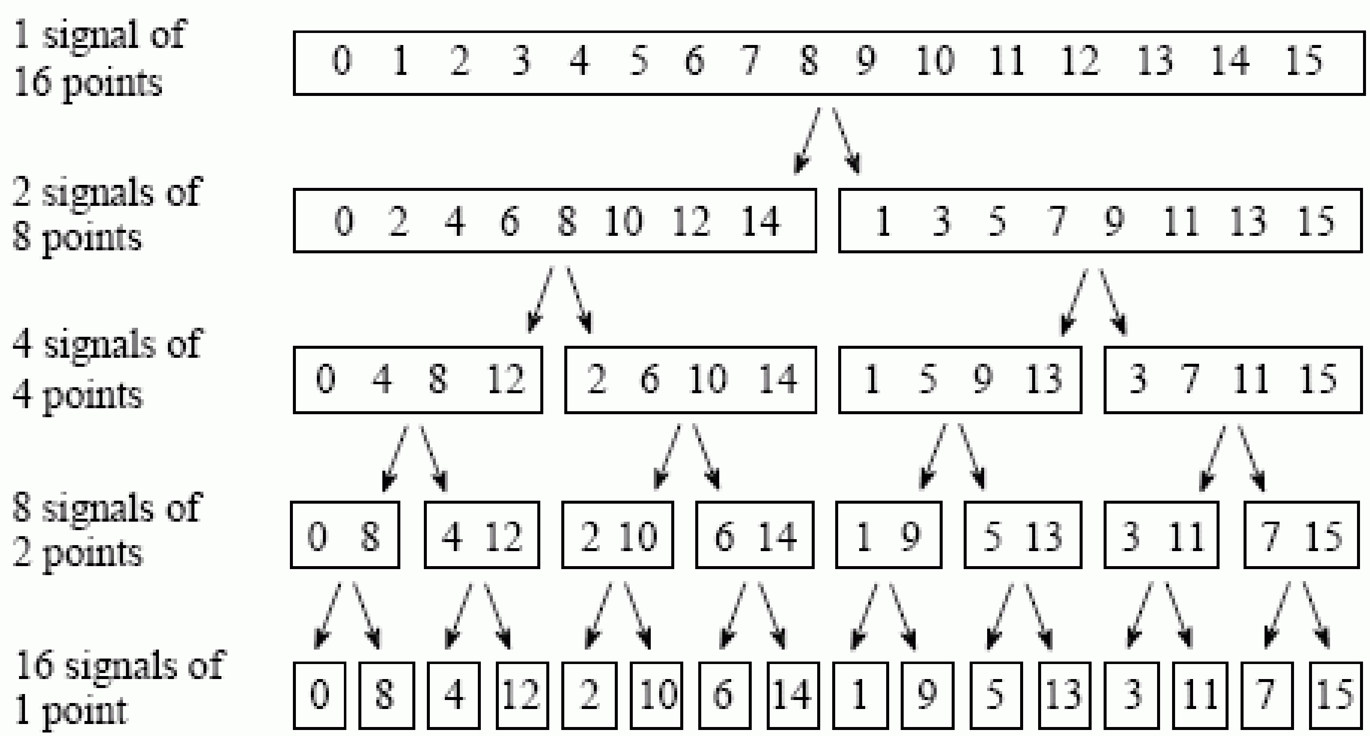


FIGURE 12-2
The FFT decomposition. An N point signal is decomposed into N signals each containing a single point. Each stage uses an *interlace decomposition*, separating the even and odd numbered samples.

http://www.dspguide.com/ch12/2.htm

Discrete Fourier Transform in 1-D

The discrete Fourier transform (DFT) is defined as

$$F_n = \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}, \mathrm{n} = 0,1,2,...\mathrm{N} - 1$$

$\mathrm{n} = 0$ corresponding to the DC component (spatial frenquency is zero)

$\mathrm{n} = 1,..., \mathrm{N}/2 - 1$ are corresponding to the positive frenquencies $0 < \mathrm{u} < \mathrm{u_c}$

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The inverse DFT is defined as

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{-\frac{j2\pi nk}{N}}, \mathrm{k} = 0,1,2,...\mathrm{N} - 1$$

Fast Fourier Transform

Number of operations are needed for an 1-D Discrete Fourier Transform (DFT) of N points:

$$F_n = \sum_{k=0}^{N-1} f_k W^{nk}$$

N × N complex multiplications and N*(N-1) complex additions!

Number of operations are needed for an 1-D Discrete FFT of N points:

$N/2 \times (\log_2 N-1) \sim N/2 \times \log_2 N$ complex multiplications and $N \times \log_2 N$ complex additions!

$N \times N \rightarrow N/2 \cdot \log_2 N$

Fast Fourier Transform

For an 1-D Discrete Fourier Transform of 1024 points:

DFT: ~1048576 ($\times$) and 1047552 (+)

Versus

FFT: 5120 ($\times$) and 10240 (+)

For convolution operation:

$$g(x,y) = f(x,y) * h(x,y) = \mathscr{F}^{-1}\left\{\mathscr{F}\left[f(x,y)\right]\cdot\mathscr{F}\left[g(x,y)\right]\right\}$$

Direct: ~1048576 ($\times$)

Versus

FFT: ~5120 $\times$ 3 ($\times$)

A factor of ~600 difference!

Properties of Discrete Fourier Transform

Linearity

$$\text{if } f(n) \Leftrightarrow F(m) \text{ and } g(n) \Leftrightarrow G(m)$$
$$\text{then } af(\mathrm{x}) + bg(\mathrm{x}) \Leftrightarrow aF(u) + bG(u)$$

Shifting

$$DFT\left[f(n-k)\right] \Rightarrow DFT\left[\mathrm{F(m)}\right] e^{-i \cdot 2\pi \cdot \frac{km}{N}}$$

Example : if k=1

there is a 2π shift as
m varies from 0 to N-1

Discrete Fourier Transform in 2-D

2-D DFT is defined as

$$F(n,m) \equiv \sum_{l=0}^{M-1}\sum_{k=0}^{N-1} f_s(k,l) e^{-\frac{j2\pi nk}{N}} e^{-\frac{j2\pi ml}{M}},$$

where

$$f(k,l) \equiv f(k\Delta x, l\Delta y)$$

$$\mathrm{n = 0,1,2,...N-1 \; and \; m = 0,1,2,...M-1}$$

Similar to the continuous case,

$$\begin{aligned} F(n,m) &= \text{FFT on Index 1 (FFT on Index 2}[f(k,l)]) \\ &= \text{FFT on Index 2 (FFT on Index 1}[f(k,l)]) \end{aligned}$$

Discrete Inverse Fourier Transform in 2-D

2-D inverse DFT is defined as

$$f_s(n,m) \equiv \frac{1}{N \cdot M} \sum_{l=0}^{M-1} \sum_{k=0}^{N-1} F(k,k) e^{-\frac{j2\pi nk}{N}} e^{-\frac{j2\pi ml}{M}},$$

where

$$f_s(n,m) \equiv f(n\Delta x, m\Delta y)$$

$$\text{n} = 0,1,2,\ldots\text{N - 1 and m} = 0,1,2,\ldots\text{M - 1}$$

Review of Key Concepts (1)

Continuous Fourier Transform

- For any square-integrable function f(x,y), a continuous Fourier transform is defined as

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$where\ j = \sqrt{-1}$$

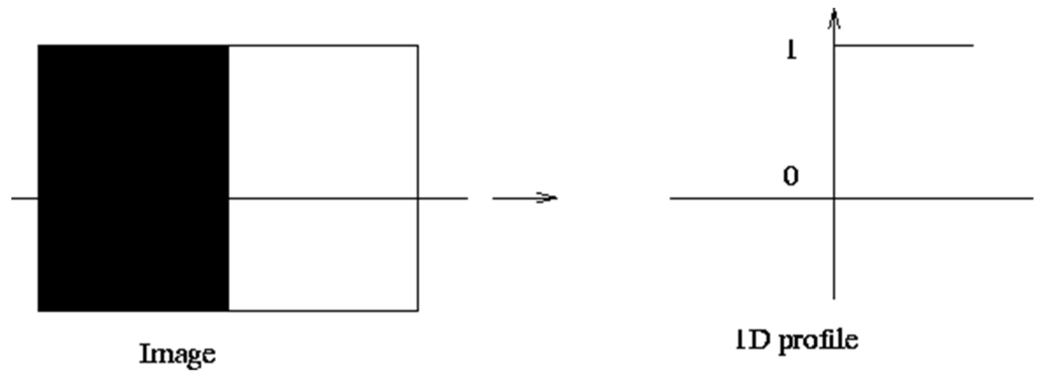
- We can also define a inverse Fourier transform as

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

Review of Key Concepts (1)

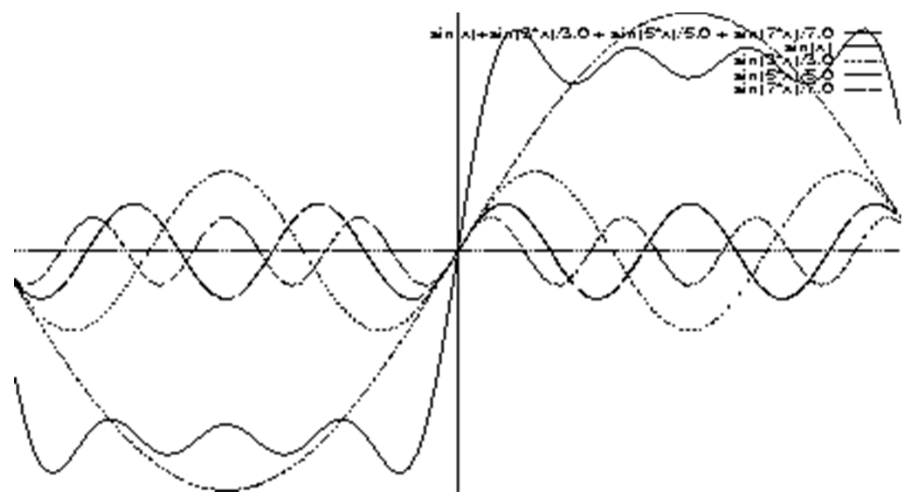
Spatial Frequency

Spatial frequency is a characteristic of any structure that is periodic across position in space. It is a measure of how often the structure repeats per unit of distance.



$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{-j2\pi(ux+vy)}dudv$$

$$e^{-j2\pi(ux+vy)}$$
$$=\cos[2\pi(ux+vy)]+j\sin[2\pi(ux+vy)]$$



Review of Key Concepts (3)

Convolution Theorem

$$\mathscr{F}\left[f(x,y)\cdot g(x,y)\right]=\mathscr{F}\left[f(x,y)\right]*\mathscr{F}\left[g(x,y)\right]$$

$$\Updownarrow$$

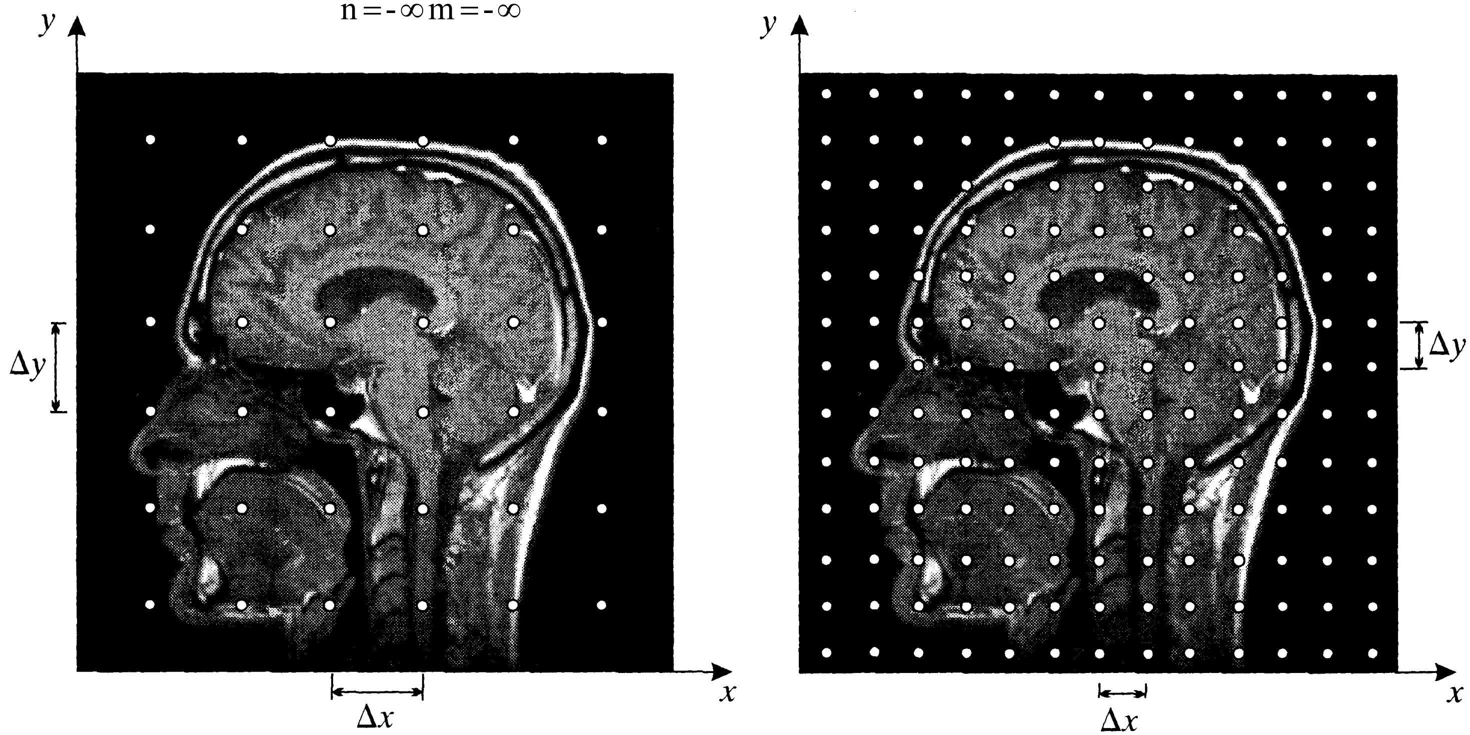
$$\mathscr{F}\left[f(x,y)*g(x,y)\right]=\mathscr{F}\left[f(x,y)\right]\cdot\mathscr{F}\left[g(x,y)\right]$$

Review of Key Concepts (4)
Two Dimensional Sampling

$$f_s(x,y) = f(x,y)\cdot\delta_s(x,y,\Delta x,\Delta y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(x,y)\cdot\delta(x-n\Delta x, y-m\Delta y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y)\cdot\delta(x-n\Delta x, y-m\Delta y)$$



Fourier Transform of Sampled Image

$$F_{f_s}(u) = \mathfrak{F}\left[f_s(x,y)\right]$$

$$= \mathfrak{F}\left[\delta_s\left(x,y,\Delta x,\Delta y\right)\cdot f(x,y)\right]$$

$$= comb\left(u\cdot\Delta x, v\cdot\Delta y\right) * \mathfrak{F}\left[f(x,y)\right]$$

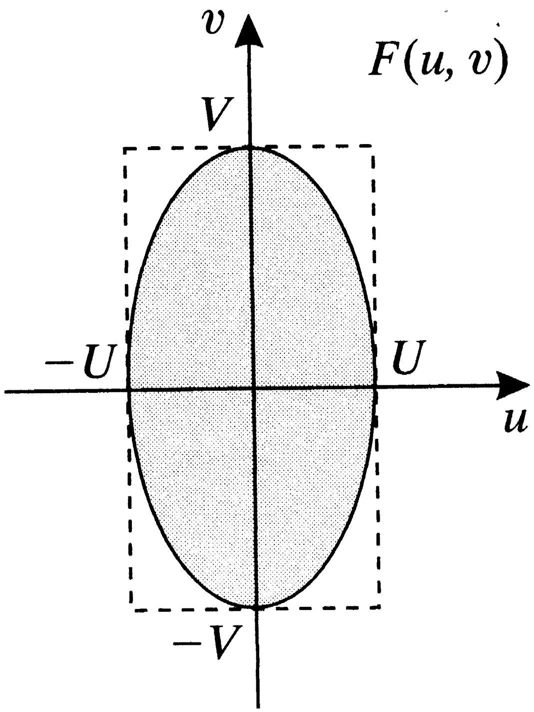
$$= \sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\delta\left[\Delta x\left(u-\frac{n}{\Delta x}\right),\Delta y\left(v-\frac{m}{\Delta y}\right)\right] * F(u,v)$$

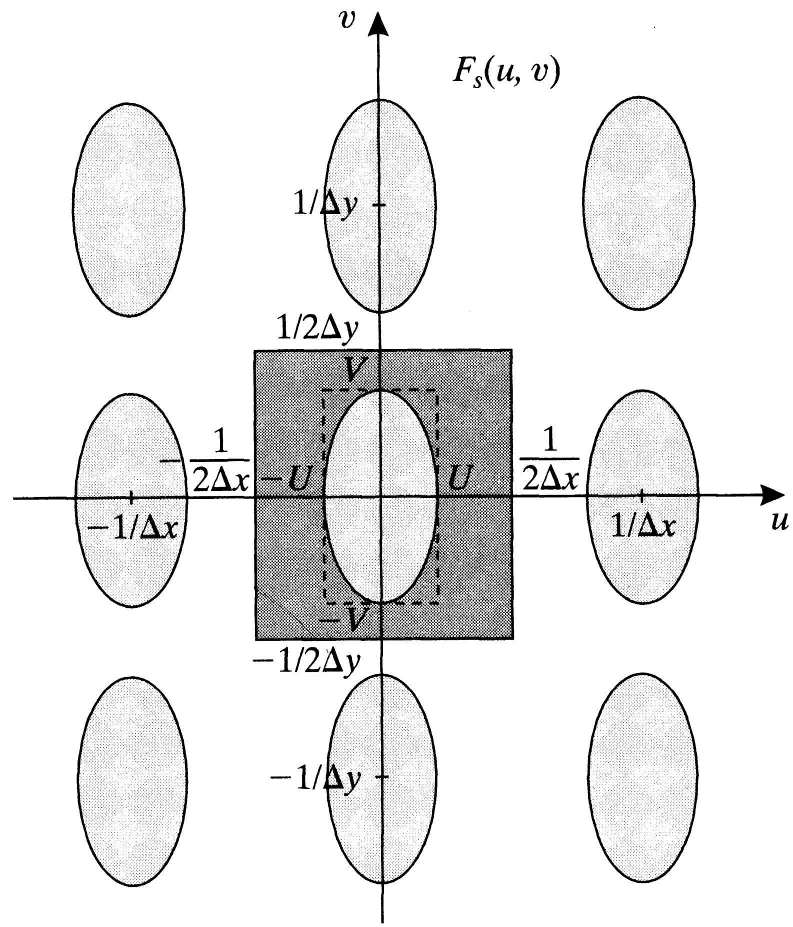
$$= \frac{1}{\Delta x\Delta y}\sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\delta(u-\frac{n}{\Delta x},v-\frac{m}{\Delta y}) * F(u,v)$$

$$= \frac{1}{\Delta x\Delta y}\sum_{n=-\infty}^{n=\infty}F(u-\frac{n}{\Delta x},v-\frac{m}{\Delta y})$$

The result: Replicated F(u,v), or "islands" every 1/ Δx in u, and 1/Δy in v.

Review of Key Concepts (5)
Two Dimensional Sampling





Nyquist/Shannon Theory:

We must sample at twice the highest frequency in x and in y (U and V) to reconstruct the original signal.

Two Dimensional Sampling

Nyquist Theorem:

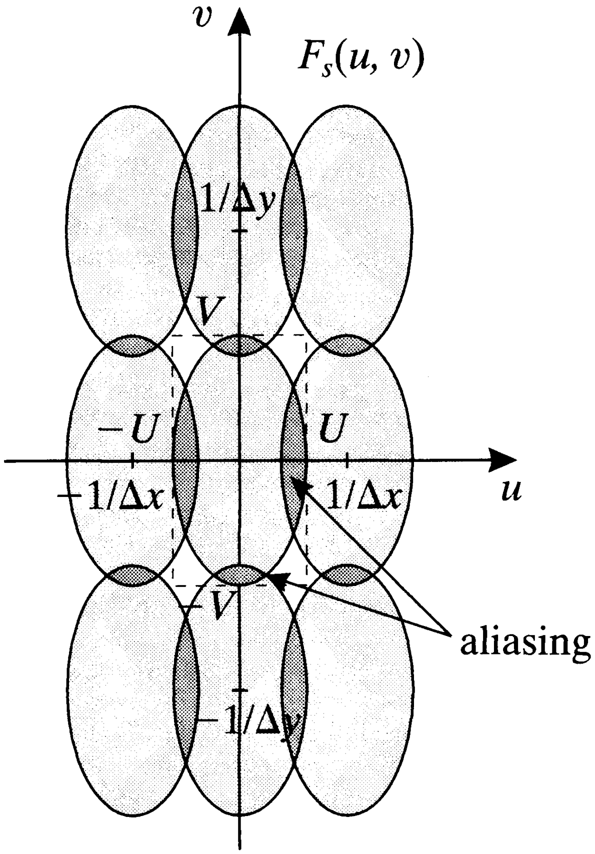
In order to restore the original function, the sampling rate must be greater than twice the highest frequency component of the function.

Nyquist Sampling Interval:

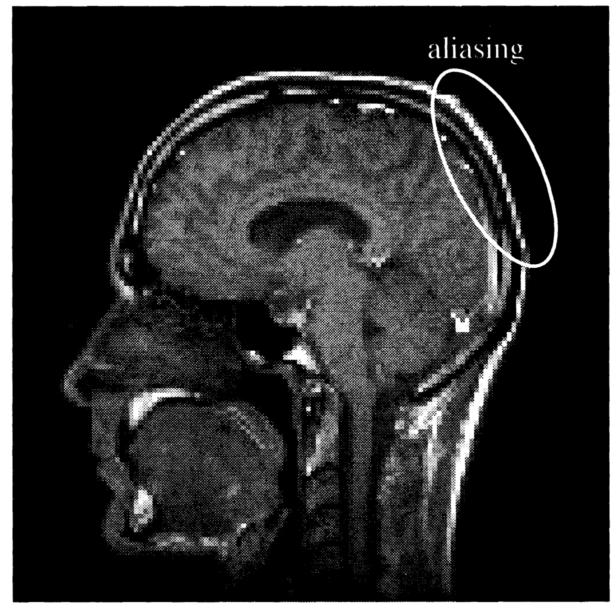
The maximum sampling interval allowed without introduce aliasing is

$$\Delta x \leq \frac{1}{2u_c}$$

Example – Fourier Transform of a Continuous Function



(a)



(b)

Aliasing due to insufficient sampling

Review of Key Concepts (6)
Discrete Fourier Transform

The discrete Fourier transform (DFT) is defined as

$$F_n = \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}, \mathrm{n} = 0,1,2,\ldots\mathrm{N-1}$$

$\mathrm{n} = 0$ corresponding to the DC component (spatial frenquency is zero)

$\mathrm{n} = 1,\ldots,\mathrm{N}/2-1$ are corresponding to the positive frenquencies $0 < \mathrm{u} < \mathrm{u_c}$

$\mathrm{n} = \mathrm{N}/2,\ldots,\mathrm{N}-1$ are corresponding to the negative frenquencies $-\mathrm{u_c} < \mathrm{u} < 0$

And the inverse DFT is defined as

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{-\frac{j2\pi nk}{N}}, \mathrm{k} = 0,1,2,\ldots\mathrm{N-1}$$

Review of Key Concepts (6)

Fast Fourier Transform

Number of operations are needed for an 1-D Discrete Fourier Transform (DFT) of N points:

$$F_n = \sum_{k=0}^{N-1} f_k W^{nk}$$

N × N complex multiplications and N*(N-1) complex additions!

Number of operations are needed for an 1-D Discrete FFT of N points:

N/2 × ($\log_2$N-1) ~ N/2 × $\log_2$N complex multiplications and N × $\log_2$N complex additions!