2.1 (20 pt) Determine whether the following signals are separable. Fully justify your answers.

(a) \( \delta_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) \)

(b) \( \delta_I(x, y) = \delta(x \cos(\theta) + y \sin(\theta) + l) \)

(c) \( e(x, y) = e^{j2\pi(u_o x + v_o y)} \)

(d) \( s(x, y) = \sin[2\pi(u_o x + v_o y)] \)

Solutions:

(a) \( \delta_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) = \sum_{m=-\infty}^{\infty} \delta(x - m) \cdot \sum_{n=-\infty}^{\infty} \delta(y - n) \), therefore it is a separable signal.

(b) \( \delta_I(x, y) \) is separable if \( \sin(2\theta) = 0 \). In this case, either \( \sin \theta = 0 \) or \( \cos \theta = 0 \), \( \delta_I(x, y) \) is a product of a constant function in one axis and a 1-D delta function in another. But in general, \( \delta_I(x, y) \) is not separable.

(c) \( e(x, y) = \exp[j2\pi(u_0 x + v_0 y)] = \exp[j2\pi u_0 x] \cdot \exp[j2\pi v_0 y] = e_{1D}(x; u_0) \cdot e_{1D}(y; v_0) \), where \( e_{1D}(x; \omega) = \exp(j2\pi \omega t) \). Therefore, \( e(x, y) \) is a separable signal.

(d) \( s(x, y) \) is a separable signal when \( u_0 v_0 = 0 \). For example, if \( u_0 = 0 \), \( s(x, y) = \sin(2\pi v_0 y) \) is the product of a constant signal in \( x \) and a 1-D sinusoidal signal in \( y \). But in general, when both \( u_0 \) and \( v_0 \) are nonzero, \( s(x, y) \) is not separable.

2.6 (5 pts) Determine whether the system \( g(x, y) = f(x, -1) + f(0, y) \) is

(a) linear?

(b) shift-invariant?

Solutions:

(a) If \( g'(x, y) \) is the response of the system to input \( \sum_{k=1}^{K} w_k f_k(x, y) \), then

\[
\begin{align*}
g'(x, y) &= \sum_{k=1}^{K} w_k f_k(x, y) + \sum_{k=1}^{K} w_k f_k(0, y) \\
&= \sum_{k=1}^{K} w_k [f_k(x, -1) + f_k(0, y)] \\
&= \sum_{k=1}^{K} w_k g_k(x, y)
\end{align*}
\]

where \( g_k(x, y) \) is the response of the system to input \( f_k(x, y) \). Therefore, the system is linear.
(b) If \( g'(x, y) \) is the response of the system to input \( f(x - x_0, y_0) \), then
\[
g'(x, y) = f(x - x_0, -1 - y_0) + f(-x_0, y - y_0);
\]
while
\[
g(x - x_0, y - y_0) = f(x - x_0, -1) + f(0, y - y_0).
\]
Since \( g'(x, y) \neq g(x - x_0, y - y_0) \), the system is not shift-invariant.

2.7 (20 pts) For each system with the following input-output equation, determine whether the system is (1) linear and (2) shift-invariant.

(a) \( g(x, y) = f(x, y)f(x - x_0, y) \).

(b) \( g(x, y) = \int_{-\infty}^{\infty} f(x, \eta)d\eta \)

Solution:

(a) If \( g'(x, y) \) is the response of the system to input \( \sum_{k=1}^{K} w_k f_k(x, y) \), then
\[
g'(x, y) = \left( \sum_{k=1}^{K} w_k f_k(x, y) \right) \left( \sum_{k=1}^{K} w_k f_k(x - x_0, y - y_0) \right)
\]
\[
= \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j f_i(x, y) f_j(x - x_0, y - y_0),
\]
while
\[
\sum_{k=1}^{K} w_k g_k(x, y) = \sum_{k=1}^{K} w_k f_k(x, y) f_k(x - x_0, y - y_0).
\]
Since \( g'(x, y) \neq \sum_{k=1}^{K} g_k(x, y) \), the system is nonlinear.

On the other hand, if \( g'(x, y) \) is the response of the system to input \( f(x - a, y - b) \), then
\[
g'(x, y) = f(x - a, y - b) f(x - a - x_0, y - b - y_0)
\]
\[
= g(x - a, y - b)
\]
and the system is thus shift-invariant.
2.9 (20 pts) Consider the 1-D system whose input-output equation is given by
\[ g(x) = f(x) \ast f(x), \]

(a) Write an integral expression that gives \( g(x) \) as a function of \( f(x) \).

(b) Determine whether the system is linear.

(c) Determine whether the system is shift-invariant.

Solution:

(a) \[ g(x) = \int_{-\infty}^{\infty} f(x-t) f(t) \, dt. \]

(b) Given an input as \( a f_1(x) + b f_2(x) \), where \( a, b \) are some constant, the output is
\[
\begin{align*}
g'(x) &= [af_1(x) + bf_2(x)] \ast [af_1(x) + bf_2(x)] \\
&= a^2 f_1(x) * f_1(x) + 2ab f_1(x) * f_2(x) + b^2 f_2(x) * f_2(x) \\
&\neq ag_1(x) + bg_2(x),
\end{align*}
\]

where \( g_1(x) \) and \( g_2(x) \) are the output corresponding to an input of \( f_1(x) \) and \( f_2(x) \) respectively. Hence, the system is nonlinear.

(c) Given a shifted input \( f_1(x) = f(x-x_0) \), the corresponding output is
\[
\begin{align*}
g_1(x) &= f_1(x) \ast f_1(x) \\
&= \int_{-\infty}^{\infty} f_1(x-t) f_1(t) \, dt \\
&= \int_{-\infty}^{\infty} f(x-t-x_0) f_1(t-x_0) \, dt.
\end{align*}
\]
Changing variable $t' = t - x_0$ in the above integration, we get

$$g_1(x) = \int_{-\infty}^{\infty} f(x - 2x_0 - t') f_1(t') dt'$$

$$= g(x - 2x_0).$$

Thus, if the input is shifted by $x_0$, the output is shifted by $2x_0$. Hence, the system is not shift-invariant.

2.10 (20 pts) Given a continuous signal $f(x, y) = x + y^2$, evaluate the following:

(a) $f(x, y)\delta(x - 1, y - 2)$

(b) $f(x, y) \ast \delta(x - 1, y - 2)$

(d) $\delta(x - 1, y - 2) \ast f(x + 1, y + 2)$

Solution:

(a)

$$f(x, y)\delta(x - 1, y - 2) = f(1, 2)\delta(x - 1, y - 2) = (1 + 2^2)\delta(x - 1, y - 2) = 5\delta(x - 1, y - 2)$$

(b)

$$f(x, y) \ast \delta(x - 1, y - 2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta)\delta(x - \xi - 1, y - \eta - 2) d\xi d\eta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - 1, y - 2)\delta(x - \xi - 1, y - \eta - 2) d\xi d\eta$$

$$= f(x - 1, y - 2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - \xi - 1, y - \eta - 2) d\xi d\eta$$

$$= f(x - 1, y - 2) = (x - 1) + (y - 2)^2$$

(c)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - 1, y - 2)f(x, 3) dx dy \quad \overset{1}{=} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - 1, y - 2)f(1, 3) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - 1, y - 2)(1 + 3^2) dx dy$$

$$= \overset{2}{10} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - 1, y - 2) dx dy$$

Equality (1) comes from the Eq. (2.7) in the text. Equality (2) comes from the fact:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - 1, y - 2) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) dx dy = 1.$$
2.19 (15 pt) Find the Fourier Transforms of the following continuous functions:

(a) \( \delta_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) \).

Solutions:

(a) See the solution to part (b) below. The Fourier transform is

\[ \mathcal{F}_2\{\delta_s(x, y)\} = \delta(u, v) \]

(b) \[
\mathcal{F}_2\{\delta_s(x, y; \Delta x, \Delta y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_s(x, y; \Delta x, \Delta y) e^{-j2\pi(ux + vy)} \, dx \, dy
\]

\( \delta_s(x, y; \Delta x, \Delta y) \) is a periodic signal with periods \( \Delta x \) and \( \Delta y \) in \( x \) and \( y \) axes. Therefore it can be written as a Fourier series expansion. (Please review Oppenheim, Willsky, and Nawab, *Signals and Systems* for the definition of Fourier series expansion of periodic signals.)

\[
\delta_s(x, y; \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{mn} e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)},
\]
where
\[ C_{mn} = \frac{1}{\Delta x \Delta y} \int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} \delta_s(x, y; \Delta x, \Delta y) e^{-j2\pi \left( \frac{mx}{\Delta x} + \frac{ny}{\Delta y} \right)} \, dx \, dy \]
\[ = \frac{1}{\Delta x \Delta y} \int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m\Delta x, y-n\Delta y) e^{-j2\pi \left( \frac{mx}{\Delta x} + \frac{ny}{\Delta y} \right)} \, dx \, dy. \]

In the integration region $-\frac{\Delta x}{2} < x < \frac{\Delta x}{2}$ and $-\frac{\Delta y}{2} < y < \frac{\Delta y}{2}$ there is only one impulse corresponding to $m = 0, n = 0$. Therefore, we have
\[ C_{mn} = \frac{1}{\Delta x \Delta y} \int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} \delta(x, y) e^{-j2\pi \left( \frac{mx}{\Delta x} + \frac{ny}{\Delta y} \right)} \, dx \, dy \]
\[ = \frac{1}{\Delta x \Delta y}. \]

We have:
\[ \delta_s(x, y; \Delta x, \Delta y) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j2\pi \left( \frac{mx}{\Delta x} + \frac{ny}{\Delta y} \right)}. \]

Therefore,
\[ \mathcal{F}_2 \{ \delta_s \} = \int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} \delta_s(x, y; \Delta x, \Delta y) e^{-j2\pi (ux + vy)} \, dx \, dy \]
\[ = \int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j2\pi \left( \frac{mx}{\Delta x} + \frac{ny}{\Delta y} \right)} e^{-j2\pi (ux + vy)} \, dx \, dy \]
\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y} e^{j2\pi \left( \frac{mx}{\Delta x} + \frac{ny}{\Delta y} \right)} e^{-j2\pi (ux + vy)} \, dx \, dy \]
\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \mathcal{F}_2 \left\{ e^{j2\pi \left( \frac{mx}{\Delta x} + \frac{ny}{\Delta y} \right)} \right\} \]
\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \delta \left( u - \frac{m}{\Delta x}, v - \frac{n}{\Delta y} \right) \]
\[ \overset{\text{(5)}}{=} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \Delta x \Delta y \delta(u \Delta x - m, v \Delta y - n) \]
\[ \mathcal{F}_2 \{ \delta_s \} = \delta_s(u \Delta x, v \Delta y) \]

Equality (5) comes from the property $\delta(ax) = \frac{1}{|a|} \delta(x)$. 

2.23 (10 pt) The PSF of a medical imaging system is given by

\[ h(x, y) = e^{-(|x| + |y|)} \]

where \( x \) and \( y \) are in millimeters.

(a) Is the system separable? Explain.

(b) What is the response of the system to the line impulse \( f(x, y) = \delta(x) \)?

(c) What is the response of the system to the line impulse \( f(x, y) = \delta(x - y) \)?

Solution:

(a) The system is separable because \( h(x, y) = e^{-(|x| + |y|)} = e^{-|x|} e^{-|y|} \).

(b) The system is not isotropic since \( h(x, y) \) is not a function of \( r = \sqrt{x^2 + y^2} \).

Additional comments: An easy check is to plug in \( x = 1 \), \( y = 1 \) and \( x = 0 \), \( y = \sqrt{2} \) into \( h(x, y) \). By noticing that \( h(1,1) \neq h(0,\sqrt{2}) \), we can conclude that \( h(x, y) \) is not rotationally invariant, and hence not isotropic.

Isotropy is rotational symmetry around the origin, not just symmetry about a few axes, e.g., the \( x \)- and \( y \)-axes. \( h(x, y) = e^{-(|x| + |y|)} \) is symmetric about a few lines, but it is not rotationally invariant.

When we studied the properties of Fourier transform, we learned that if a signal is isotropic then its Fourier transform has a certain symmetry. Note that the symmetry of the Fourier transform is only a necessary, but not sufficient, condition for the signal to be isotropic.

(c) The response is

\[
g(x, y) = h(x, y) \ast f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) f(x - \xi, y - \eta) d\xi \, d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(|\xi| + |\eta|)} \delta(x - \xi) d\xi \, d\eta
\]

\[
= \int_{-\infty}^{\infty} e^{-(|\xi| + |\eta|)} d\eta
\]

\[
= e^{-|x|} \left[ \int_{-\infty}^{0} e^{\eta} d\eta + \int_{0}^{\infty} e^{-\eta} d\eta \right]
\]

\[
= 2e^{-|x|}.
\]