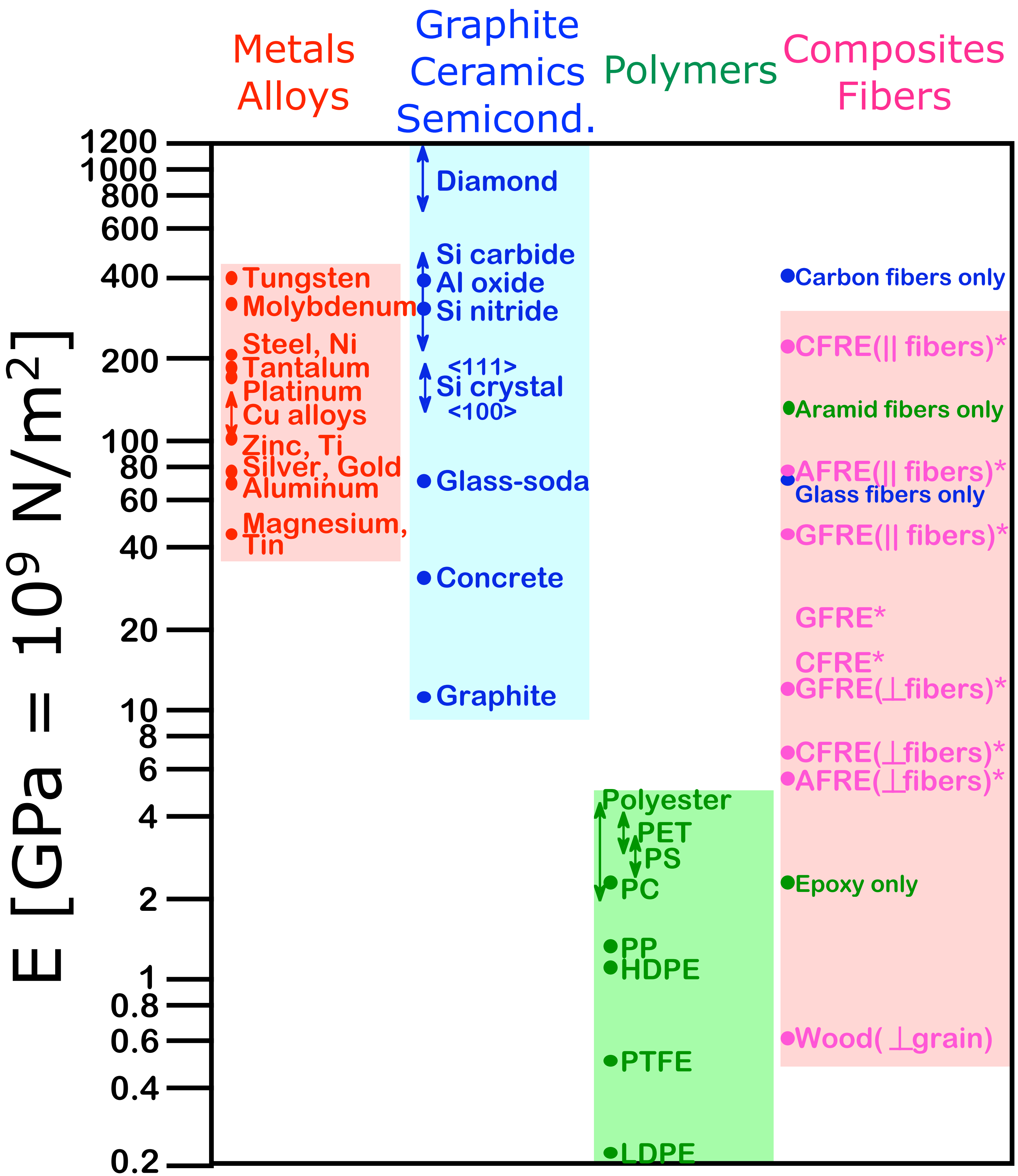
Isotropic linear elastic response

Range of Young's modulus



What is the source of both **universality** and **range** in modulus?

Based on data in Table B2, Callister 6Ed.

Composite data based on reinforced epoxy with 60 vol% of aligned carbon (CFRE), aramid (AFRE), or glass (GFRE) fibers.

From Callister, Intro to Eng. Matls., 6Ed

Universality of linear elastic response

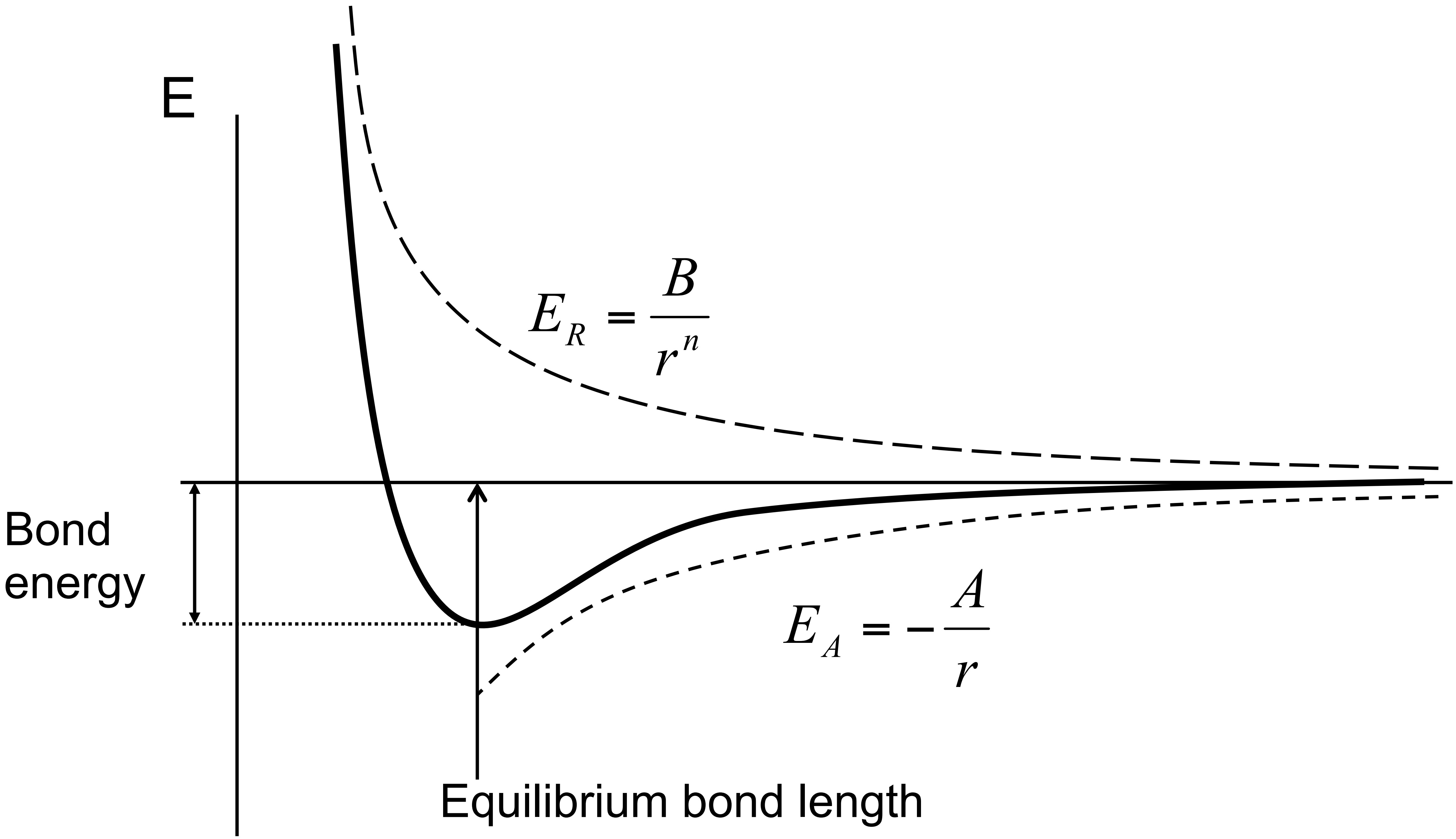
Materials are made of atoms, held together by atomic interactions

- covalent and ionic bonding: ceramics, semiconductors (~200 N/m)
- metallic bonding: metals (~ 20 N/m)
- van der Waals interaction: polymers (~ 0.5 N/m)

Materials are made of many atoms, governed by thermodynamics

- materials choose structures, phase variables (such as density) that **minimize free energy**: $A = U - TS$
- *A:* Helmholtz free energy
- *U:* internal energy (bonding)
- *T:* (absolute) temperature
- *S:* entropy (disorder: $k_B \log \Omega$)

Thermodynamic “equation of state”



$$P(V) = \frac{\partial A}{\partial V}\bigg|_V$$

Universality of linear elastic response

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$$P(V) = \left.\frac{\partial A}{\partial V}\right|_{V}$$

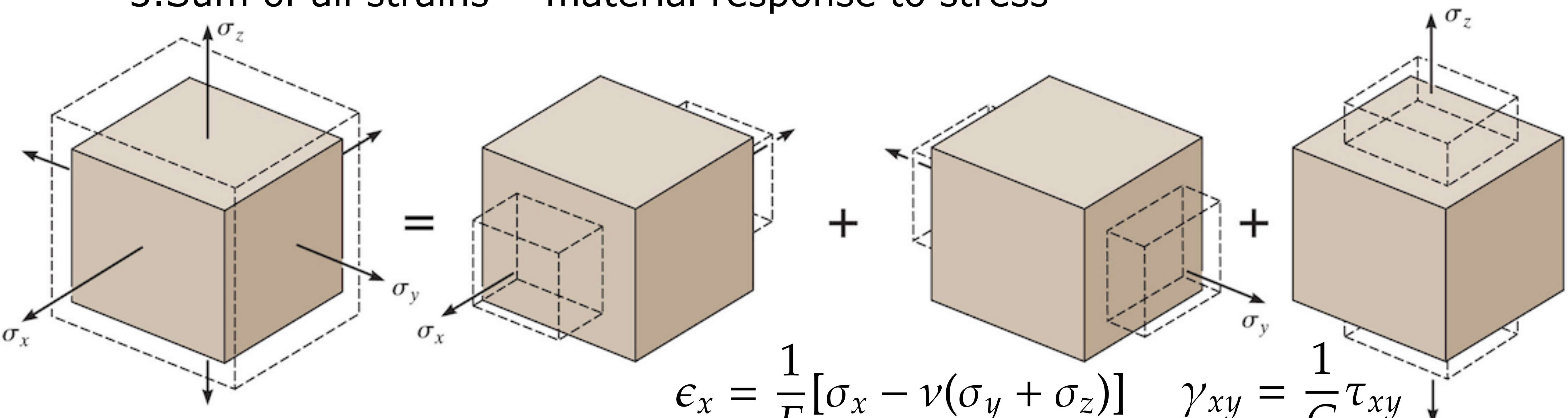
$$\begin{aligned} P(\delta V + V_0) &= \left.\frac{\partial A}{\partial V}\right|_{V_0} + \delta V \left.\frac{\partial^2 A}{\partial V^2}\right|_{V_0} + \frac{1}{2}\delta V^2 \left.\frac{\partial^3 A}{\partial V^3}\right|_{V_0} + \cdots \\ &= 0 + \frac{\delta V}{V_0}\left(V_0 \left.\frac{\partial^2 A}{\partial V^2}\right|_{V_0}\right) + \cdots \\ &= \epsilon_V K \end{aligned}$$

Superposition principle

- For **small stresses** the strains are linearly related to stresses:

$$\epsilon_{\parallel} = \frac{1}{E}\sigma_{\parallel} \qquad \epsilon_{\perp} = -\frac{\nu}{E}\sigma_{\parallel} \qquad \gamma = \frac{1}{G}\tau$$

- We can generalize these results by considering **superposition**
  1. Each stress component (σ_x σ_y σ_z τ_{xy} τ_{xz} τ_{yz}) is considered individually
  2. All of the strains from each stress component computed
  3. Sum of all strains = material response to stress



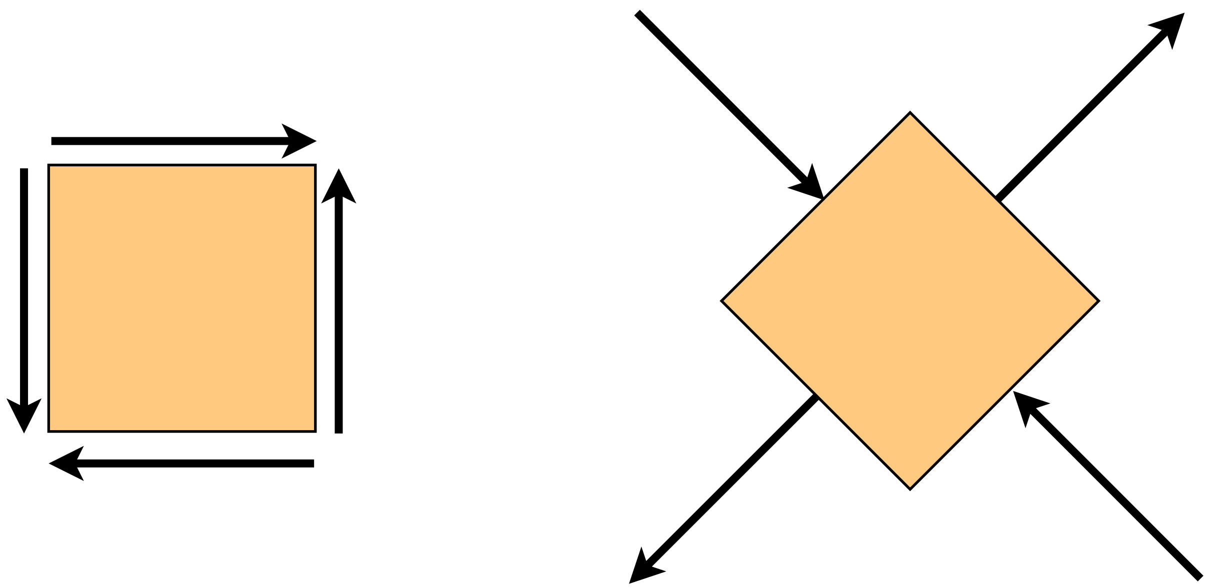
$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

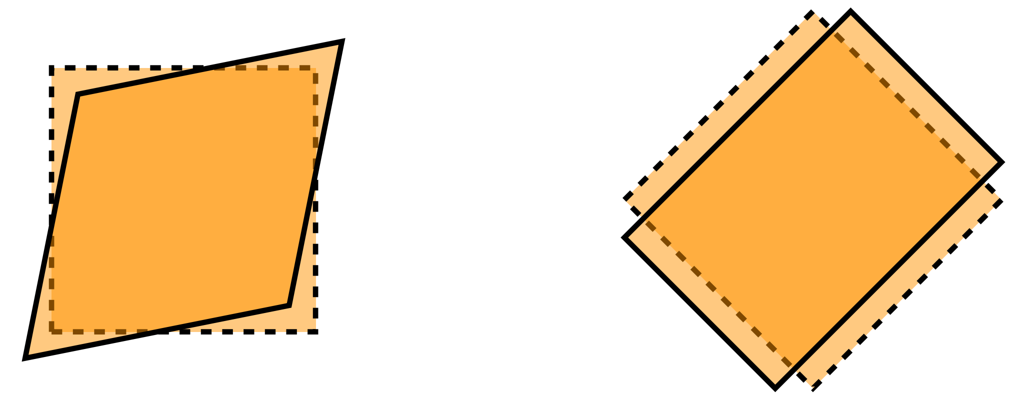
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

Material property relationships

- The superposition principle can relate our elastic moduli:
  - *E*: Young's modulus (normal strain from uniaxial stress)
  - *ν*: Poisson's ratio (perpendicular normal strain from uniaxial stress)
  - *G*: shear modulus (shear strain from shear stress)
  - *K*: bulk modulus (volume change from hydrostatic pressure)





$$\begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix} \qquad \begin{pmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} \gamma/2 & 0 \\ 0 & -\gamma/2 \end{pmatrix}$$

$$\gamma = \frac{1}{G}\tau$$

$$\frac{\gamma}{2} = \frac{1}{E}\tau - \frac{\nu}{E}(-\tau)$$

$$\gamma = \frac{2(1+\nu)}{E}\tau$$

$$G = \frac{E}{2(1+\nu)}$$

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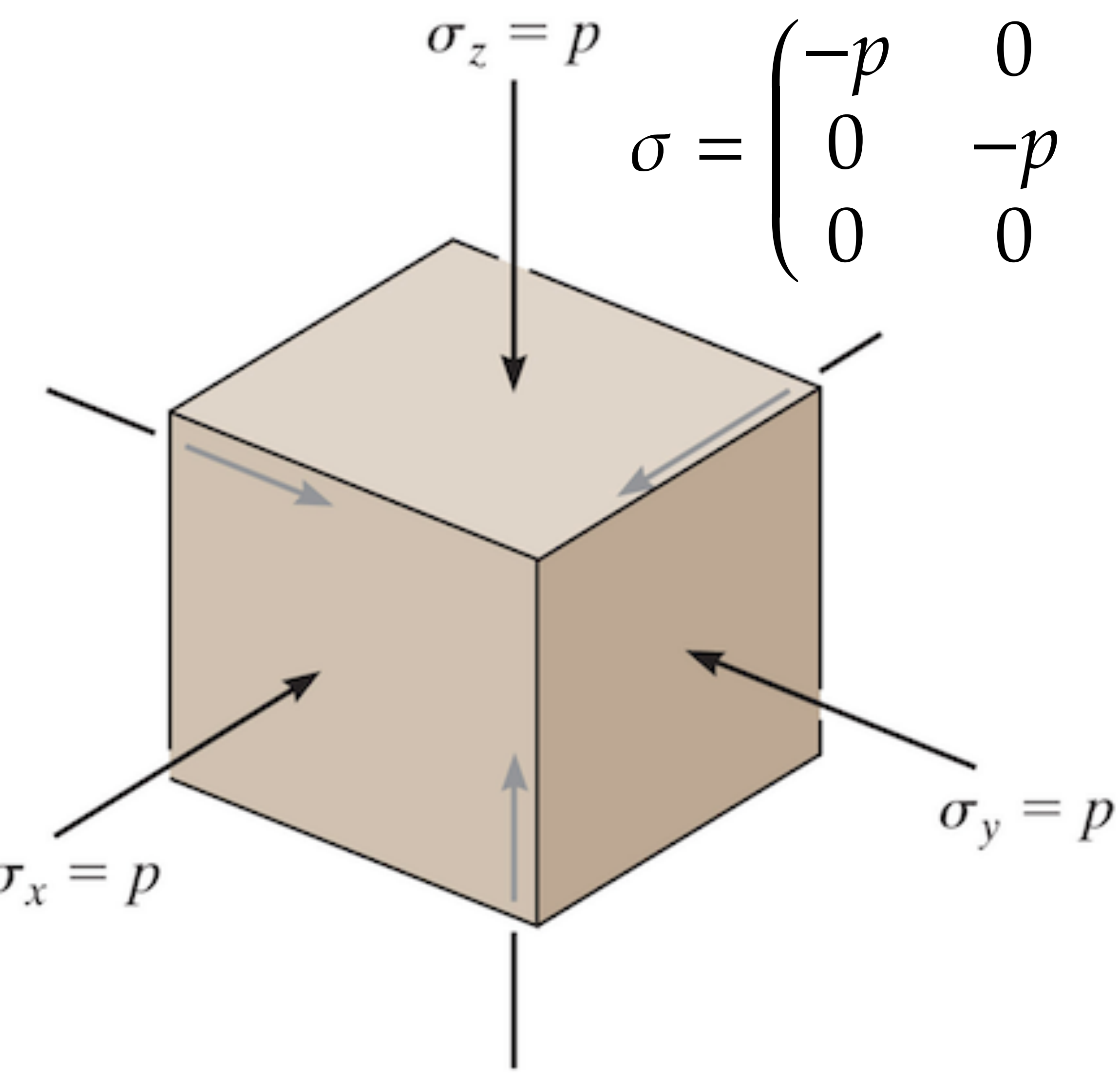


$$\sigma = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \quad \epsilon = \begin{pmatrix} -\frac{p}{E} + 2\nu\frac{p}{E} & 0 & 0 \\ 0 & -\frac{p}{E} + 2\nu\frac{p}{E} & 0 \\ 0 & 0 & -\frac{p}{E} + 2\nu\frac{p}{E} \end{pmatrix}$$

$$\begin{aligned} \Delta V &= V(1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z) - V \\ &= V(1 + (\epsilon_x + \epsilon_y + \epsilon_z) + \cdots) - V \end{aligned}$$

$$\begin{aligned} \frac{\Delta V}{V} &\approx \epsilon_x + \epsilon_y + \epsilon_z \\ &= -\frac{3(1-2\nu)}{E}p = -\frac{p}{K} \end{aligned}$$

$$\boxed{K = \frac{E}{3(1-2\nu)}}$$

Anisotropic linear elastic response

Isotropic stress/strain relations

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$
$$\gamma_{xz} = \frac{1}{G}\tau_{xz}$$
$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

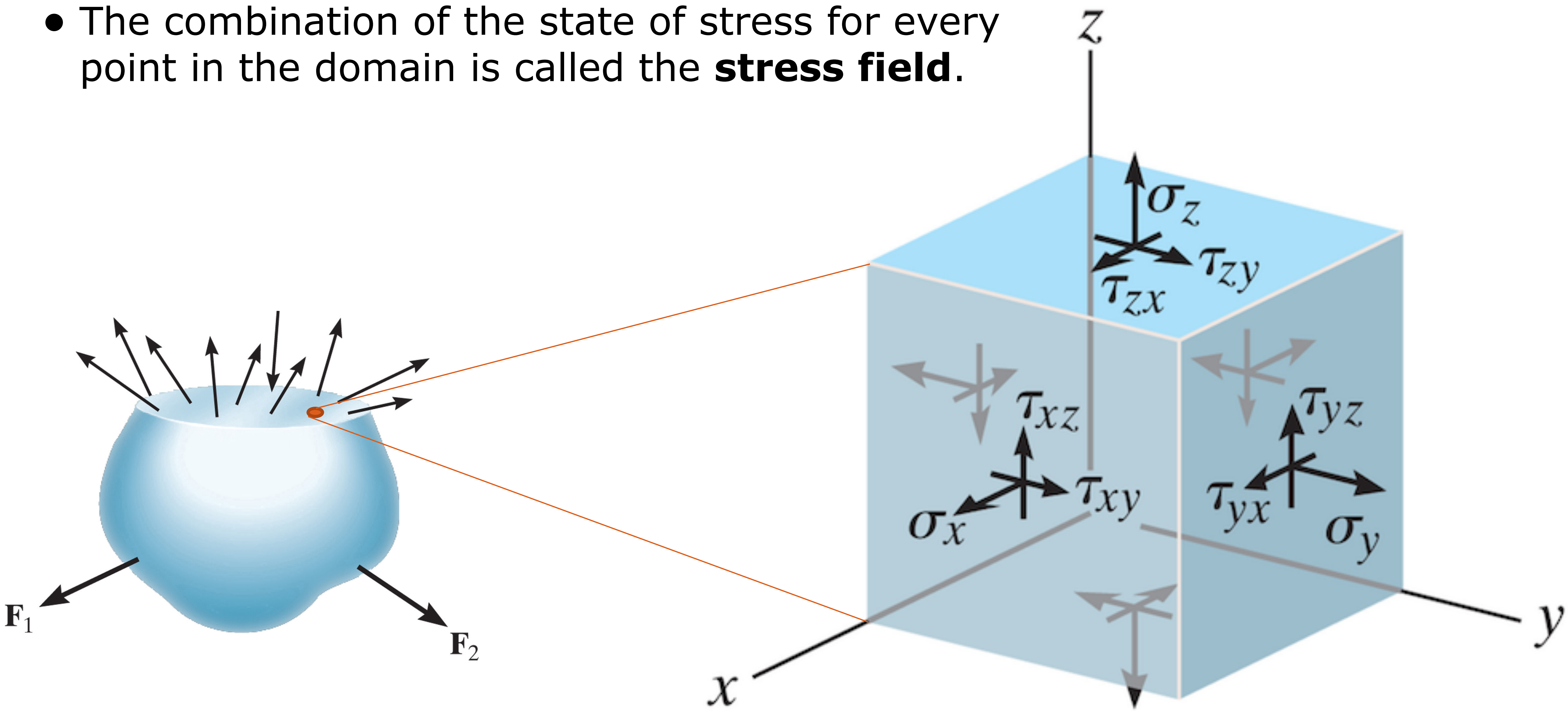
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)\right]$$
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_y + \nu(\epsilon_x + \epsilon_z)\right]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_z + \nu(\epsilon_x + \epsilon_y)\right]$$

$$\tau_{xy} = G\gamma_{xy}$$
$$\tau_{xz} = G\gamma_{xz}$$
$$\tau_{yz} = G\gamma_{yz}$$

Is there a way to extend this to anisotropic response?

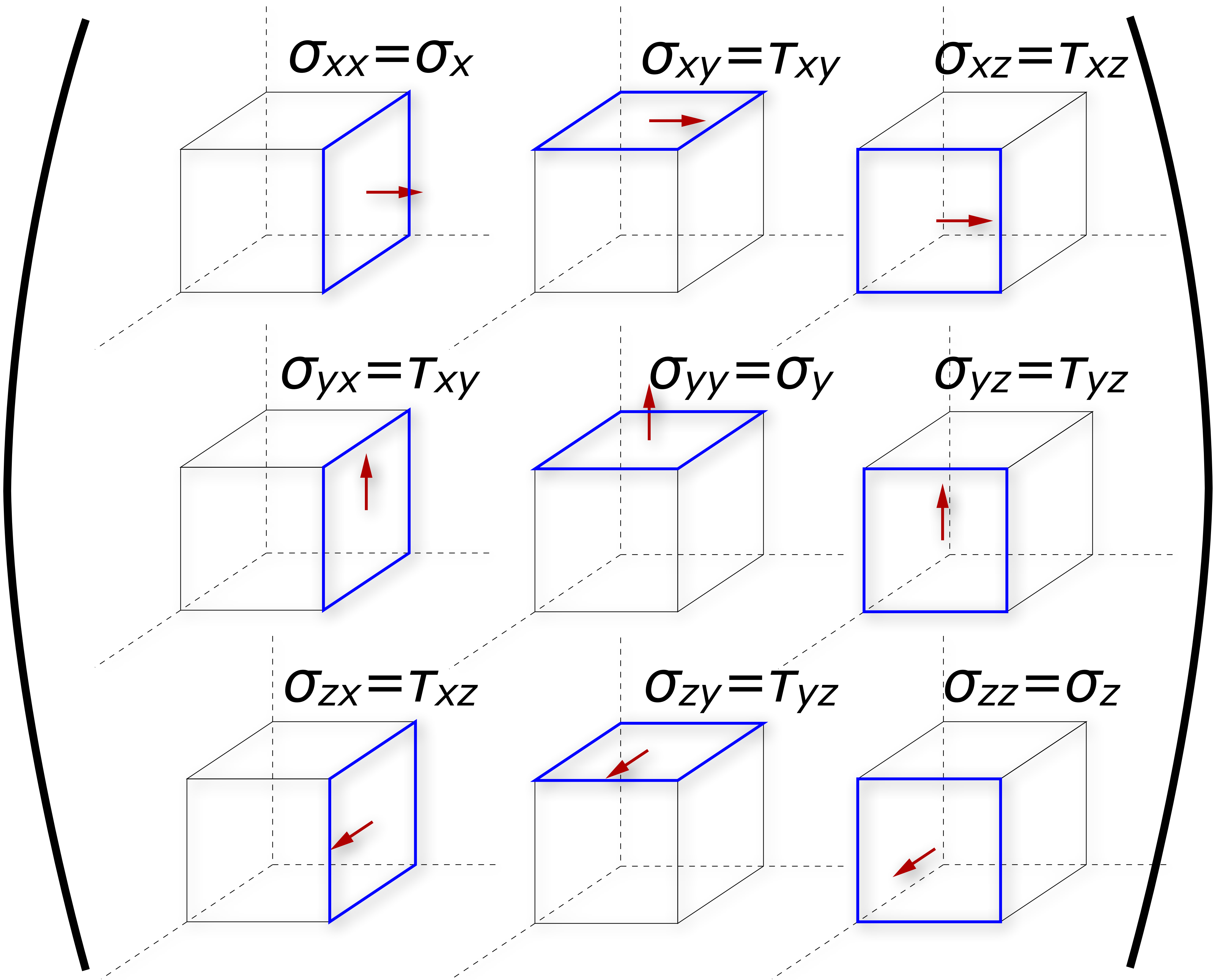
General state of stress

- Each point in a body has *normal* and *shear* stress components.
- We can section a *cubic volume* of material that represents the **state of stress** acting around the chosen point.
- As the cube is at equilibrium the **total forces and moments** are zero:
  - Infinitesimal cube = equal and opposite forces on opposite sides of cube
- Note also: the values of all the components depend on how the cube is oriented in the material (we'll talk later about relating those values)
- The combination of the state of stress for every point in the domain is called the **stress field**.



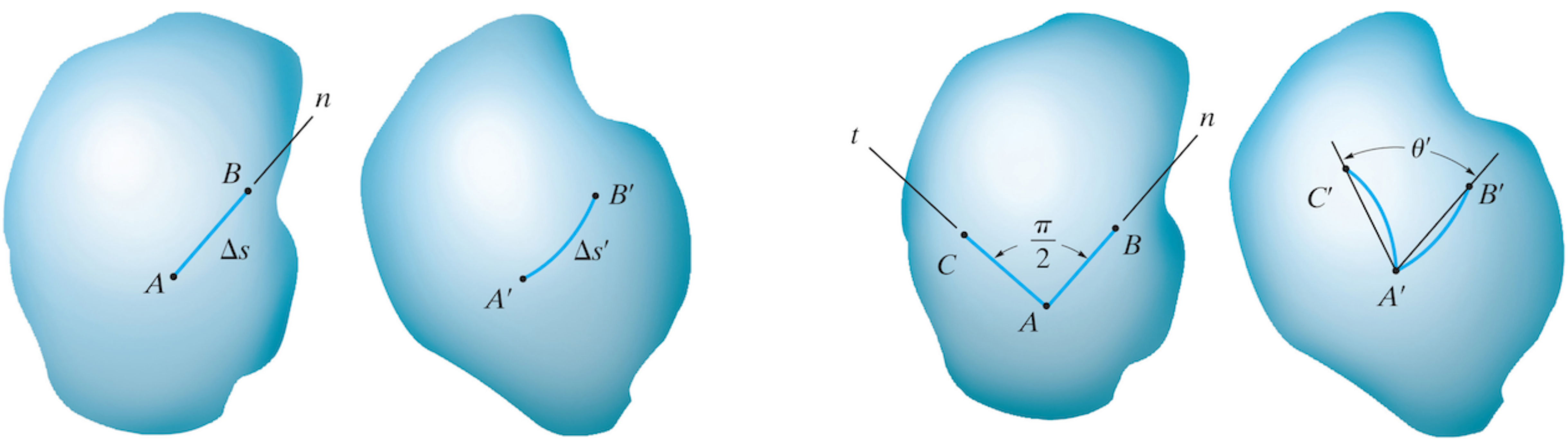
Graphical stress tensor components

- Stress × area = force $F_i = \sum_{j=xyz} \sigma_{ij} A_j$



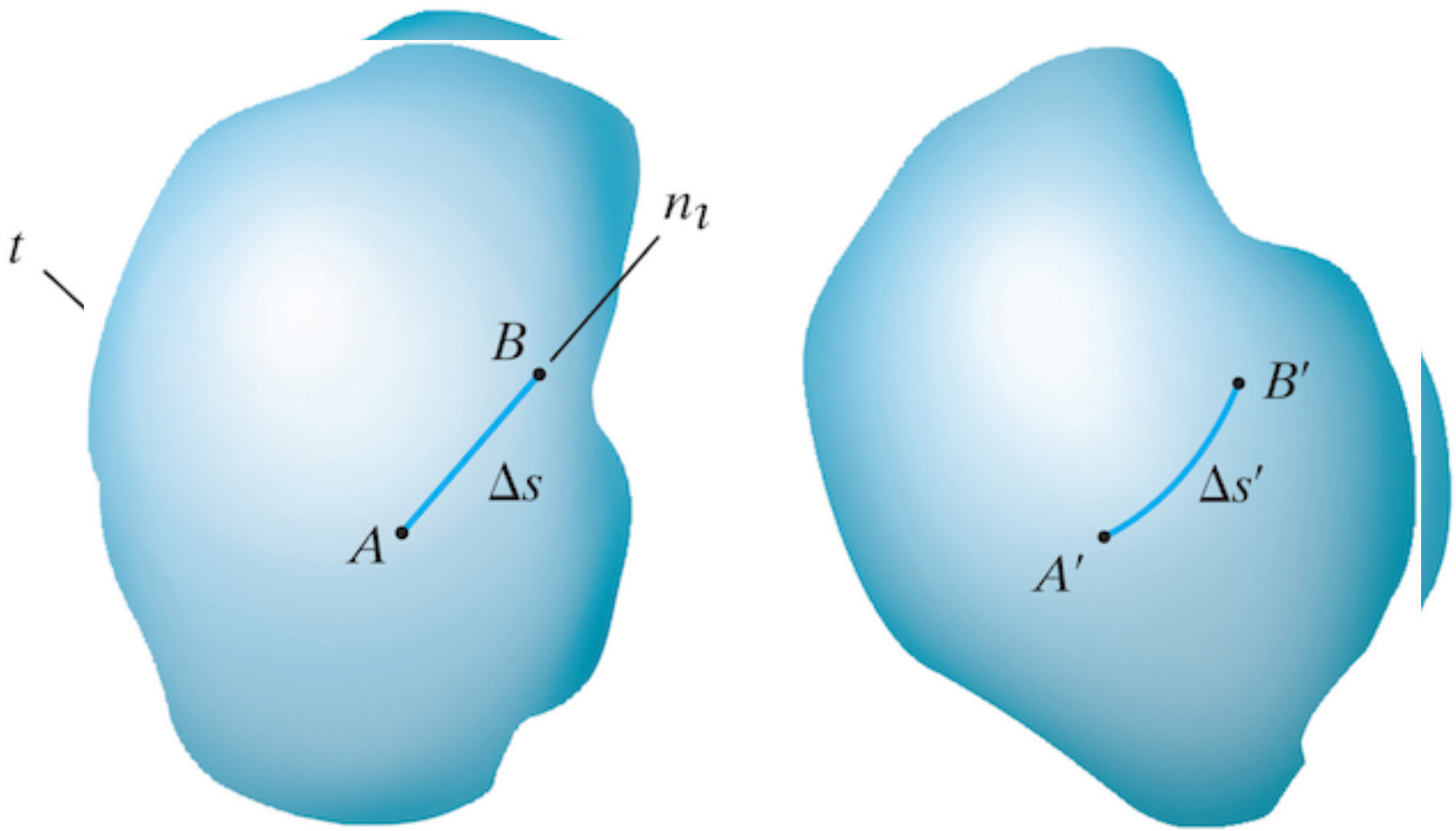
Defining strain

- We want to describe the *dimension and shape change* in a *continuous cohesive* body
- In a sufficiently small element, deformations of the element are all proportional to the size of the element
- length / length = unitless, % (10^{-2}), mm/mm, µm/m (10^{-6}), or in/in
- Requires that we capture **both** the orientation of *original vector* and *change in that vector*
  - Original relative position is a *vector*: *one index* = 3 numbers to describe
  - New relative position is a *vector*: *one index* = 3 numbers to describe
  - Strain is a *tensor*: *two indices* (coordinates) = 3×3 numbers to describe



Normal strain and shear strain

- Normal strain describes a length change in a vector
- Shear strain describes an orientation change in a vector
  - **Be aware:** whether deformation *changes length* or *changes orientation* also depends on the *original orientation*

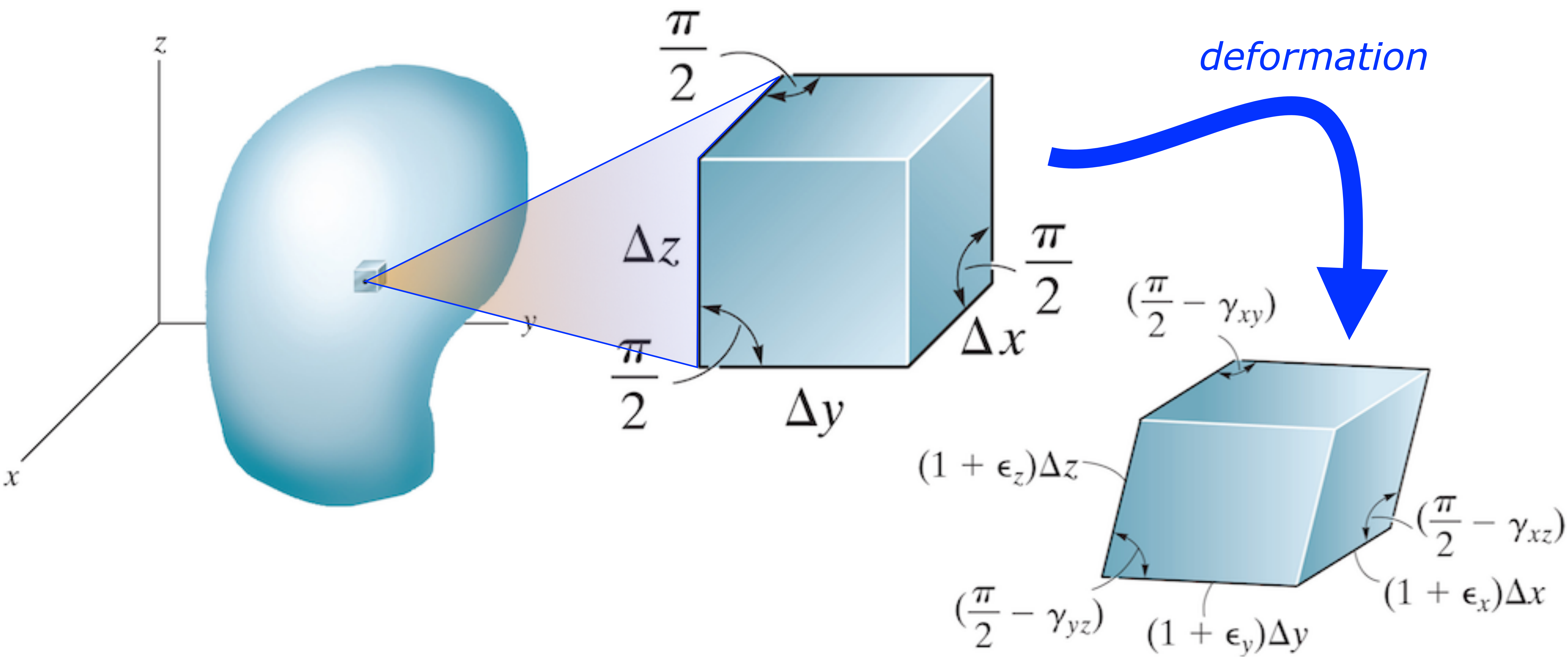


$$\varepsilon \equiv \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

$$\gamma \equiv \lim_{AB, AC \to 0} \left(\frac{\pi}{2} - \theta' \right)$$

Normal strain and shear strain

- We can describe all the strains on an *element* in a body:



$$\varepsilon_{ij} = \begin{pmatrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \epsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \epsilon_z \end{pmatrix}$$

shear strains (off-diagonal entries)

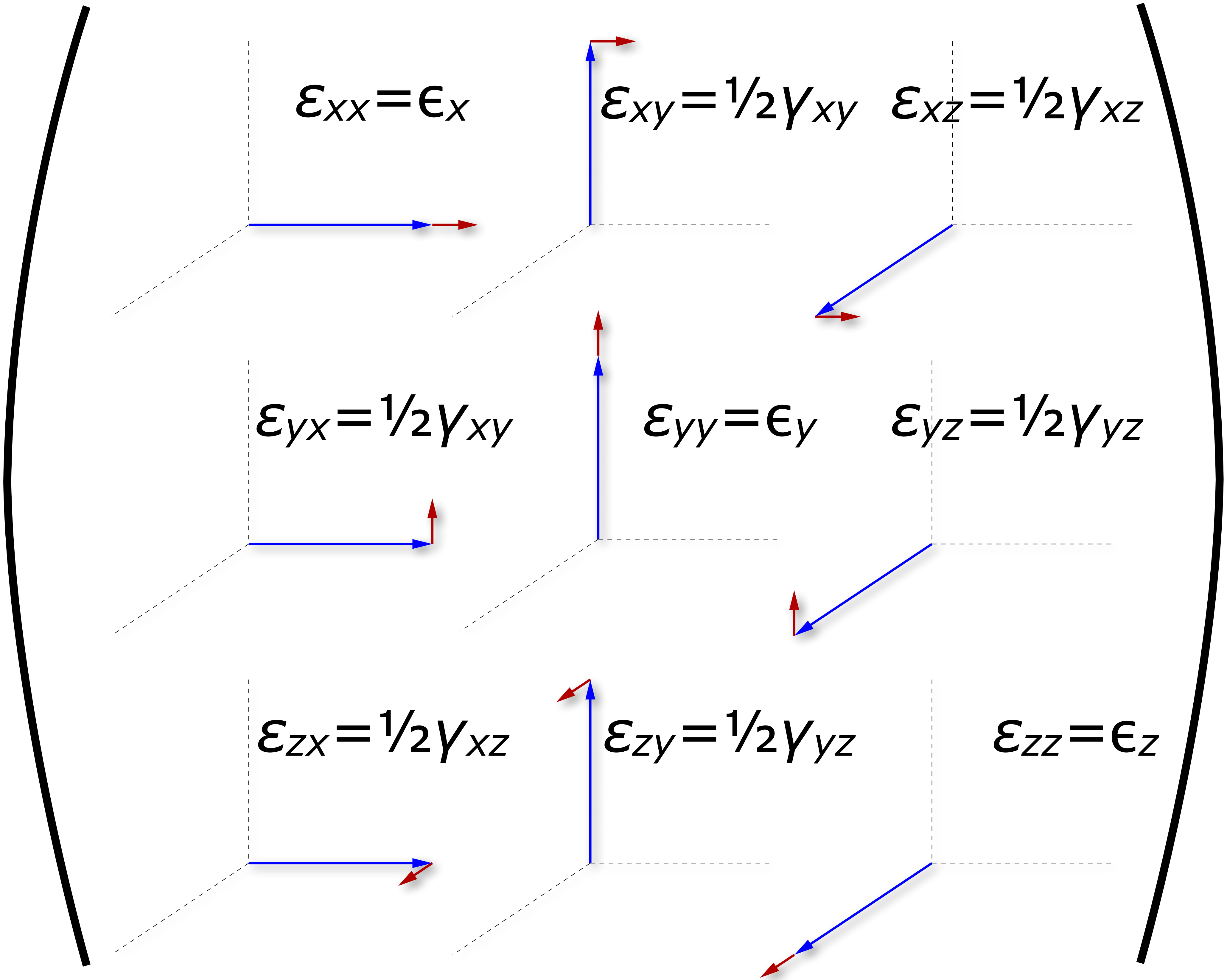
normal strains (diagonal entries)

$$\gamma_{xy} = (\text{angle change in } xy \text{ plane}) = \varepsilon_{xy} + \varepsilon_{yx}$$

$$\theta_{xy} = (\text{rotation in } xy \text{ plane}) = \varepsilon_{yx} - \varepsilon_{xy}$$

Graphical strain tensor components

- Strain × length = length change $\delta\ell_i = \sum_{j=xyz} \varepsilon_{ij}\, \ell_j$



Elastic constants: stiffnesses and compliances

- Just as stress relates a vector (area) to another vector (force), and strain relates a vector (position) to another vector (change in position), our elastic constants relate **stresses** to **strains**: 4th rank tensors

$$\epsilon_{ij} = \sum_{kl} S_{ijkl}\sigma_{kl} \qquad \sigma_{ij} = \sum_{kl} C_{ijkl}\epsilon_{kl}$$

compliance [GPa^{-1}] — stiffness [GPa]

- 3×3×3×3 = 81 components!
- But first two and last two are **symmetric**: *xyzz* = *yxzz* and *zzxy* = *zzyx*
- And first pair and last pair can be swapped: *xyzz* = *zzyx*
  - Stiffness is a second derivative of energy: $C_{ijkl} = d^2U/d\epsilon_{ij}\, d\epsilon_{kl}$
- Results in 21 unique elastic constants. Better written with Voigt notation:

$$\begin{pmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{pmatrix} \qquad \begin{pmatrix} e_1 & \frac{1}{2}e_6 & \frac{1}{2}e_5 \\ \frac{1}{2}e_6 & e_2 & \frac{1}{2}e_4 \\ \frac{1}{2}e_5 & \frac{1}{2}e_4 & e_3 \end{pmatrix}$$

$$e_i = \sum_{j=1}^{6} S_{ij}\sigma_j \qquad \sigma_i = \sum_{j=1}^{6} C_{ij}e_j$$

1	2	3
xx	yy	zz
4	5	6
yz	xz	xz

Elastic constants: stiffnesses and compliances

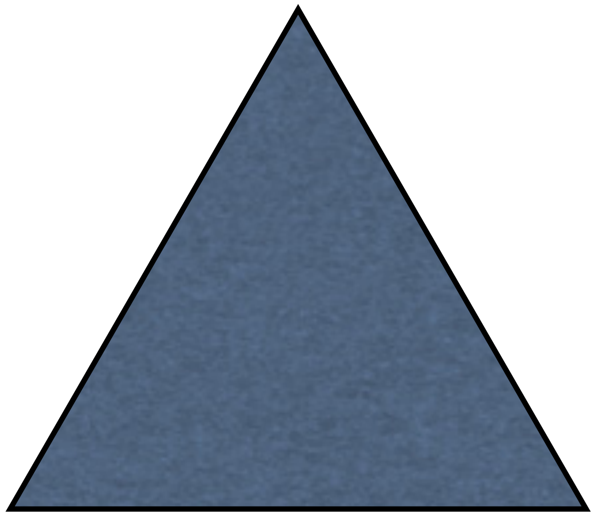
- The S and C matrices are **inverses** of each other
- The 21 stiffness and compliance matrix entries have factors of 2 and 4 to convert to tensor components:
  - $C_{ab} = C_{ijkl}$ for a=1..6, b=1..6
  - $S_{ab} = S_{ijkl}$ for a=1..3 and b=1..3
  - $S_{ab} = 2S_{ijkl}$ for a=1..3 and b=1..6 or a=4..6 and b=1..3 or
  - $S_{ab} = 4S_{ijkl}$ for a=4..6 and b=4..6
- Crystalline symmetry reduces the number of unique and nonzero entries

Stiffness / Compliance symmetry

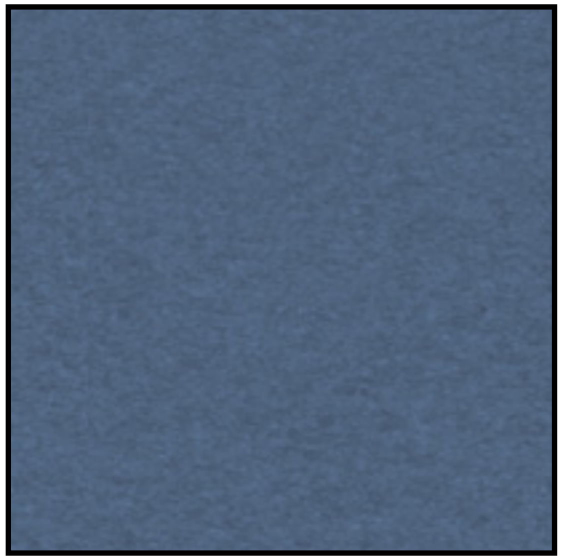
$$
\begin{pmatrix}
xx|xx & \begin{pmatrix} xx|yy \\ yy|xx \end{pmatrix} & \begin{pmatrix} xx|zz \\ zz|xx \end{pmatrix} & \begin{pmatrix} xx|yz & xx|zy \\ yz|xx & zy|xx \end{pmatrix} & \begin{pmatrix} xx|zx & xx|xz \\ zx|xx & xz|xx \end{pmatrix} & \begin{pmatrix} xx|xy & xx|yx \\ xy|xx & yx|xx \end{pmatrix} \\
\cdot & yy|yy & \begin{pmatrix} yy|zz \\ zz|yy \end{pmatrix} & \begin{pmatrix} yy|yz & yy|zy \\ yz|yy & zy|yy \end{pmatrix} & \begin{pmatrix} yy|zx & yy|xz \\ zx|yy & xz|yy \end{pmatrix} & \begin{pmatrix} yy|xy & yy|yx \\ xy|yy & yx|yy \end{pmatrix} \\
\cdot & \cdot & zz|zz & \begin{pmatrix} zz|yz & zz|zy \\ yz|zz & zy|zz \end{pmatrix} & \begin{pmatrix} zz|zx & zz|xz \\ zx|zz & xz|zz \end{pmatrix} & \begin{pmatrix} zz|xy & zz|yx \\ xy|zz & yx|zz \end{pmatrix} \\
\cdot & \cdot & \cdot & \begin{pmatrix} yz|yz & yz|zy \\ zy|yz & zy|zy \end{pmatrix} & \begin{pmatrix} yz|zx & yz|xz & zx|yz & xz|yz \\ zy|zx & zy|xz & zx|zy & xz|zy \end{pmatrix} & \begin{pmatrix} yz|xy & yz|yx & xy|yz & yx|yz \\ zy|xy & zy|yx & xy|zy & yx|zy \end{pmatrix} \\
\cdot & \cdot & \cdot & \cdot & \begin{pmatrix} zx|zx & zx|xz \\ xz|zx & xz|xz \end{pmatrix} & \begin{pmatrix} zx|xy & zx|yx & xy|zx & yx|zx \\ xz|xy & xz|yx & xy|xz & yx|xz \end{pmatrix} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \begin{pmatrix} xy|xy & xy|yx \\ yx|xy & yx|yx \end{pmatrix}
\end{pmatrix}
$$

Symmetry operations

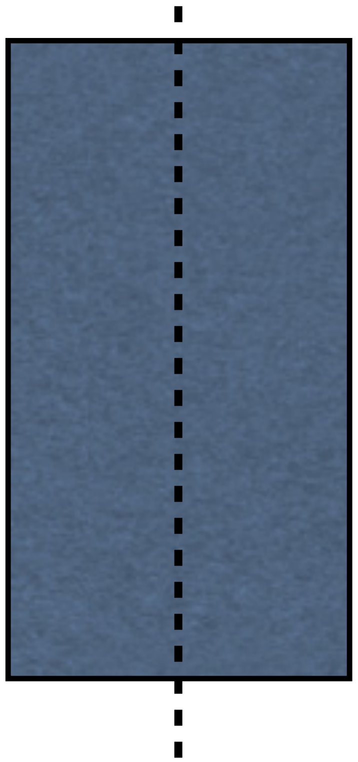
2-fold axis



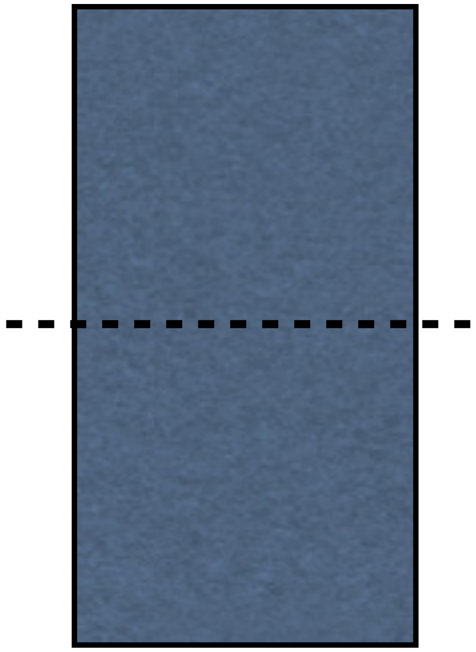
3-fold axis



4-fold axis

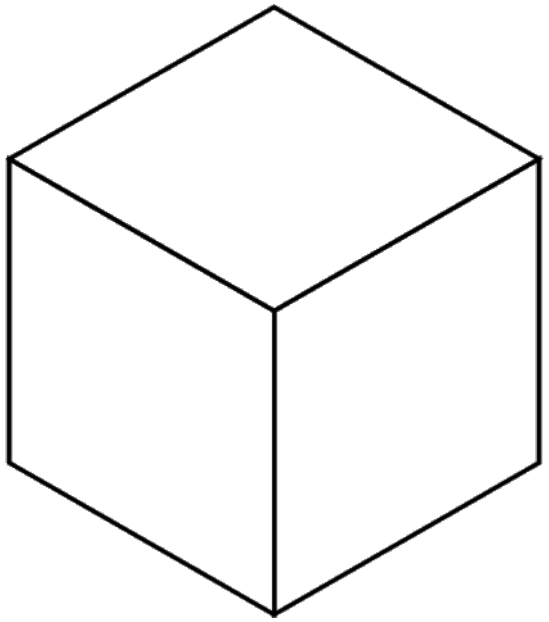


mirror plane



mirror plane

Rotating a cube around the body diagonal ⟨111⟩?



3-fold axis

Elastic constants: stiffnesses and compliances

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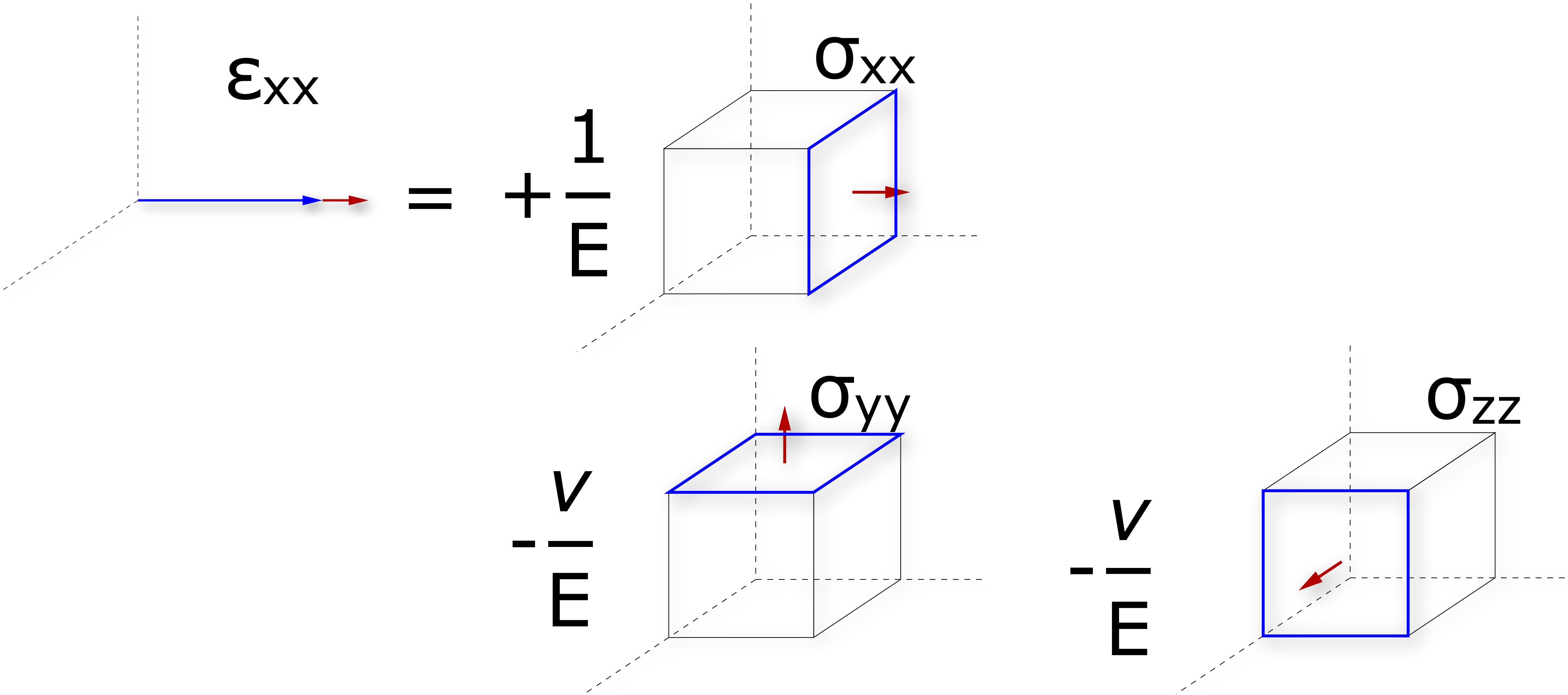
  - $S_{ab} = S_{ijkl}$ for a=1..3 and b=1..3
  - $S_{ab} = 2S_{ijkl}$ for a=1..3 and b=1..6 or a=4..6 and b=1..3 or
  - $S_{ab} = 4S_{ijkl}$ for a=4..6 and b=4..6
- Crystalline symmetry reduces the number of unique and nonzero entries
- **Cubic** symmetry is the most common for structural materials:
  - $C_{11} = C_{22} = C_{33}$
  - $C_{12} = C_{13} = C_{23}$
  - $C_{44} = C_{55} = C_{66}$
  - all others zero
- Isotropic materials are **cubic** *and* $C_{11} - C_{12} = 2C_{44}$ (or $S_{11} - S_{12} = S_{44}/2$)
- Hexagonal materials and aligned fiber composites have lower symmetry:
  - $C_{11} = C_{22} \neq C_{33}$; $C_{12} \neq C_{13} = C_{23}$; $C_{44} = C_{55} \neq C_{66}$
  - Isotropic in basal plane: $2C_{66} = C_{11} - C_{12}$
  - all others zero

Graphical compliance components

$$\varepsilon_{ij} = \Sigma_{k=xyz} \Sigma_{l=xyz} S_{ijkl} \sigma_{kl}$$



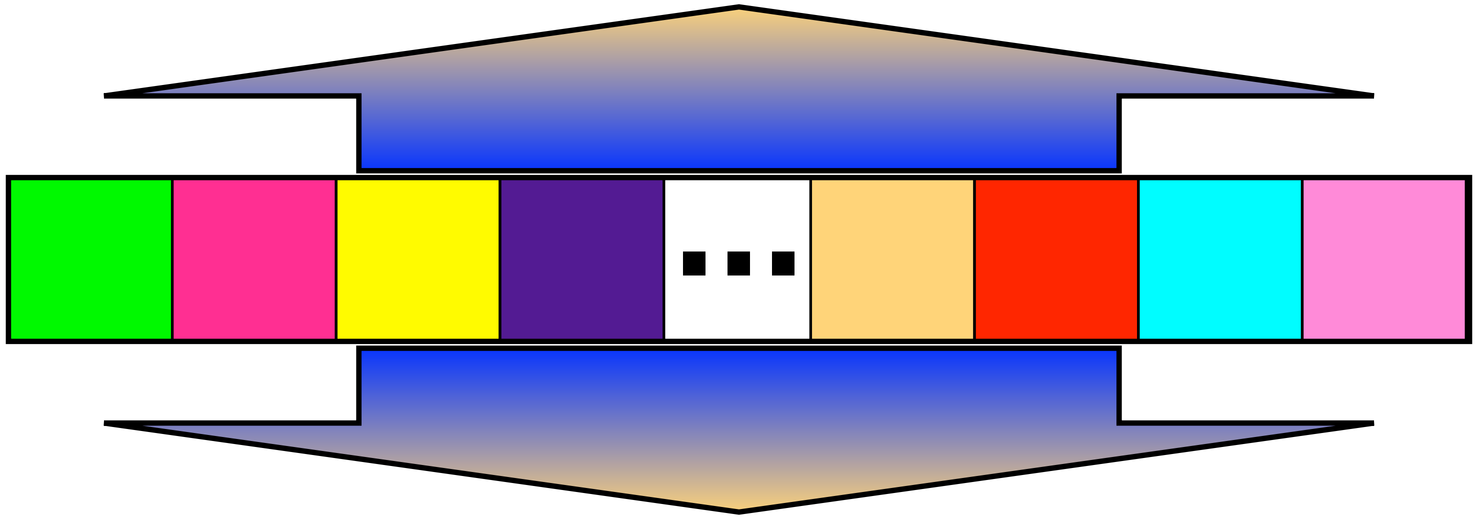
$$S_{xxxx} = 1/E \quad S_{xxyy} = -\nu/E \quad S_{xxzz} = -\nu/E$$

$S_{xxkl} = 0$ for all other kl

Voigt and Reuss averages

Voigt average = **isostrain**

Randomly oriented grains



$$E_{\text{Voigt}} = \frac{\left(\overline{C}_{11} - \overline{C}_{12} + 3\overline{C}_{44}\right)\left(\overline{C}_{11} + 2\overline{C}_{12}\right)}{2\overline{C}_{11} + 3\overline{C}_{12} + \overline{C}_{44}}$$

$$\overline{C}_{11} = \frac{1}{3}\left(C_{11} + C_{22} + C_{33}\right)$$

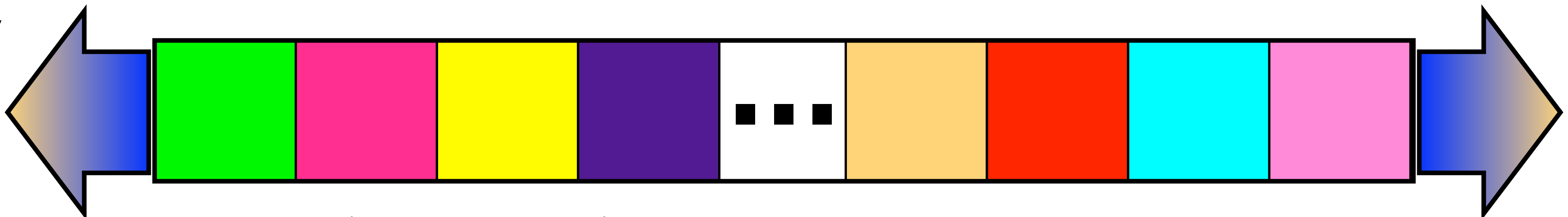
$$\overline{C}_{12} = \frac{1}{3}\left(C_{12} + C_{13} + C_{23}\right)$$

$$\overline{C}_{44} = \frac{1}{3}\left(C_{44} + C_{55} + C_{66}\right)$$

Voigt and Reuss averages

Reuss average = **isostress**

Randomly oriented grains



$$\frac{1}{E_{\text{Reuss}}} = \frac{1}{5}\left(3\overline{S}_{11} + 2\overline{S}_{12} + \overline{S}_{44}\right)$$

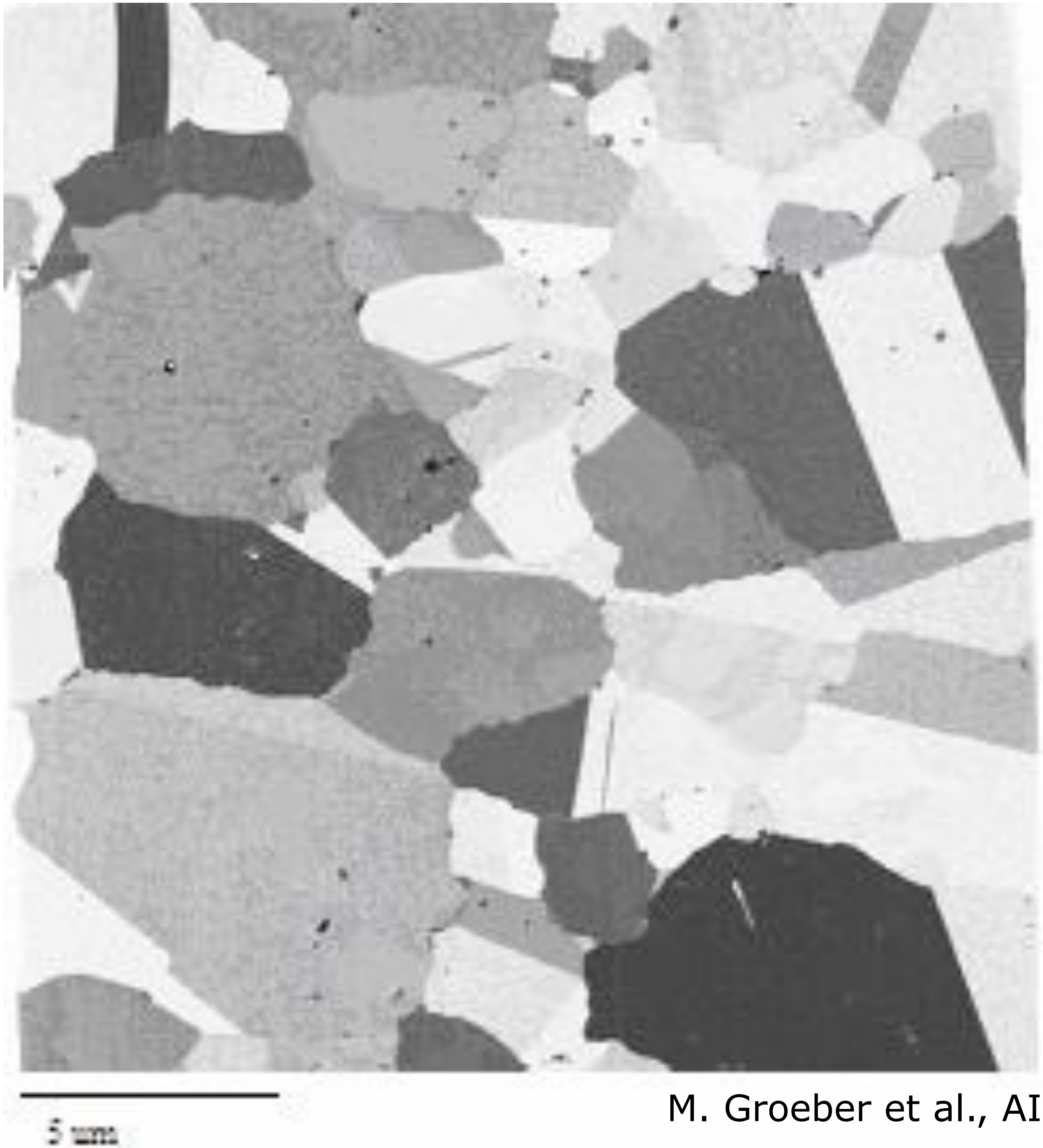
$$\overline{S}_{11} = \frac{1}{3}(S_{11} + S_{22} + S_{33})$$

$$\overline{S}_{12} = \frac{1}{3}(S_{12} + S_{13} + S_{23})$$

$$\overline{S}_{44} = \frac{1}{3}(S_{44} + S_{55} + S_{66})$$

$E_{\text{Voigt}} > E_{\text{random polycrystal}} > E_{\text{Reuss}}$

Grain structure and texture



Ni alloy grain structure

Each grain has a different orientation, and responds differently to applied stress

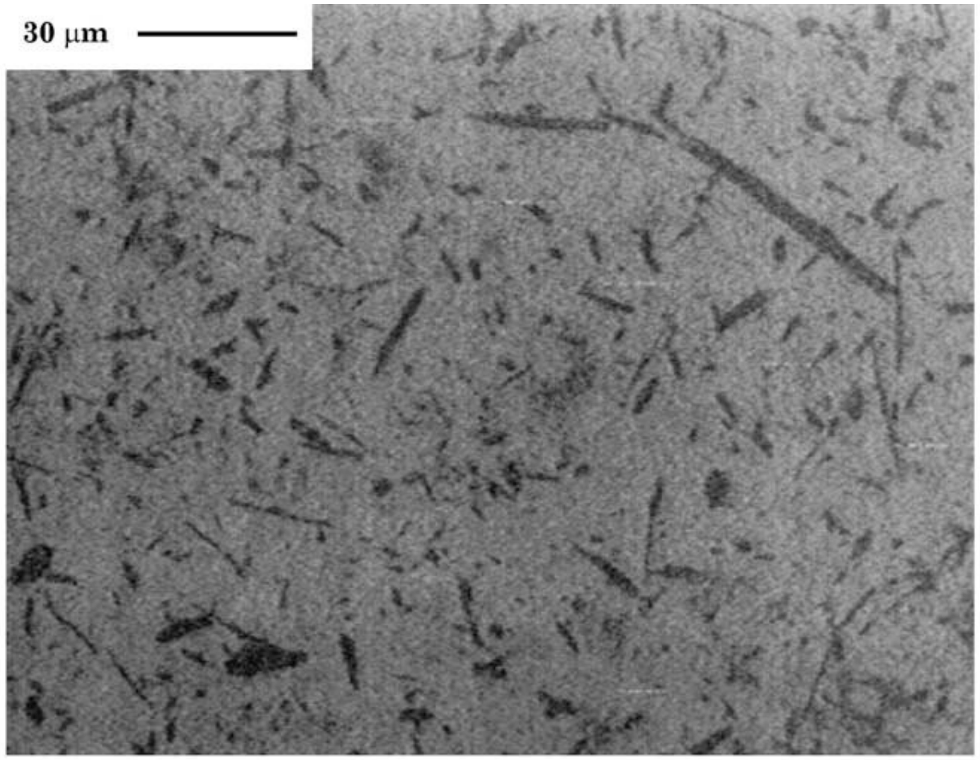
Polycrystalline response is an average of individual grain responses.

Texture is a preferential orientation of grains.

M. Groeber et al., AIP Conf. Proc. **712**, 1712 (2004).

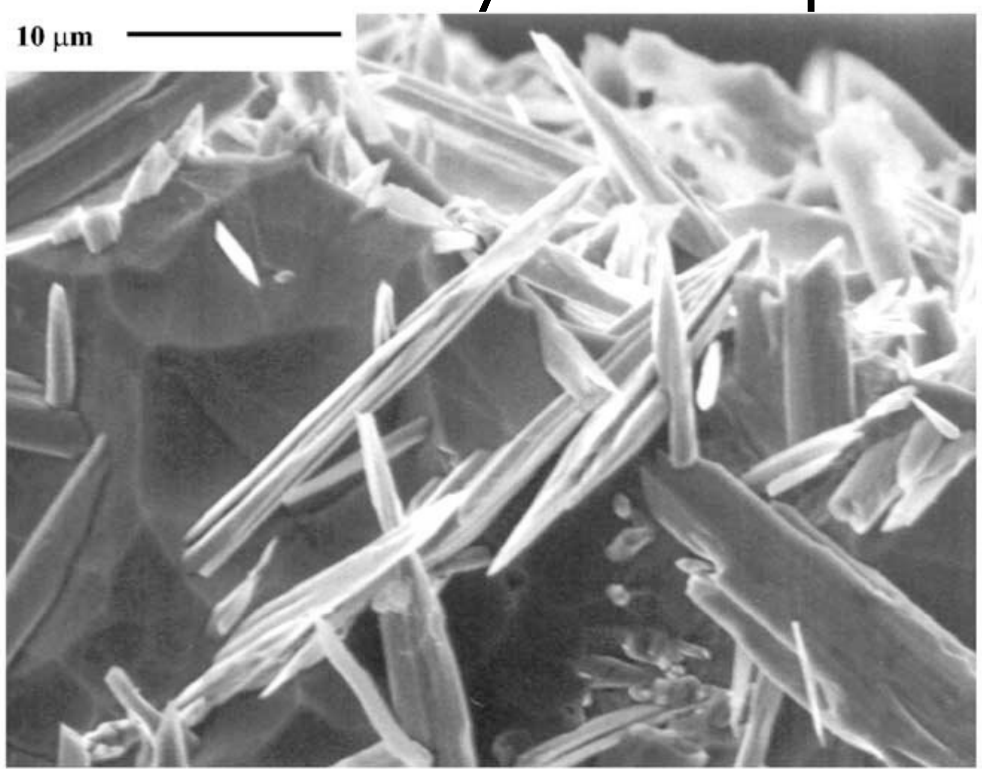
Ti / TiB metal-matrix composite

SEM backscatter: polish



(a)

SEM secondary e⁻: deep etch



(b)

TiB: orthorhombic crystal

$$C_{ij} = \begin{pmatrix} 419 & 92 & 113 & 0 & 0 & 0 \\ 92 & 523 & 63 & 0 & 0 & 0 \\ 113 & 63 & 418 & 0 & 0 & 0 \\ 0 & 0 & 0 & 196 & 0 & 0 \\ 0 & 0 & 0 & 0 & 179 & 0 \\ 0 & 0 & 0 & 0 & 0 & 220 \end{pmatrix} \text{GPa}$$

$$E_{\text{Voigt}} = 442\text{GPa}$$

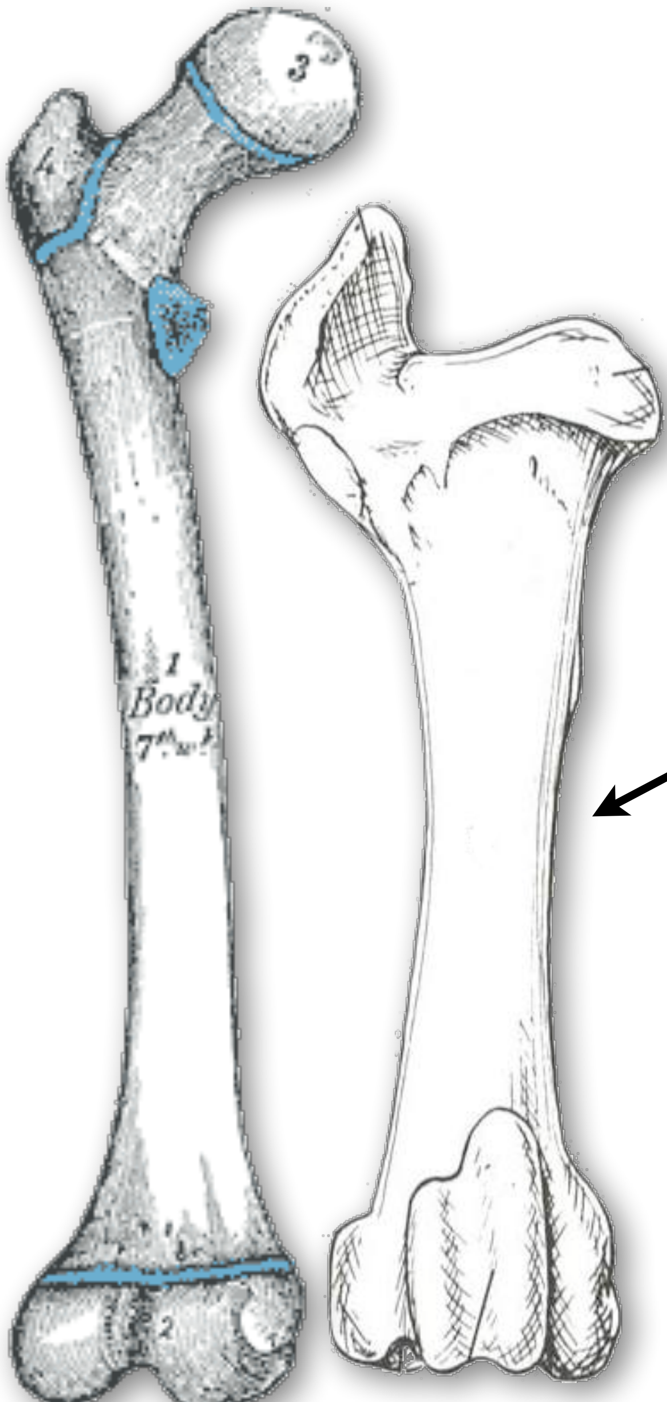
$$E_{\text{Reuss}} = 435\text{GPa}$$

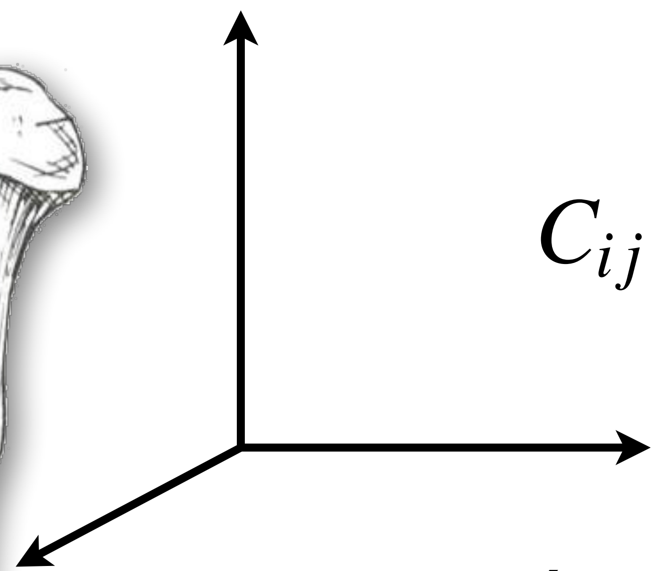
$$E_{\text{Ti}} = 110\text{GPa}$$

$$\boxed{E_{\text{Ti+20\%vol TiB}} = 153\text{GPa}}$$

S. Gorsse et al., Mat. Sci. Eng. A**340**, 80-87 (2003)
D. R. Trinkle, Scripta Mater. **56**, 273-276 (2007)

Bovine femural bone: elastic constants





bone orthotropic stiffnesses:

$$C_{ij} = \begin{pmatrix} 14 & 6.3 & 4.8 & 0 & 0 & 0 \\ 6.3 & 18.4 & 7.0 & 0 & 0 & 0 \\ 4.8 & 7.0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.3 \end{pmatrix} \text{GPa}$$

along length: E_3 = 21.7GPa

transverse: E_1 = 11.6GPa

Extracted from sound-speed measurements:

$$C_{11} = \rho(v_{11})^2 \qquad C_{22} = \rho(v_{22})^2 \qquad C_{44} = \rho(v_{23})^2$$

W. C. Buskirk et al., J. Biomech. Eng. **103**, 67-72 (1981)

Composite behavior

Composite = matrix + reinforcement

Matrix: continuous phase

- transfers load to reinforcement
- protects reinforcement from environment

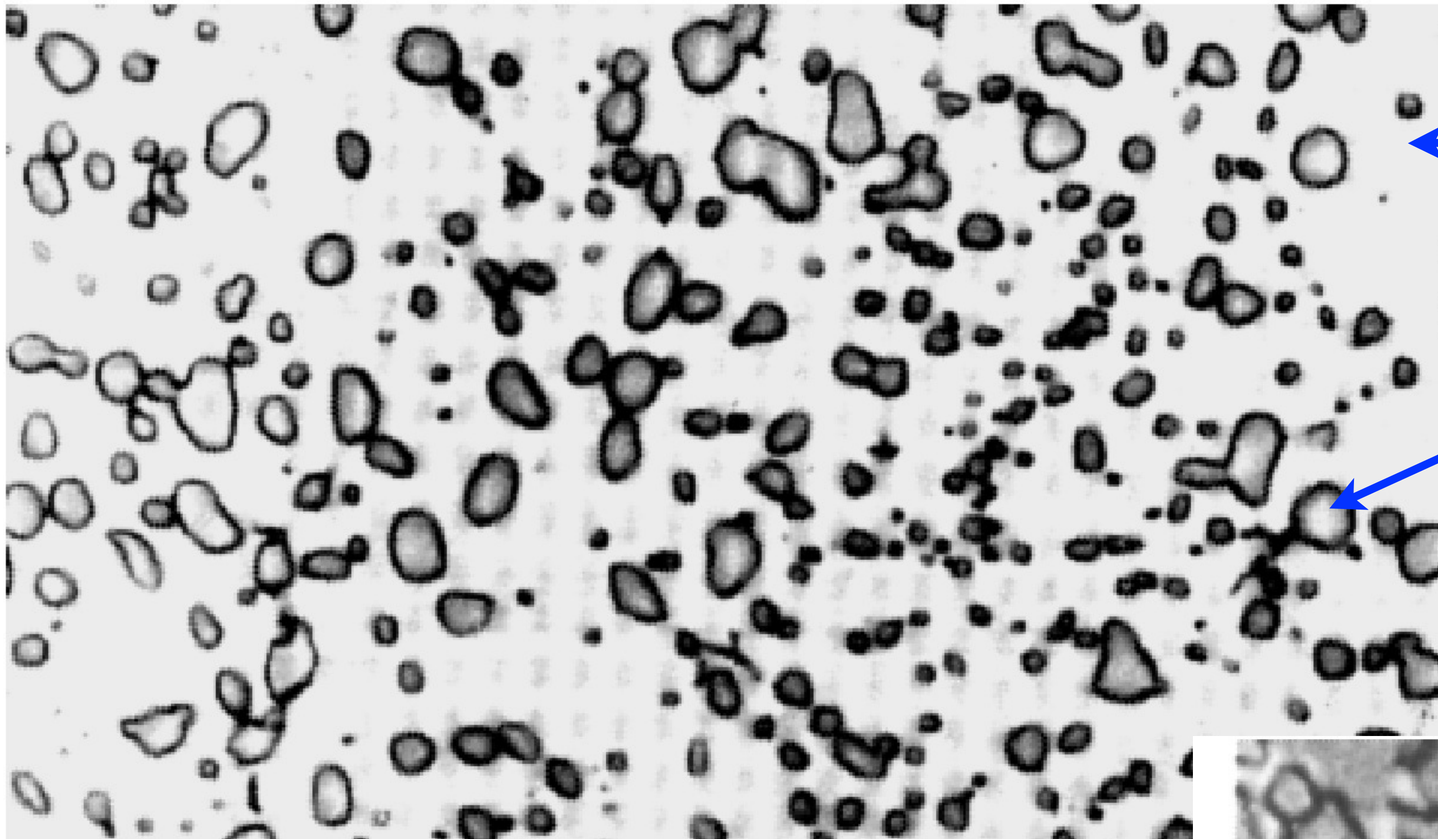
Types of matrix:

- **MMC** metal matrix composite: designed for **plastic strain**
  - better yield stress, tensile strength, creep resistance
- **CMC** ceramic matrix composite: designed for **fracture**
  - better toughness
- **PMC** polymer matrix composite: designed for **elastic and plastic strain**
  - better modulus, yield stress, tensile strength, creep
  - inexpensive, temperature range limited by polymer decomposition

Reinforcement: stronger, discontinuous phase

- carries significant portion of load
- classified by geometry

Particle reinforcements



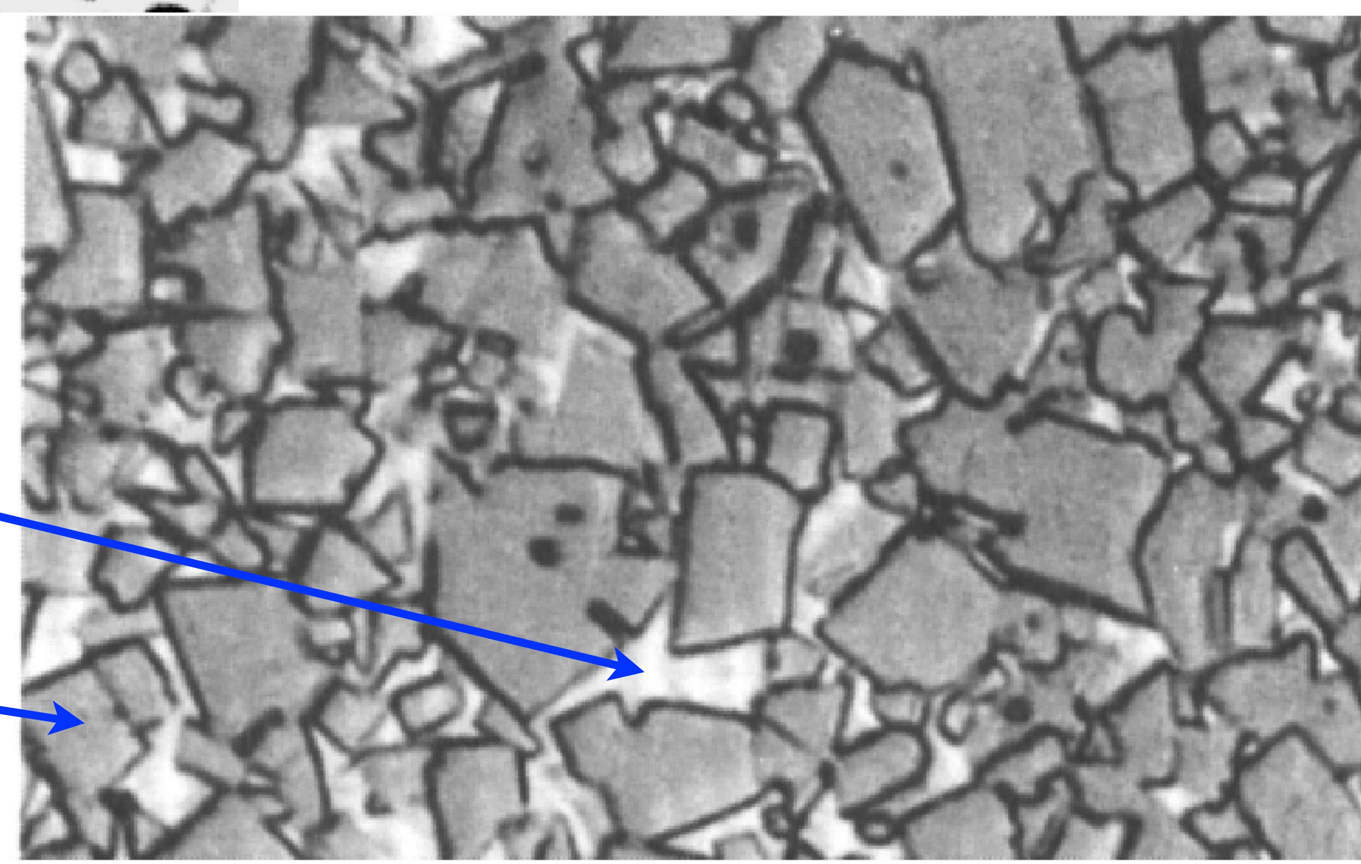
ferrite (bcc-Fe)

cementite (Fe_3C)

10μm

Spheroidite steel

cemented carbide



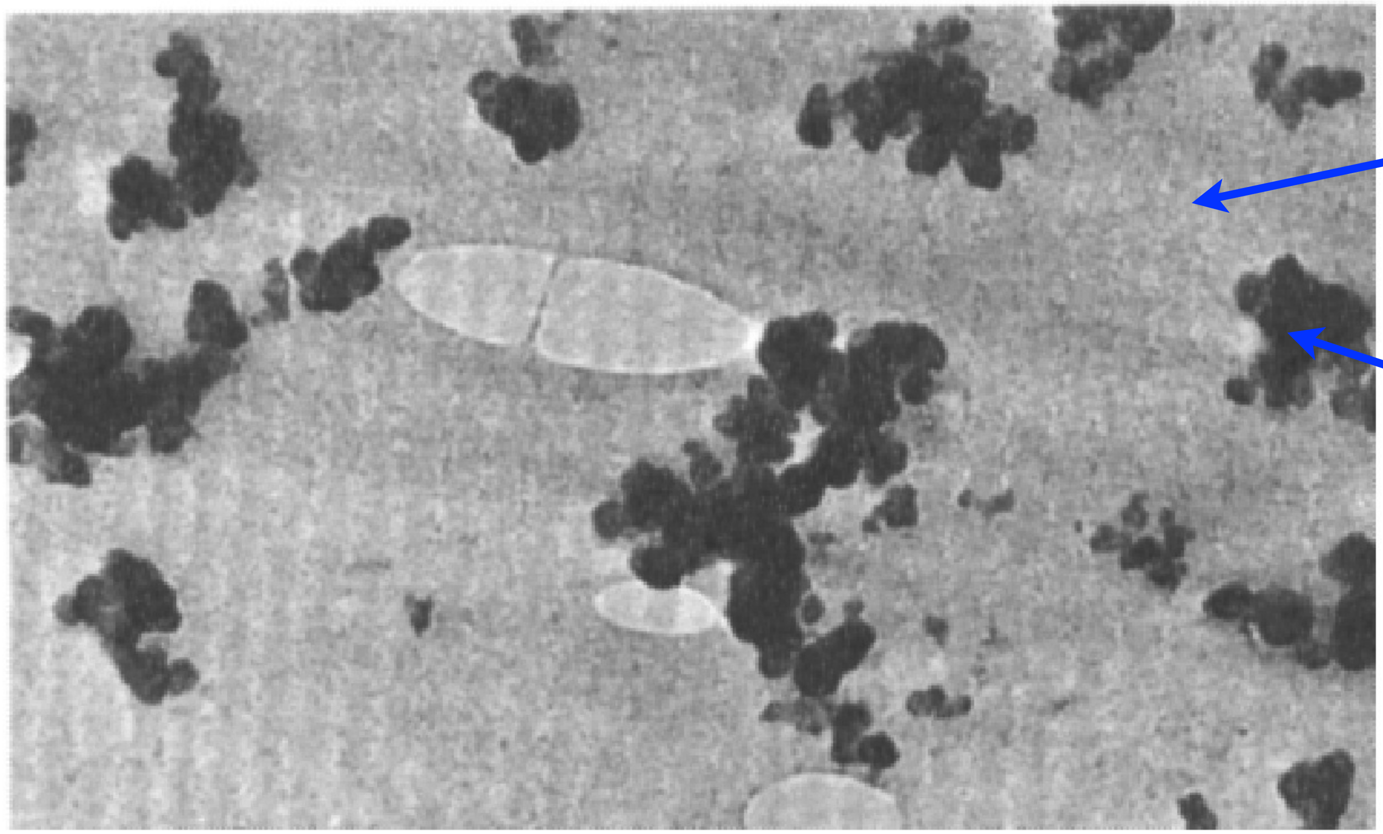
Co matrix (V_m=10-15%)

WC particles

100μm

From Callister,
Intro to Eng. Matls., 6Ed

Particle reinforcements



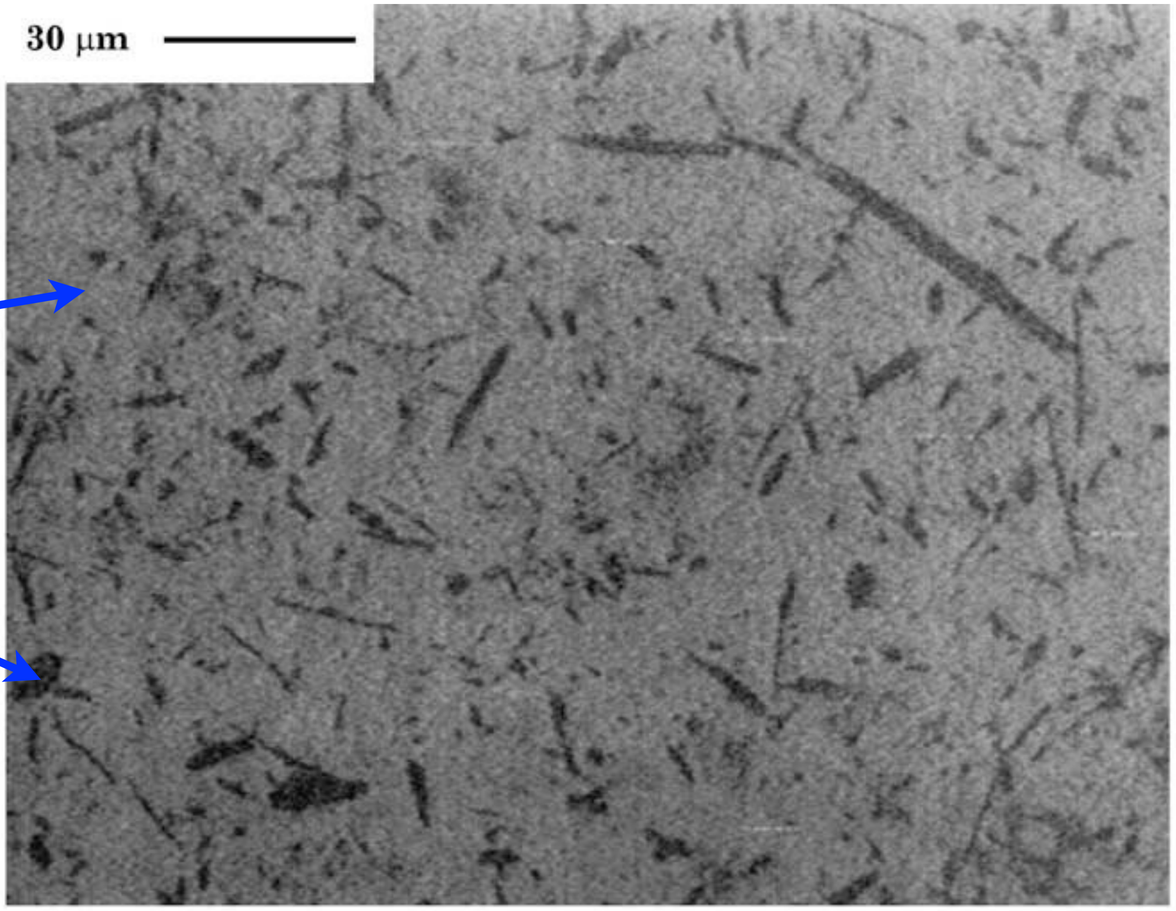
rubber

carbon particles

100nm

Tire rubber

Ti/TiB MMC



alpha-Ti (hcp)

TiB needles

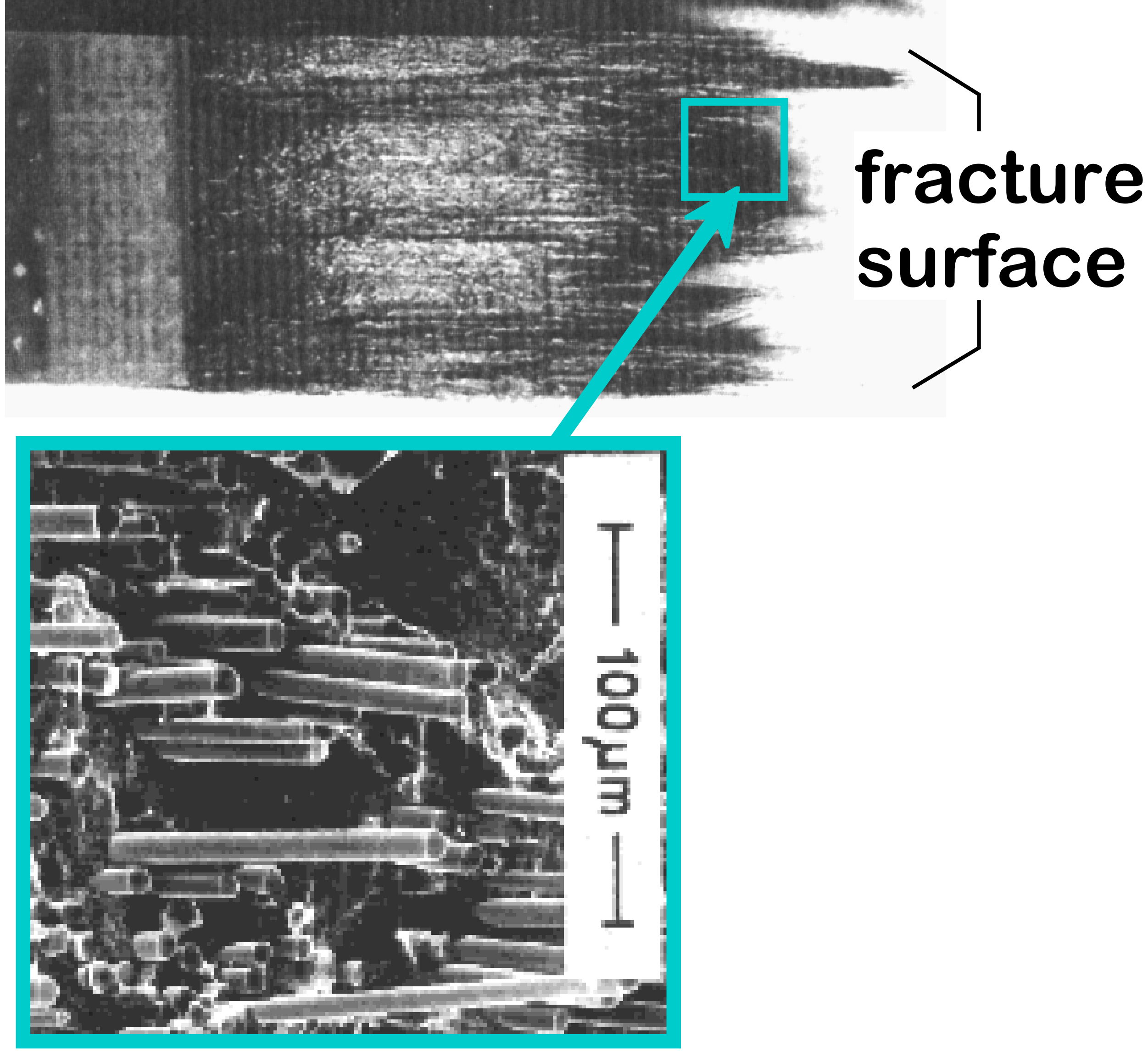
(a)

From Callister,
Intro to Eng. Matls., 6Ed

S. Gorsse et al., Mat. Sci. Eng. A**340**, 80-87 (2003).

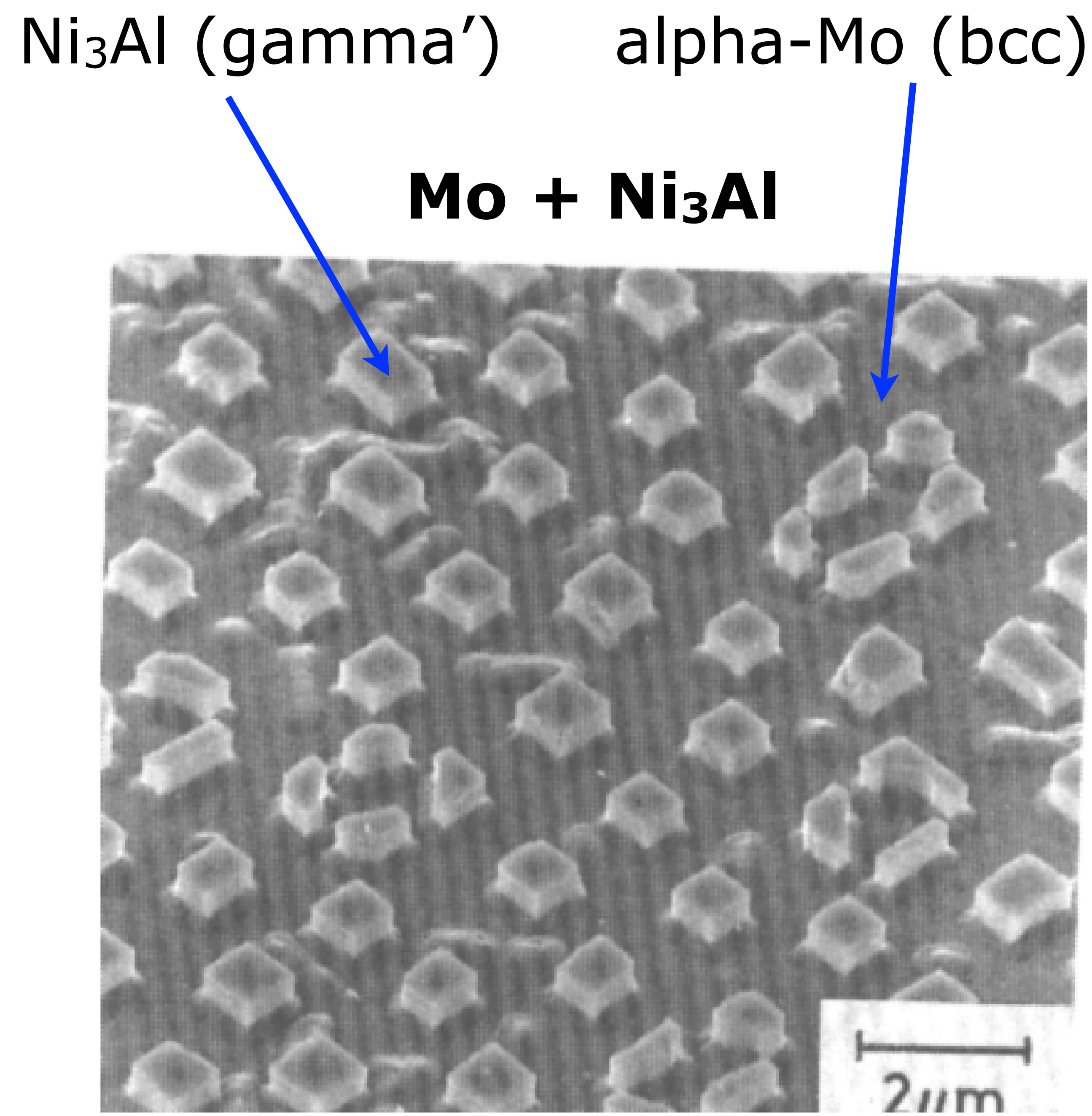
Fiber reinforcements: continuous, aligned

Glass (E=76GPa) with
SiC fibers (E=400GPa)



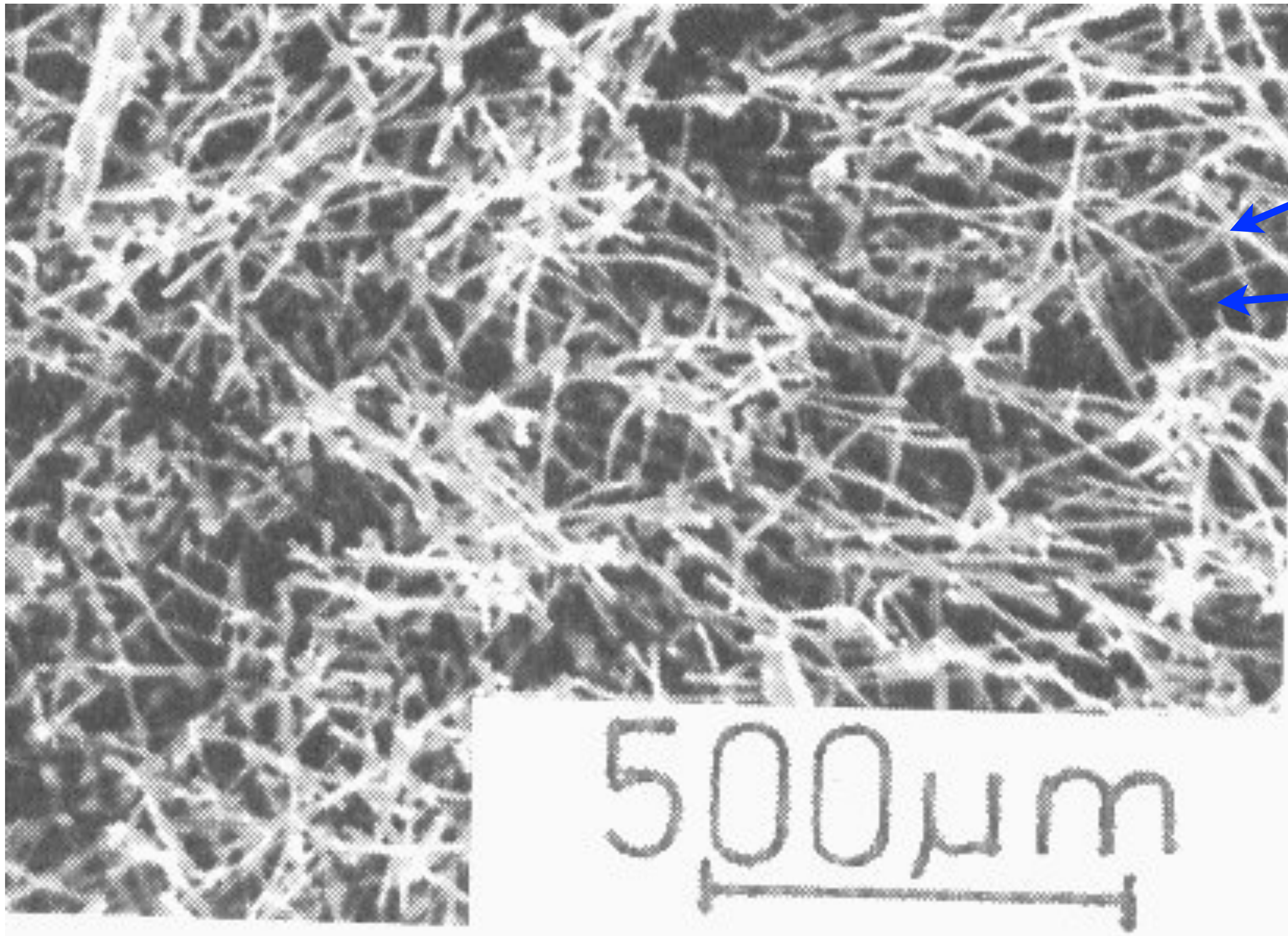
From F.L. Matthews and R.L. Rawlings, Composite Materials; Engineering and Science, Reprint ed., CRC Press, Boca Raton, FL, 2000. (a) Fig. 4.22, p. 145 (photo by J. Davies); (b) Fig. 11.20, p. 349 (micrograph by H.S. Kim, P.S. Rodgers, and R.D. Rawlings).

Mo + Ni_3Al



W. Funk et al., Met. Trans A**19**, 987-998 (1988).

Fiber reinforcements: discontinuous, random

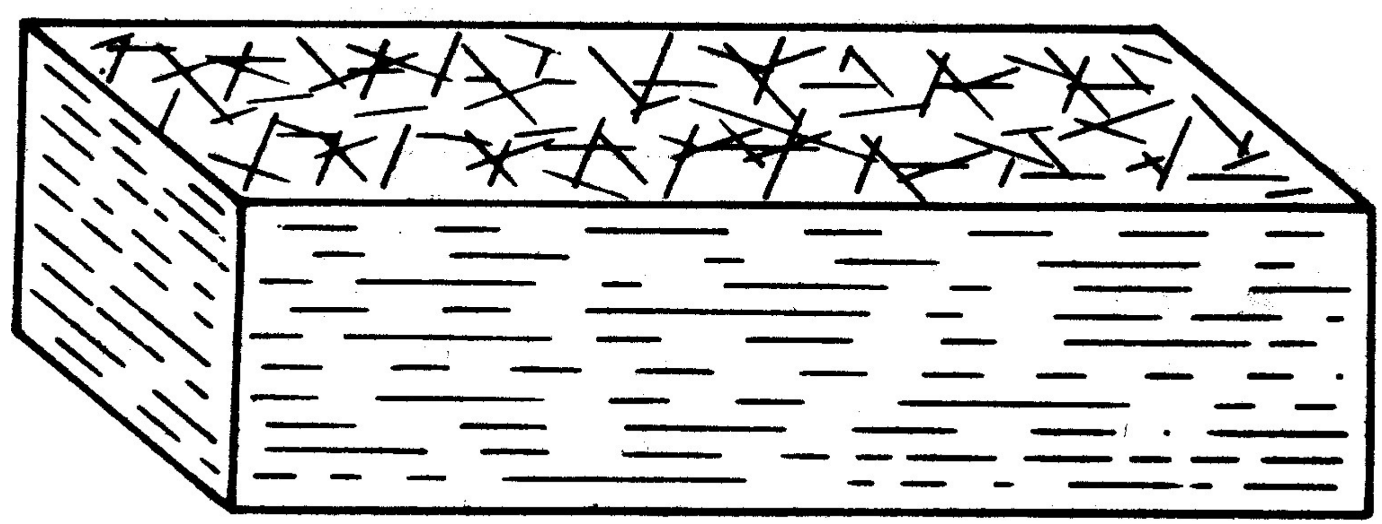


C fibers

C matrix

processed by laying down fibers in binder (pitch); high heat converts binder to C matrix

carbon-carbon composite



Randomly oriented fibers layered in 2D, not continuous with composite

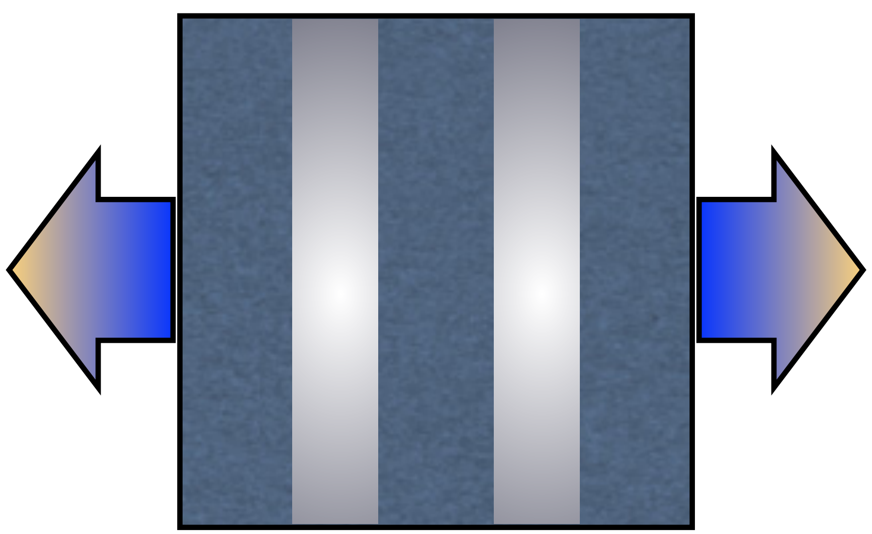
Determining mechanical behavior

Stress / strain response depends on

- material properties (E, TS, σ_Y) of matrix + reinforcement
- amount of matrix + reinforcement (V_m, $V_r = 1 - V_m$)
- orientation of reinforcement relative to load
- size and distribution of reinforcement
- geometry (length of fibers, cross-sectional shape, aspect ratio)

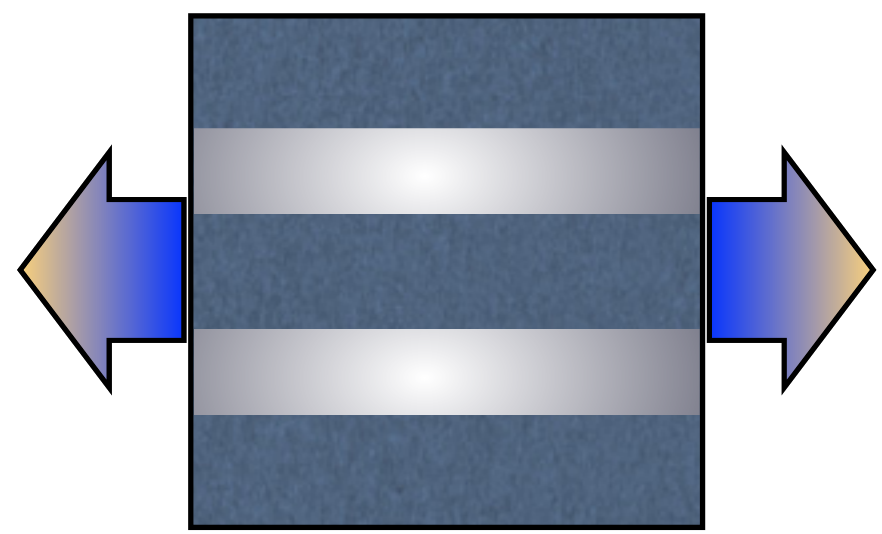
Two limiting cases for analysis:

isoload/isostress



equal load in phases

isostrain



equal strain in phases

Isostrain

equal length/strain in phases

$$\ell_{\text{reinforcement}} = \ell_{\text{matrix}} = \ell_{\text{composite}}$$

$$\varepsilon_{\text{reinforcement}} = \varepsilon_{\text{matrix}} = \varepsilon_{\text{composite}}$$

shared load: $$F_c = F_m + F_r$$

$$\frac{F_c}{A} = \frac{F_m}{A} + \frac{F_r}{A}$$

$$\sigma_c = \frac{F_m}{A_m}\frac{A_m}{A} + \frac{F_r}{A_r}\frac{A_r}{A}$$

$$\sigma_c = V_m \sigma_m + V_r \sigma_r$$ **ROM for stresses**

Similar to Voigt average

Isoload / isostress

equal load/stress in phases

$$F_{reinforcement} = F_{matrix} = F_{composite}$$

$$\sigma_{reinforcement} = \sigma_{matrix} = \sigma_{composite}$$

shared length: $\ell'_c = \ell'_m + \ell'_r$

$$\ell_c(1+\varepsilon_c) = \ell_m(1+\varepsilon_m) + \ell_r(1+\varepsilon_r)$$

$$1+\varepsilon_c = V_m(1+\varepsilon_m) + V_r(1+\varepsilon_r)$$

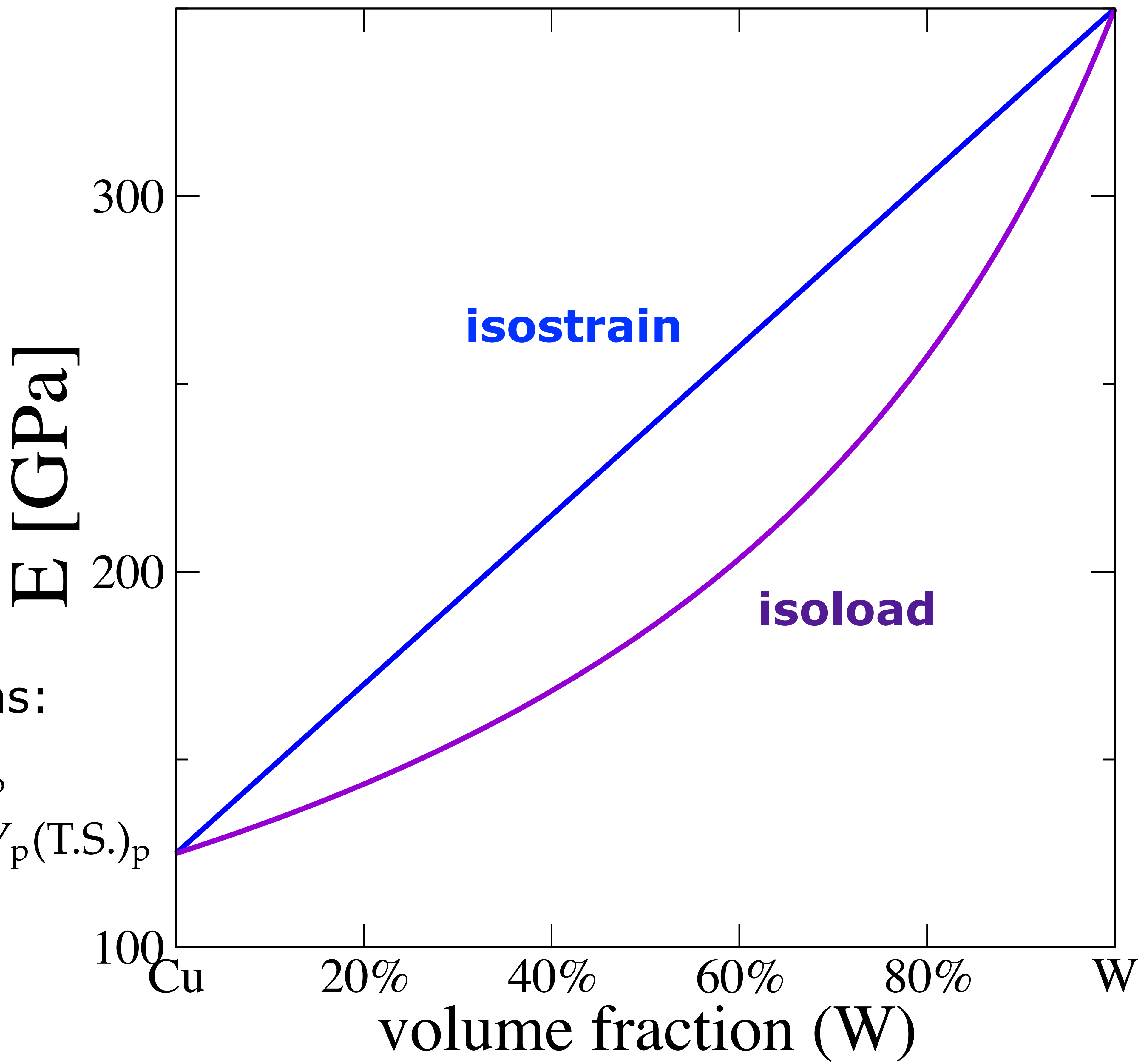
$$\varepsilon_c = V_m\varepsilon_m + V_r\varepsilon_r$$

ROM for strains

Similar to Reuss average

Rule of mixtures: Cu particles + W matrix

Elastic moduli of the composite are constrained by two limits: **isostrain** and **isoload**



Empirical relations:

$$E_c = V_m E_m + K_c V_p E_p$$

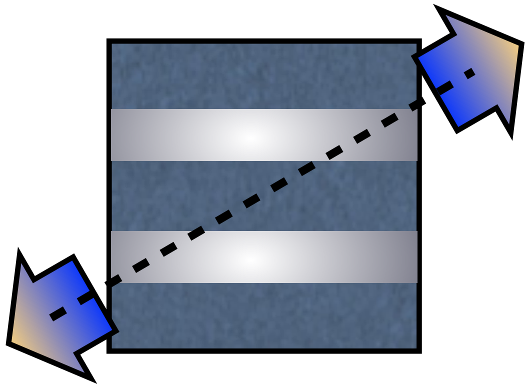
$$(T.S.)_c = V_m (T.S.)_m + K_s V_p (T.S.)_p$$

$$K_c \neq K_s < 1$$

Orientation effects on tensile strength

Tensile stress not parallel to fibers has complex stress state:

3 limiting cases:



$$\underline{\sigma} = \begin{pmatrix} \sigma\cos^2\theta & \sigma\cos\theta\sin\theta \\ \sigma\cos\theta\sin\theta & \sigma\sin^2\theta \end{pmatrix}$$

1.small misorientation: limited by fiber failure ($\sigma_{\parallel} = \sigma\cos^2\theta$)

$$(\mathrm{T.S.})_c = \frac{\sigma_{\parallel}^{\star}}{\cos^2\theta}$$

2.large misorientation: limited by matrix tensile failure ($\sigma_{\perp} = \sigma\sin^2\theta$)

$$(\mathrm{T.S.})_c = \frac{\sigma_{\perp}^{\star}}{\sin^2\theta}$$

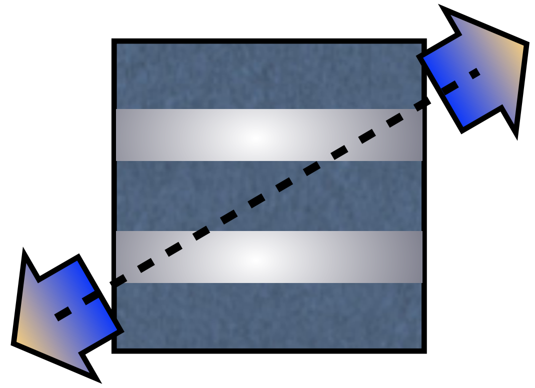
3.medium misorientation: limited by matrix shear failure ($\tau = \sigma\cos\theta\sin\theta$)

$$(\mathrm{T.S.})_c = \frac{\tau_{m,y}}{\cos\theta\sin\theta}$$

Orientation effects on tensile strength

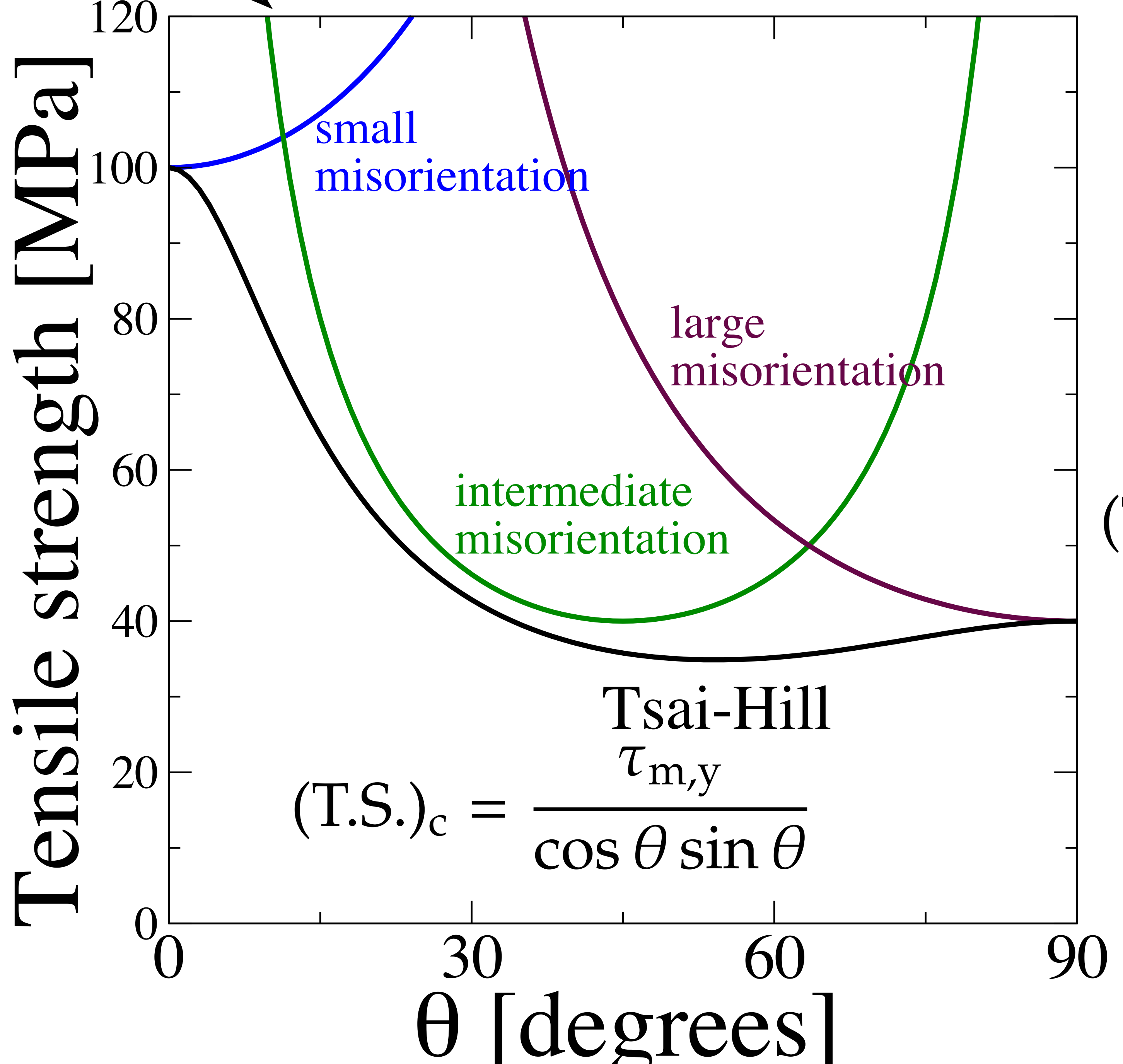
Tensile stress not parallel to fibers has complex stress state:

3 limiting cases:



$$\underline{\sigma} = \begin{pmatrix} \sigma\cos^2\theta & \sigma\cos\theta\sin\theta \\ \sigma\cos\theta\sin\theta & \sigma\sin^2\theta \end{pmatrix}$$

$$(\mathrm{T.S.})_c = \frac{\sigma^\star_\parallel}{\cos^2\theta}$$

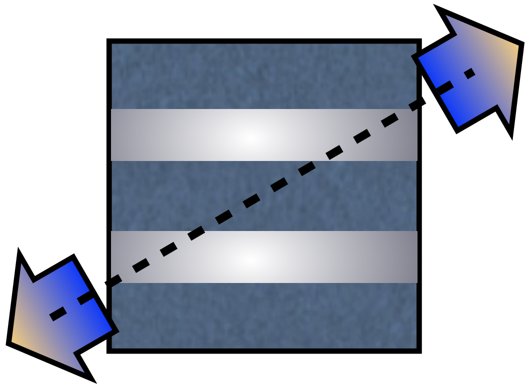


$$(\mathrm{T.S.})_c = \frac{\sigma^\star_\perp}{\sin^2\theta}$$

Orientation effects on tensile strength

Tensile stress not parallel to fibers has complex stress state:

Some limitations:



$$\underline{\sigma} = \begin{pmatrix} \sigma\cos^2\theta & \sigma\cos\theta\sin\theta \\ \sigma\cos\theta\sin\theta & \sigma\sin^2\theta \end{pmatrix}$$

1. Predicts that tensile strength *increases* for small misorientation.
2. Predicts "cusps" in strength vs. misorientation angle.
3. Doesn't account for multiaxial loading effects.

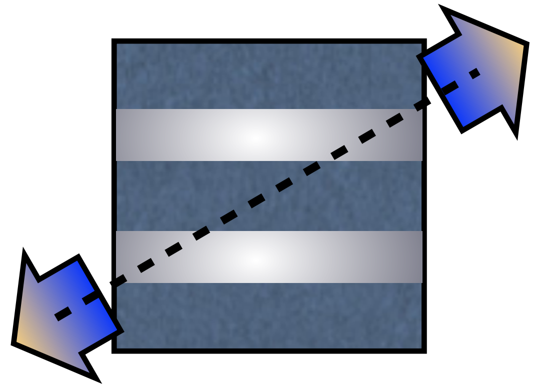
Solution: Tsai-Hill failure criterion:

$$\left(\frac{\sigma_{\|}}{\sigma_{\|}^{\star}}\right)^2 - \left(\frac{\sigma_{\|}\sigma_{\perp}}{\sigma_{\perp}^{\star\,2}}\right) + \left(\frac{\sigma_{\perp}}{\sigma_{\perp}^{\star}}\right)^2 + \left(\frac{\tau}{\tau_{\mathrm{m,y}}}\right)^2 = 1$$

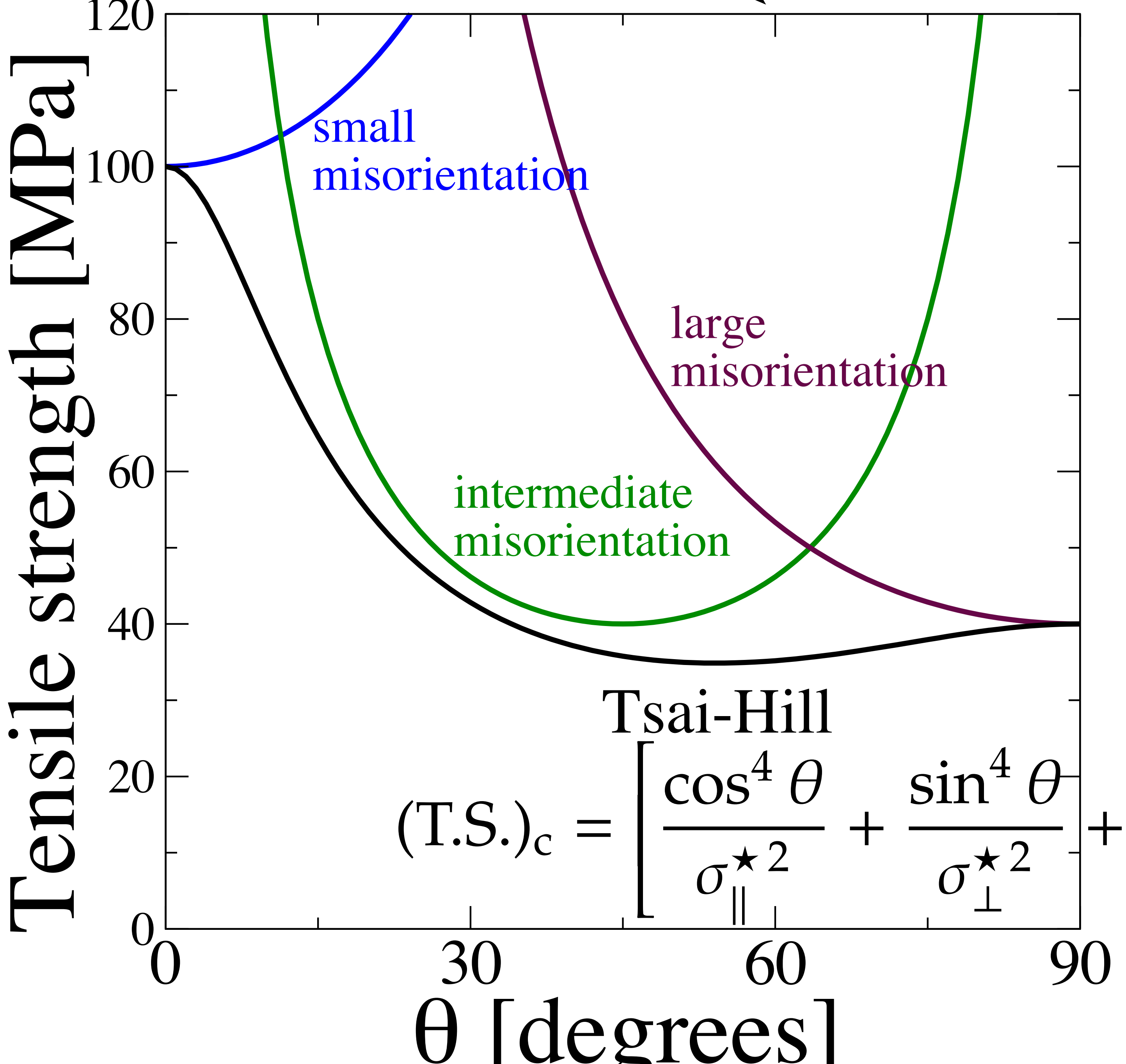
$$(\mathrm{T.S.})_{\mathrm{c}} = \left[\frac{\cos^4\theta}{\sigma_{\|}^{\star\,2}} + \frac{\sin^4\theta}{\sigma_{\perp}^{\star\,2}} + \cos^2\theta\sin^2\theta\left(\frac{1}{\tau_{\mathrm{m,y}}^2} - \frac{1}{\sigma_{\|}^{\star\,2}}\right)\right]^{-1/2}$$

Orientation effects on tensile strength

Tensile stress not parallel to fibers has complex stress state:



$$\underline{\sigma} = \begin{pmatrix} \sigma\cos^2\theta & \sigma\cos\theta\sin\theta \\ \sigma\cos\theta\sin\theta & \sigma\sin^2\theta \end{pmatrix}$$



$$(\mathrm{T.S.})_c = \left[\frac{\cos^4\theta}{\sigma_{\parallel}^{\star 2}} + \frac{\sin^4\theta}{\sigma_{\perp}^{\star 2}} + \cos^2\theta\sin^2\theta\left(\frac{1}{\tau_{\mathrm{m,y}}^2} - \frac{1}{\sigma_{\parallel}^{\star 2}}\right)\right]^{-1/2}$$

Tsai-Hill smooths out cusps
Never exceeds aligned T.S.