ECE 586GT: Problem Set 5: Problems and Solutions
Games with incomplete information, multistage games, VCG mechanisims

Due: Wednesday, November 11, 11:59 p.m.
Reading: Course notes, Section 5.3, Chapters 6, & Section 7.1

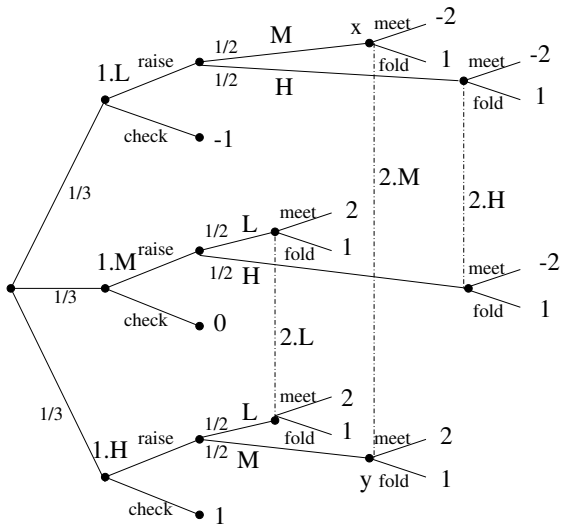
1. **[Half Kuhn poker]**
Consider the following two-player zero-sum game. There is a deck of three cards, $\{L, M, H\}$ with the ordering $L < M < H$. The players are randomly assigned one card each with each possibility having probability 1/6. The players see their own cards but not the card of the other player. If the players simply compared cards, the player with the larger valued card would win 1 unit of money. Instead, the players follow the following protocol. Player 1 first says "raise" or "check." If player 1 says check the two players show their cards and the player with the larger valued care wins one unit. If player 1 says check then player 2 can either meet or fold. If player 2 meets then the stakes are doubled and the player with the higher card wins 2 units. If player 2 folds then player 1 wins one unit.

   (a) Which player has an advantage over the other and why?

   Solution: Player 1 has an advantage. If player 1 always selected "check" then the game reduces to simply comparing cards and the expected payoff of player 1 is zero. But player 1 can do better by raising with H and sometimes raising with L (as we will see below).

   (b) Make a sketch of the game tree. (Hint: Try making it as useful as you can by giving meaningful names to nodes or information sets. To avoid clutter, since it is a zero sum game you could label the leaves with the payoff of player 1 which is equal to the loss of player 2, instead of giving the payoffs of both players. It might be simpler to first have nature select a card for player 1 and then, if needed, later select a card for player 2, equally likely to be either card not selected for player 1.

   Solution: See the figure below. The draw of cards is modeled sequentially. Given the card drawn by player 1, there are two equally likely possibilities for the card drawn by player 2. In case of check the subtree for the card drawn by player 2 is omitted.



(c) Identify a subgame perfect equilibrium for the game and the corresponding value of the game. Is the subgame perfect equilibrium unique?

Solution: Consider a sequential equilibrium – one is guaranteed to exist for this game with perfect recall. For information set 2.H of player 2, "meet" is a dominant strategy and must be used in any sequential equilibrium. For information set 2.L of player 2, "fold" is a dominant strategy and must be used in any sequential equilibrium. Thus, the behavioral strategyies for player 2 that can possibly be part of a sequential equilibria can be completely specified by q, where q is the probability of selecting "meet" in information set 2.M.

Turning to player 1, first consider decision node 1.M. Since, as noted above, player 2 always folds on L and always meets on H, the expected payoff of player 1 using "raise" at 1.M is (1-2)/2 = -0.5, which is smaller than the expected payoff, namely 0, of using "check." Thus, player 1 must use "check" at 1.M. Next consider decision node 1.H. Observe that "raise" is a weakly dominant strategy. Also, raise will give a strictly larger payoff than "check" if player 2 uses "meet" with positive probability in information set 2.M. So suppose for the time being that player 1 uses "raise" in state 1.H. Then the behaviorial strategy of player 1 is completely specified by p, where p is the probability of selecting "raise" in information state 1.L.

The continuation payoff at decision node 1.L (for player 1) for action "raise" is $(1/2)(q(-2)+(1-q)(1))+(1/2)(-2) = -1/2-(3/2)q$ and the continuation payoff for action "check" is -1. Therefore,
—If $q < 1/3$ then $p = 1$
—If $q > 1/3$ then $p = 0$

State x is reached with probability $(1/3)p(1/2) = p/6$ and state y is reached with probability $(1/3)(1)(1/2) = 1/6$, so Bayes rule implies that the belief vector for information set 2.M is $\mu(2.M) = (\mu(x|2.M), \mu(y|2.M)) = (\frac{p}{1+p}, \frac{1}{1+p})$. Therefore, the expected continuation loss from 2.M for player 2 is $\frac{p}{1+p}(-2)+\frac{1}{1+p}(2) = (2-2p)/(1+p)$ for "meet" and 1 for "fold." Thus, the best response at 2.M (i.e. minimizing the continuation cost of player 2) is "meet" if $p > 1/3$ and "fold" if $p < 1/3$. If $p = 1/3$ either "meet" or "fold" is a best response at 2.M. So we have:
—If $p > 1/3$ then $q = 1$
—If $p < 1/3$ the $q = 0$

So if $q < 1/3$ then $p = 1$ then $q = 1$, which is impossible. Similarly, if $q > 1/3$ then $p = 0$ then $q = 0$ which is also impossible. So the only solution is $p = q = 1/3$. That gives the following sequential equilibrium:

Player 1 raises on H, folds on M, and raises on L with probability 1/3.
Player 2 (assuming player 1 raises) folds on L, meets on M with probability 1/3, and meets on H. The expected payoff for player 1 (expected loss for player 2) is 1/18, which can be found using the probabilities and leaf payoffs on the resulting game tree.

Finally, let's go back and suppose player 1 in state 1.H uses "raise" with some probability $r < 1$ and "check" with some positive probability $1 - r$. As noted above, by sequential rationality at state 1.H this requires $q = 0$. Also as noted above, sequential rationality at decision node 1.L requires:
If $q < 1/3$ then $p = 1$
If $q > 1/3$ then $p = 0$
So it must be that $p = 1$.
State x is still reached with probability $(1/3)p(1/2) = p/6$ but now state y is reached with probability $(1/3)r(1/2) = r/6$, so Bayes rule implies that the belief vector for information set 2.M is $\mu(2.M) = (\mu(x|2.M), \mu(y|2.M)) = (\frac{p}{r+p}, \frac{r}{r+p})$. Therefore, the expected continuation loss from 2.M for player 2 is $\frac{p}{r+p}(-2)+\frac{r}{r+p}(2) = 2(r-p)/(r+p)$ for "meet" and 1 for "fold." Thus, the best response at 2.M (i.e. minimizing the continuation loss of player 2) is "meet" if $p > r/3$ and "fold" if $p < r/3$. If $p = r/3$ either meet or fold is a best response at 2.M. So we have:
If $p > r/3$ then $q = 1$
If $p < r/3$ then $q = 0$
So we must have $q = 1$ which is a contradiction. We conclude that for any sequential equilibrium action "raise" must be used at decision node 1.H (by player 1). Thus, above we have found the unique sequential equilibrium.

2. **[The lemon problem]**
A seller has a used car to sell and a buyer is interested in buying it. Suppose the value θ of the car to seller is random and uniformly distributed on the interval $[0, 1]$. The seller knows θ, which can be thought of as the type of the seller. The buyer does not know θ but it is assumed that the value of the car to the buyer is $a + b\theta$, for some constants a and b such that $a \geq 0$ and $1 < a + b < 2$. These constraints ensure that the car is worth more to the buyer than to the seller for any θ, so, payment aside, the socially optimal outcome would be for the car to be transferred from seller to buyer. But can they agree on a price? Suppose the seller, knowing θ, calculates a reserve price $r = r(\theta)$, which is the minimum amount the seller would sell the car for. Suppose the buyer selects a price $p \in [0, 1]$ and offers to pay p for the car. If $p \geq r$, the car is sold for price p. Otherwise the car is not sold. The payoff for the seller is $u_s(p, r, \theta) = \theta \mathbf{1}_{\{p<r\}} + p\mathbf{1}_{\{p\geq r\}}$ and the payoff for the buyer is $u_b(p, r, \theta) = (a + b\theta - p)\mathbf{1}_{\{p\geq r\}}$.

   (a) Identify a weakly dominant strategy for the seller.
   Solution: For any θ, p, $u_s(p, r, \theta) \leq \max\{\theta, p\}$, with equality for $r = \theta$. Thus, $r = \theta$ is a weakly dominant strategy.

(b) Assuming seller uses its weakly dominant strategy, find the price p the buyer should offer to maximize her/his expected payoff. (Suppose the buyer knows a, b, and the distribution of θ).

Solution: Taking $r(\theta) = \theta$,

$$\mathbb{E}\left[u_b(p, r, \theta)\right] = \mathbb{E}\left[(a + b\theta - p)\mathbf{1}_{\{p \geq \theta\}}\right] = \int_0^p (a + b\theta - p)d\theta = ap + \frac{bp^2}{2} - p^2,$$

and $\frac{d\mathbb{E}[u_b(p,r,\theta)]}{dr} = a + (b-2)p$. So the best response for the buyer is $p = \frac{a}{(2-b)}$. (This value is in $[0, 1]$ by the assumptions on a, b.)

(c) What happens in the special case $a = 0$ and $1 < b < 2$. Give a simple explanation.

Solution: The optimal price offered is $p = 0$, so no sale is made with probability one. If buyer were to instead bid p with $0 < p \leq 1$, given the seller accepts, that is, given $\theta \leq p$, the conditional mean value of the car to player 2 would be $\frac{bp}{2}$, which would be less than the price p paid.

3. **[Repeated play for the Bertrand equilibrium problem]**
Suppose n players, for some $n \geq 2$, represent firms that can each produce a common divisible good at a cost c per unit of good. Suppose the action of each player is to declare a price p_i per unit of good. Suppose there is an aggregate demand of consumers such that if the lowest price offered by any player is p_{min} then the consumers purchase a total quantity $(a - p_{\min})_+$ of good, where a is a constant with $a > c$, and they purchase an equal amount from each player offering the minimum price. The game is among the players offering prices; the consumers are not considered to be part of the game. As we saw in problem set 1, problem 7(a), p is a Nash equilibrium if and only if $p_{\min} = c$ and $p_i = c$ for at least two players. The resulting payoff vector is the all zero vector. Basically, the competition among the players drives the sum of payoffs to zero at Nash equilibrium.

(a) Let's see if positive payoffs can be sustained in the infinite repeated game with the Bertrand based game as a stage game, with weight $(1 - \delta)\delta^{t-1}$ on stage t. To begin, fix $\bar{p}$ with $c < \bar{p} < \frac{a+c}{2}$, and consider the trigger strategy based on the script that every player bids price $\bar{p}$ in every stage. Players follow the script, with switching to playing c forever if any player deviated from the script in an earlier stage. Each player will receive payoff $(a - \bar{p})(\bar{p} - c)/n$ per stage. Find the minimum δ so that such strategy profile is subgame perfect.

Solution: By the one-step deviation principle for infinite-horizon repeated games, it suffices to consider one player deviating from the trigger strategy in one stage. By symmetry and time invariance we can consider without loss of generality a change for player 1 in the first stage. The player should select an action to maximize the payoff in the first stage, because the payoffs in all subsequent stages will be zero. Thus, the player should bid $\bar{p} - \epsilon$ for a very small value of ϵ, resulting in payoff $(a - \bar{p} + \epsilon)(\bar{p} - c - \epsilon)$, which is smaller, but can be made arbitrarily close to, $(a - \bar{p})(\bar{p} - c)$. Thus, player 1 has no incentive to deviate if and only if

$$(a - \bar{p})(\bar{p} - c) \leq \left(\sum_{t=1}^{\infty} \delta^{t-1}\right)(a - \bar{p})(\bar{p} - c)/n$$

or equivalently, since the infinite sum is $\frac{1}{1-\delta}$, if and only if $1 - \delta \leq \frac{1}{n}$, or $\delta \geq \frac{n-1}{n}$. Interestingly, this condition doesn't depend on $\bar{p}$. The payoffs would be maximized by taking $\bar{p} = \frac{a+c}{2}$.

(b) Identify the set of payoff vectors v in the Nash realization region for the infinite repeated game with stage game based on the Bertrand selling mechanism.

Solution: Let $\bar{p} = \frac{a+c}{2}$ and $P = \frac{(a-c)^2}{4}$. Note that P is the maximum sum of payoffs. If player i bids $\bar{p}$ and all other players make larger bids, then player gets payoff P and all other players get 0 payoff. Therefore, the vectors $(P, 0, 0, \ldots, 0), (0, P, 0, \ldots, 0), \ldots, (0, \ldots, 0, P)$ as well as $(0, 0, \ldots, 0)$, are feasible. The convex hull is $\{v \in \mathbb{R}^n : v \geq 0, \sum_{i\in[n]} v_i \leq P\}$, and the Nash realization region is therefore $\{v \in (0, +\infty)^n : \sum_{i\in[n]} v_i \leq P\}$. (Since the players revert to a Nash equilibrium, the points in the Nash realization region are realizable by subgame perfect equilibria as well, by Friedman's theorem.)

4. **[VCG applied to combinatorial auction]**
Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a, b, c\}$ to three bidders, who have submitted the following bids:

$$\begin{array}{lll} v_1\{a\} = 12, & v_1\{a, b\} = 14 & \\ v_2\{b, c\} = 6, & v_2\{a, b, c\} = 12 & \\ v_3\{b\} = 2, & v_3\{c\} = 5, & v_3\{b, c\} = 6. \end{array}$$

Assume the value of any bundle to any bidder, if not listed above, is zero. Also, assume that for any bidder, at most one of the bidder's bids may be used. In particular, by bidding such that $v_3\{b, c\} < v_3\{b\} + v_3\{c\}$, bidder 3 communicates that b and c are partial substitutes for the bidder. Suppose also that the seller has a reserve price, or value, of 1, for each item that is not sold. Determine the assignment of objects to bidders and the payments of the bidders, for the VCG mechanism under truthful bidding. Use the individually rational version of payments.

Solution: The VCG allocation is unique, and is given by $A^* = (\{a, b\}, \emptyset, \{c\})$, with total value 19. To find m_1, note that if bidder 1 were not present, the maximum welfare assignment would be $(-, \{a, b, c\}, \emptyset)$ with total value 12, whereas bidders 2 and 3 receive total value 5 under A^*, so $m_1 = 12 - 5 = 7$. To find m_3, note that if bidder 3 were not present, the maximum welfare assignment would be $(\{a\}, \{b, c\}, \emptyset, -)$, for a total value 12+6 = 18, whereas bidders 1 and 2 receive total value 14 under A^*, so $m_3 = 18 - 14 = 4$. Also, $m_2 = 0$ because bidder 2 is assigned no objects under A^*. So the payment profile is $m = (7, 0, 4)$.

Another way to identify m_1 is to replace player 1 by a single-minded version as shown (see Example 6.6 of the notes):

$$\begin{array}{lll} v_1'\{a, b\} = m_1 & & \\ v_2\{b, c\} = 6, & v_2\{a, b, c\} = 12 & \\ v_3\{b\} = 2, & v_3\{c\} = 5, & v_3\{b, c\} = 6. \end{array}$$

Then note that the smallest value of m_1 such that A^* continues to be the maximum welfare assignment is $m_1 = 7$.

5. **[Supporting a public good with private information]**
Consider n players, any of whom could perform a particular job that would benefit all the players. For example, the players could be participating in some venture and the job is to make preparations to ensure safety in case of some unanticipated event. Suppose each player derives value 1 if at least one player does the job. Suppose θ_i with $0 \leq \theta_i \leq 1$ is private information of player i, which represents the cost for player i to do the job. So player i knows the value of θ_i but neither the coordinator nor any other player knows the value of θ_i.

(a) Explain how this situation could be handled by a VCG mechanism. In particular, describe the bids made by the player to the coordinator, how the coordinator selects which player to do the job, and how much the player should be paid by the coordinator.

Solution: If player i is the one to due the job the maximum social welfare is $n - \theta_i$, which is maximized if $\theta_i = \min\{\theta_j : 1 \leq j \leq n\}$. Suppose each player i sends a bid $\hat{\theta}_i$ to the coordinator. If these bids are truthful (i.e. $\hat{\theta}_i = \theta_i$ for all i.) then the coordinator maximizes welfare by assigning the player with the lowest bid to do the job, with ties broken arbitrarily. Suppose the i^* is the one selected. The sum of welfares of the other $n-1$ players in that case ie $n-1$, while the maximum welfare that could be achieved by the other $n-1$ players in the absence of player i^* would be $n-1-\min\{\hat{\theta}_j : j \neq i^*\}$. Thus, in essence, by doing the job, player i increases the welfare of the other $n-1$ players by $\sigma := \min\{\hat{\theta}_j : j \neq i^*\}$. By the VCG principle, we thus have the coordinator pay the amount σ to player i^*.

In summary, the player with the minimum bit $\hat{\theta}_i$ is paid an amount equal to the second smallest bid to do the job.

(b) Your solution to part (a) does not need to be budget balanced in the sense that the coordinator may need to subsidize the game by paying a player to do the job. A simple attempt at fixing the problem would be for the coordinator impose a fee on each player in the amount of σ/n. In such way the coordinator has a balanced budget. Are the other important properties of the VCG mechanism preserved: maximum social welfare, optimality of truthful bidding, nonnegative total payoffs for all players?

Solution: No. If there were no ties and player i were to have the second largest θ_i value and if all players report truthfully then $\sigma = \hat{\theta}_i = \theta_i$. If the player were to slightly decrease $\hat{\theta}_i$ so that it was still the second smallest bid, then the fee σ/n to the player would be slightly reduced, providing an incentive for the player to deviate from truthful bidding. (If the coordinator had a good *a priori* estimate σ_{approx} of what σ was going to be, the coordinator could impose a charge of σ_{approx}/n to make the payment approximately budget balanced.)