University of Illinois **Fall 2026**

ECE 586GT: Problem Set 2
Uniqueness of Nash equilibria, zero sum games, evolutionary dynamics

Due: Wednesday, September 23, 11:59 p.m.
Reading: Course notes, Sections 1.5-1.7 and Chapter 2

1. **[Some continuous games]**
Consider a normal form game with a finite number of players n and strategy sets $S_i = [0, 1]$ for $1 \le i \le n$.

   (a) Suppose $u_i(x) = x_i + \frac{b}{2}x_{tot}^2$ for some constant b, where $x_{tot} = x_1 + \cdots + x_n$. Identify all the pure Nash equilibria (i.e. all Nash equilibria in pure strategies). Hint: Consider cases $b \ge 0$, $-\frac{1}{n} \le b < 0$ and $b \le -\frac{1}{n}$ separately.

   (b) More generally, suppose $u_i(x) = x_i + \frac{1}{2}x^T Ax$ for a given symmetric matrix A while still assuming $S_i = [0, 1]$ for all i. Under what conditions on A does the Debreu-Glicksberg-Fan theorem imply that there exists a pure Nash equilibrium? (Find the most general condition possible.)

   (c) For the payoff functions of part (b), under what conditions on A does Rosen's theorem imply that there is at most one pure Nash equilibrium?

2. **[Min-max and max-min strategies for rock-scissors-paper game]**
Consider the rock–scissors–paper game described in Example 1.17. Let $\ell(p, q)$, with $p, q \in \{R, S, P\}$, denote the payoff function of player 2, which player 1 seeks to minimize.

   (a) Assuming both players use pure strategies, identify the set of min–max strategies of player 1 and the set of max–min strategies of player 2. Determine whether there is a duality gap. If no gap, determine the set of all saddle points in pure strategies.

   (b) Repeat part (a) but now consider p, q in the space of mixed strategies.

3. **[Karush-Kuhn-Tucker Debreu-Glicksberg-Fan]**
Consider the convex optimization problem: $\min x$ over $x \in \mathbb{R}$ s.t. $g(x) \le 0$ where $g(x) = x^2 - \epsilon$.

   (a) Suppose $\epsilon > 0$. Is the relaxed Slater condition satisfied? What is the Lagrangian $L(x, \lambda)$? Find (x^*, λ) satisfying the KKT conditions, if possible. Does strong duality hold in the sense that $\inf_x \sup_{\lambda \ge 0} L(x, \lambda) = \sup_{\lambda \ge 0} \inf_x L(x, \lambda)$? If $L(x, \lambda)$ is viewed as the objective function for a two-person zero-sum game between primal and dual players, does the DGF theorem for existence of Nash equilibrium apply?

   (b) Repeat part (a) for $\epsilon = 0$.

   (c) If $\epsilon = 0$ and we were to impose the constraints $x, \lambda \in [-1, 1]$ then would there exist a Nash equilibrium for the two-person zero-sum game based on the Lagrangian? Briefly explain your answer.

4. **[Dual of a transport problem]**
Let $n \ge 1$ and let W be an $n \times n$ matrix with nonnegative elements. Consider the following

linear optimization problem:

$$
\begin{aligned}
\min_{X} \sum_{i,j\in[n]} & X_{i,j}W_{i,j} \\
\text{subject to:} \quad & \sum_{j\in[n]} X_{i,j} = 1 && \text{for } i \in [n] \\
& \sum_{i\in[n]} X_{i,j} = 1 && \text{for } j \in [n] \\
& X_{i,j} \geq 0 && \text{for } i,j \in [n]
\end{aligned}
$$

An interpretation is that there are n sources of some good, n destinations, $W_{i,j}$ is the cost of transporting a unit of good from i to j, and $X_{i,j}$ is the amount of good transported from i to j. Each source provides one unit of good and each destination receives one unit of good. The problem in vector matrix form can be written as:

$$
\begin{aligned}
&\min_{X} \langle X, W \rangle \\
&\text{subject to: } X\mathbf{1} = \mathbf{1},\ X^T\mathbf{1} = \mathbf{1},\ X \geq 0.
\end{aligned}
$$

(a) Derive the dual problem by first finding the Lagrangian function. Find a simple version of the dual problem. (Hint: You can either eliminate the Lagrange multipliers for the constraints $X_{i,j} \geq 0$, or don't handle those constraints with Lagrange multipliers in the first place.

(b) Find the common value of the primal and dual, and solutions, for $W = \begin{pmatrix} 3 & 8 \\ 2 & 4 \end{pmatrix}$.

5. **[Evolutionarily stable strategies and states]**

Consider the following symmetric, two-player game::

	1	2
1	4,4	1,6
2	6,1	0,0

(a) Does either player have a (weakly or strongly) dominant strategy?

(b) Identify all the pure strategy and mixed strategy Nash equilibria.

(c) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

(d) The replicator dynamics based on this game represents a large population consisting of type 1 and type 2 individuals. Show that the evolution of the population share vector $\theta(t)$ under the replicator dynamics for this model reduces to a one dimensional ordinary differential equation for $\theta_t(1)$, the fraction of the population that is type 1.

(e) Identify the steady states of the replicator dynamics.

(f) Of the steady states identified in the previous part, which are asymptotically stable states of the replicator dynamics? Justify your answer.