ECE 586GT: Problem Set 1
Analysis of static games

Due: Wednesday, September 9, 11:59 p.m.
Reading: Course notes, Sections 1.1 - 1.4

1. **[A two player game]**
Consider the two-player strategic game with the following payoff matrix, where player 1 selects a row and player 2 selects a column:

	L	M	R
A	8,9	5,7	2,8
B	6,6	7,4	4,8
C	4,3	3,6	3,8

.

   (a) Identify all dominant and all weakly dominant strategies by either player.
   (b) Identify any pairs of strategies for either player such that one strategy dominates the other.
   (c) Identify all pure Nash equilibria.
   (d) Identify the NE involving at least one nondegenerate mixed strategy, if any.

2. **[Iterative elimination of strategies]**
Consider finite games in normal form.

   (a) Give an example of such a game such that iterated elimination of weakly dominated strategies (IEWDS) removes a pure Nash equilibrium and after no more iterations are possible there is no remaining pure Nash equilibrium. (If no such examples exist, explain why.)
   (b) Give an example of such a game such that iterated elimination of dominated strategies (IEDS) removes a Nash equilibrium and after no more iterations are possible there is no pure Nash equilibrium. (If no such examples exist, explain why.)

3. **[Use of a binding contract]**
Consider the two player game shown:

	L	R
T	9,1	7,2
B	8,5	6,6

The first player selects T or B and the second player selects L or R. Suppose the numbers in the table represent multiples of one unit such that one unit is a substantial amount of money.

   (a) Note that T is a dominant strategy for player 1, but action T has a substantial negative externality for player 2. So suppose player 2 could propose a binding offer to player 1 promising to pay player 1 x units of money and also promise to play strategy L if player 1 agrees to play strategy B. (If x is negative it means player 1 has to pay $-x$ units to player 2.) What is the range of possible values of x that player 2 could offer such that neither player would have incentive to reject the proposal? For such values of x, what is the set of possible final reward vectors that would result if player 1 agrees to the proposal, after taking into account the transfer payment. Suppose that if the proposal is not used the players play a Nash equilibrium strategy.

(b) Find the polytope of reward vectors for correlated equilibria for this game (without transfer payments). Are any of them as good or better for the players than what was found in part (a).

4. **[Provisioning a public good]**
Suppose n players are invited to contribute payments for a public good, such as pavement for a road, a well for water, or a fireworks display, that will be valued by all players. Each player i decides an amount p_i to pay. A strategy profile is denoted by $p = (p_1, \ldots, p_n)$, and the total sum of payments is denoted by $P = p_1 + \cdots + p_n$. Suppose the total sum is used to invest in a public good that is worth $a\ln(1+P)$ to all players for some fixed and known $a > 0$, so the payoff function of each player i is $u_i(p) = a\ln(1+P) - p_i$.

   (a) Identify all the pure Nash equilibrium points for this n-player game. Also, find the value of the total welfare, $\sum_{i=1}^{n} u_i(p)$, at Nash equilibrium. (Hint: The form of your answer may be different for different values of a.)

   (b) Identify the maximum possible social welfare, which is the maximum over p of $\sum_i u_i(p)$. Compare to the welfare found in part (a) for $a = 2$ and large n.

5. **[Weierstrass extreme value theorem and existence of minima and maxima]**
For each of the following expressions, determine whether the minimum or maximum indicated exists and also say whether the Weierstrass extreme value theorem *directly* applies and if not, why not.

   (a) $\min\{x^2 + y^2 : x, y \in (0, 1]\}$.

   (b) $\max\left\{\frac{x}{1+x^2} : 0 \leq x < \infty\right\}$

   (c) $\min\left\{\lfloor x \rfloor + (x-1)^2 : 0 \leq x \leq 10\right\}$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

6. **[Guessing 2/3 of the average]**
Consider the following game for n players. Assume $n \geq 2$. Each of the players selects a number from the set $\{1, \ldots, 100\}$, and a cash prize is split evenly among the players whose numbers are closest to two-thirds the average of the n numbers chosen.

   (a) Show that the problem is solvable by iterated elimination of *weakly* dominated strategies, meaning the method can be used to eliminate all but one strategy for each player, which necessarily gives a Nash equilibrium. (A strategy μ_i of a player i is called weakly dominated if there is another strategy μ_i' that always does at least as well as μ_i, and is strictly better than μ_i for some vector of strategies of the other players.)

   (b) Give an example of a two player game, with two possible actions for each player, such that iterated elimination of weakly dominated strategies can eliminate a Nash equilibrium. (Hint: The eliminated Nash equilibrium might not be very good for either player.)

   (c) Show that the Nash equilibrium found in part (a) is the unique mixed strategy Nash equilibrium (as usual we consider pure strategies to be special cases of mixed strategies). (Hint: Let k^* be the largest integer such that there exists at least one player choosing k^* with strictly positive probability. Show that $k^* = 1$.)

7. **[Bertrand equilibrium]**
Suppose n players, for some $n \geq 2$, represent firms that can each produce a common good

at a cost c per unit of good. Suppose the action of each player is to declare a price p_i per unit of good. Suppose there is an aggregate demand of consumers such that if the lowest price offered by any firm is p_{min} then the consumers purchase a total quantity $(a - p_{\min})_+$ of goods, where a is a constant with $a > c$, and they purchase an equal amount from each player offering the minimum price. The game is among the players offering prices; the consumers are not considered to be part of the game.

(a) Find the set of all Nash equilibrium profiles $(p_1, \ldots, p_n)$. The form of your answer may depend on the values of a and c

(b) Suppose the production costs vary by player, with the per unit production cost of player i given by some $c_i > 0$. For simplicity, suppose $c_1 < c_2 < \cdots < c_n$ and suppose $c_1 < a$. Find the set of all Nash equilibrium profiles $(p_1, \ldots, p_n)$.