Solution Set 8

57. Let us first prove the following Lemma, which was given as a hint:

Lemma. *Let $x \in C[0,1]$ have a unique maximum at $t_0 \in (0,1)$, which is also the unique maximum of $|x(t)|$, and let h be an arbitrary element of $C[0,1]$. Then,*

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \left\{ \|x + \alpha h\| - |x(t_0) + \alpha h(t_0)| \right\} \quad = \quad 0.$$

Toward this end assume, without any loss of generality, that $x(t_0) > 0$, and let $\epsilon > 0$ be a sufficiently small scalar (in particular, $\epsilon < t_0$, $t_0 + \epsilon < 1$). Then, for sufficiently small α,

$$\begin{aligned} \|x + \alpha h\| &= \max_{0 \le t \le 1} |x(t) + \alpha h(t)| \\ &= \max_{t_0 - \epsilon \le t \le t_0 + \epsilon} |x(t) + \alpha h(t)| \end{aligned}$$

which follows because of the continuity of x and h. Furthermore, since $x(t_0) > 0$, for sufficiently small ϵ and α,

$$x(t) + \alpha h(t) > 0 \qquad \forall t \in [t_0 - \epsilon, t_0 + \epsilon].$$

Case 1. $\alpha > 0$

Then,

$$\begin{aligned} \max_{t_0 - \epsilon \le t \le t_0 + \epsilon} |x(t) + \alpha h(t)| &= \max_{t \in \mathcal{N}_\epsilon(t_0)} [x(t) + \alpha h(t)] \\ &\le \max_{t \in \mathcal{N}_\epsilon(t_0)} x(t) + \alpha \max_{t \in \mathcal{N}_\epsilon(t_0)} h(t) \\ &= x(t_0) + \alpha \max_{t \in \mathcal{N}_\epsilon(t_0)} h(t). \end{aligned}$$

Similarly, replacing the maximum of h with minimum at the 2nd line:

$$\max_{t \in \mathcal{N}_\epsilon(t_0)} |x(t) + \alpha h(t)| \quad \ge \quad x(t_0) + \alpha \min_{t \in \mathcal{N}_\epsilon(t_0)} h(t).$$

Hence,

$$x(t_0) + \alpha \min_{t \in \mathcal{N}_\epsilon(t_0)} h(t) \quad \le \quad \max_{t \in \mathcal{N}_\epsilon(t_0)} [x(t) + \alpha h(t)| \le x(t_0) + \max_{t \in \mathcal{N}_\epsilon(t_0)} h(t) \cdot \alpha.$$

This now leads to

$$\begin{aligned} \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \frac{\|x + \alpha h\| - |x(t_0) + \alpha h(t_0)|}{\alpha} \quad &= \quad \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \frac{\|x + \alpha h\| - x(t_0) - \alpha h(t_0)}{\alpha} \\ \le \frac{\not{\alpha}}{\not{\alpha}} \left[\max_{t \in \mathcal{N}_\epsilon(t_0)} h(t) - h(t_0) \right] \quad &= \quad \max_{t \in \mathcal{N}_\epsilon(t_0)} h(t) - h(t_0). \end{aligned}$$

Likewise, the limit above can be bounded below by

$$\geq \min_{t\in\mathcal{N}_\epsilon(t_0)} h(t) - h(t_0).$$

But since $\epsilon > 0$ was arbitrary, we can let it go to zero, which makes both bounds go to zero, since h is continuous. This then verifies the desired result as $\alpha \to 0$ on the positive side.

Case 2. $\alpha < 0$.

Let $\beta = -\alpha$, $\bar{h} = -h$ in $[0,1]$. Then

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}} \frac{\|x+\alpha h\| - |x(t_0)+\alpha h(t_0)|}{\alpha} \quad = \quad -\lim_{\beta\to 0} \frac{\|x+\beta\bar{h}\| - |x(t_0)+\beta\bar{h}(t_0)|}{\beta}$$

$\to 0$ by the argument of Case 1.

Remark. If t_0 is a boundary point, say $t_0 = 0$, then replace $\mathcal{N}_\epsilon(t_0)$ by $\{t : 0 \leq t \leq \epsilon\}$; and observe that similar arguments as above apply to prove an extension of the Lemma to this more general case. ◇

Now, coming back to the original problem, where we are given the function

$$f(x) = \max_{0\leq t\leq 1} |x(t)| = \|x\|, \quad x \in C[0,1]\ ,$$

let us take $D \subset C[0,1]$ to be the class of continuous functions x with the property that $|x(t)|$ has a unique maximum on $[0,1]$. Given $x \in D$, let t_0 be the point where $|x(t)|$ achieves its unique maximum on $[0,1]$. We now claim that,

$$\delta f(x;h) = \operatorname{sgn}[x(t_0)]\, h(t_0)$$

To show this, simply note that

$$\lim_{\alpha\to 0} \frac{1}{\alpha}[\|x+\alpha h\| - |x(t_0)| - \alpha\delta f(x;h)] = \lim_{\alpha\to 0} \frac{1}{\alpha}[\|x+\alpha h\| - |x(t_0)+\alpha h(t_0)|] = 0$$

where the first equality follows because for sufficiently small α (since clearly $x(t_0) \neq 0$), the sign of $x(t_0) + \alpha h(t_0)$ is determined by $\operatorname{sgn}[x(t_0)]$, and the equality to *zero* follows from the Lemma just proved.

The $\delta f(x;h)$ given above is clearly <u>linear</u> in h, since $\operatorname{sgn}[x(t_0)]$ is a constant.

58. i) Let

$$\Gamma := \{s \in [0,1] : \quad x(s) = \max_{0\leq t\leq 1} x(t)\}$$

Then it follows from the analysis of Problem 48 above that if Γ is a singleton, say $\Gamma = \{t_0\}$, then

$$\delta f(x;h) = h(t_0)\,.$$

If Γ is not a singleton, then $\delta f(x;h)$ does not exist for all $h \in C[0,1]$, since it would be possible to have

$$\lim_{\alpha\to 0^+} \frac{1}{\alpha}[f(x+\alpha h) - f(x)] \neq \lim_{\alpha\to 0^-} \frac{1}{\alpha}[f(x+\alpha h) - f(x)]$$

ii)

$$f(x) \quad = \quad \int_0^1 |x(t)|dt.$$

Here, let D be the class of continuous functions which do not vanish at any point on $[0,1]$. If $x \in D$, then it is either positive or negative. Then, given any $h \in C[0,1]$, we can find $\alpha_0 > 0$ such that $\text{sgn}(x+\alpha h) = \text{sgn}(x)$, $\forall\ \alpha, \quad |\alpha| < \alpha_0$.

Hence,

$$\delta f(x;h) \quad = \quad \lim_{\substack{\alpha\to 0\\ |\alpha|<\alpha_0}} \frac{1}{\alpha}\int_0^1 |x+\alpha h - x|\,\text{sgn}(x)\text{sgn}(\alpha h(t))dt = \text{sgn}(x)\int_0^1 h(t)dt$$

$$\therefore \quad \boxed{\delta f(x;h) = \text{sgn}(x)\int_0^1 h(t)dt} \to \text{clearly linear in } h.$$

59. In each case, I show that a Frechet derivative exists, which immediately implies that a linear Gateaux derivative exists.

i) For $h, x \in C[0,1]$, $\quad \delta T(x;h) \equiv T'(x)h = 2xh$, because with $\|y\|_\infty := \max\limits_{0\le t\le 1}|y(t)|$,

$$\frac{\max\limits_{0\le t\le 1}|(x+h)^2(t) - x^2(t) - 2(xh)(t)|}{\|h\|_\infty} = \frac{\max\limits_{0\le t\le 1}|(x(t)+h(t))^2 - x(t)^2 - 2x(t)h(t)|}{\|h\|_\infty}$$

$$= \frac{\max\limits_{0\le t\le 1}|h(t)^2|}{\|h(t)\|_\infty} = \|h\|_\infty$$

which clearly goes to zero as $\quad \|h\|_\infty \to 0$. Hence,

$$T'(x) = 2x\ , \quad \in B(C[0,1], C[0,1])$$

ii) In this case, For $h, x \in C[0,1]$, $\quad \delta T(x;h) \equiv T'(x)h = 2x(\tfrac{1}{2})h(\tfrac{1}{2})$, because

$$\frac{|(x+h)^2(\frac{1}{2}) - x^2(\frac{1}{2}) - 2(xh)(\frac{1}{2})|}{\|h\|_\infty} = \frac{|(x(\frac{1}{2})+h(\frac{1}{2}))^2 - x(\frac{1}{2})^2 - 2x(\frac{1}{2})h(\frac{1}{2})|}{\|h\|_\infty} \le \frac{\|h^2\|_\infty}{\|h\|_\infty} = \|h\|$$

Hence,

$$T'(x)h = \int_0^1 h(t)\, d\nu(t)$$

where ν is a function of bounded variation: piecewise-constant and with a single jump at $t = \frac{1}{2}$ of magnitude $x(\frac{1}{2})$. Note that , for each $x \in C[0,1]$,

$$T'(x) \in B(C[0,1],\ \mathbf{R})$$

which is isomorphic to the space of functions on $[0,1]$ with bounded variation.

iii) We claim that, for $h, x \in L_2[0,1]$, $\quad \delta T(x;h) \equiv T'(x)h = 2\int_0^t x(s)h(s)\,ds$. To show this, let $\|\cdot\|$ denote the standard $L_2[0,1]$ norm, and simply note:

$$\frac{\|T(x+h) - T(x) - 2\int_0^t x(s)h(s)\,ds\|}{\|h\|} = \frac{\|\int_0^t h(s)^2\,ds\|}{\|h\|} \le \frac{\|h\|^2}{\|h\|} \to_{\|h\|\to 0} 0$$

where the inequality follows from the natural bound

$$\int_0^t h(s)^2\,ds \le \int_0^1 h(s)^2\,ds \equiv \|h\|^2\ .$$

Hence, $T'(x) \in B(L_2[0,1], L_2[0,1])$ is given by

$$T'(x)h = 2\int_0^t x(s)h(s)\,ds$$

iv) Since T is a functional on $L_2[0,1]$, $\quad T'(x) \in B(L_2[0,1],\ \mathbf{R}) \equiv L_2^*[0,1] = L_2[0,1]$, given by $T'(x) = 2x$. It is a simple matter to show that

$$\frac{\|T(x+h) - T(x) - 2(x,h)\|}{\|h\|} = \frac{\|h\|^2}{\|h\|} \to_{\|h\|\to 0} 0\ ,$$

where $(\cdot,\cdot)$ denotes the standard inner product on $L_2[0,1]$.

60.

$$J(x) \quad = \quad \int_0^1 f[t, x(t), \dot{x}(t), \ddot{x}(t)]dt$$

The constraint set is:

$$\Omega \quad = \quad \left\{x \in D^2[0,1] : x(0) = a_0, \dot{x}(0) = b_0, x(1) = a_1, \dot{x}(1) = b_1\right\}$$

and the class of admissible variations is:

$$H \quad = \quad \left\{h \in D^2[0,1] : h(0) = \dot{h}(0) = h(1) = \dot{h}(1) = 0\right\}$$

Condition. Let f be continuously differentiable w.r.t. to x, $\dot{x}$, $\ddot{x}$, and continuous in t.

Then, a first-order necessary condition for $x_0 \in \Omega$ to provide an extremum is

$$\delta J(x_0; h) = 0 \;\; \forall h \in H$$

$\Leftrightarrow$

$$\delta J(x_0; h) = \int_0^1 [f_x h(t) + f_{\dot{x}} \dot{h}(t) + f_{\ddot{x}} \ddot{h}(t)]_{x=x_0(t)}\, dt = 0 \quad \forall h \in H.$$

Let

$$F_1(t) = -\int_0^t f_{\dot{x}}(s, x_0(s), \dot{x}_0(s), \ddot{x}_0(s)) ds$$
$$F_2(t) = \int_0^t ds \int_0^s f_x(\sigma, x_0(\sigma), \dot{x}_0(\sigma), \ddot{x}_0(\sigma))\, d\sigma.$$

Then, by integrating the 1st and 2nd terms of $\delta J(x_0; h)$ by parts, and making use of the property that for $h \in H$ both h and $\dot{h}$ vanish at both end points, we obtain

$$\delta J(x_0; h) = \int_0^1 [F_2(t) + F_1(t) + f_{\ddot{x}}(t, x_0(t), \dot{x}_0(t), \ddot{x}_0(t))]\ddot{h}(t) dt \;\; \forall h \in H.$$

Let

$$\boxed{G(t) \triangleq F_2(t) + F_1(t) + f_{\ddot{x}}(t, x_0(t), \dot{x}_0(t), \ddot{x}_o(t))}$$

which is continuous in t. Then,

$$\int_0^1 G(t)\ddot{h}(t) dt = 0 \qquad \forall h \in H$$

$\Rightarrow G(t) = c_1 t + c_0$, where c_0, c_1 are constants. Hence, the first order necessary condition is

$$\boxed{\begin{array}{l} F_2(t) + F_1(t) + f_{\ddot{x}}(t, x_0(t), \dot{x}_0(t), \ddot{x}_0(t)) = c_1 t + c_0 \\ x_0(0) = a_0, \dot{x}_0(0) = b_0, x_0(1) = a_1, \dot{x}_0(1) = b_1. \end{array}}$$

which is an integral equation, to be solved under the given end-point restrictions.

If f is twice continuously differentiable in t, and $x_0 \in D^4[0, 1]$, then by differentiating G twice we obtain an alternate representation:

$$f_x(t, x_0(t), \dot{x}_0(t), \ddot{x}_0(t)) - \frac{d}{dt} f_{\dot{x}}(t, x_0(t), \dot{x}_0(t), \ddot{x}_0(t)) + \frac{d^2}{dt^2} f_{\ddot{x}}(t, x_0(t), \dot{x}_0(t), \ddot{x}_0(t)) = 0$$

which is the Euler-Lagrange equation. This is a differential equation to be solved subject to the given end-point conditions.

61.

$$J(x) = \int_0^{\ln 2} x(t)[1+\dot{x}(t)^2]^{\frac{1}{2}} dt$$

$$x(0)=1, \quad x(\ln 2) = \frac{5}{4},$$

The E-L equation is

$$[1+\dot{x}^2]^{\frac{1}{2}} = \frac{d}{dt}\left[\frac{x}{2}[1+\dot{x}^2]^{-\frac{1}{2}}2\dot{x}\right]$$

$\Leftrightarrow$

$$[1+\dot{x}^2]^{\frac{1}{2}} = \frac{\dot{x}^2 + x\ddot{x}}{[1+\dot{x}^2]^{\frac{1}{2}}} - \frac{\dot{x}^2 x\ddot{x}}{[1+\dot{x}^2]^{\frac{3}{2}}}$$

$\Leftrightarrow$

$$[1+\dot{x}^2]^2 = [1+\dot{x}^2][\dot{x}^2 + x\ddot{x}] - \dot{x}^2 x\ddot{x}$$

$\Leftrightarrow$

$$\boxed{\begin{array}{rcl} x\ddot{x} - \dot{x}^2 - 1 & = & 0 \\ x(0)=1, x(\ln 2) & = & \frac{5}{4} \end{array}} \Leftrightarrow \begin{array}{c}\text{admitting the solution} \\ \boxed{x(t) = \cosh t}.\end{array}$$

Note that

$$x_0(0) = \cosh(0) = 1; \qquad x_0(\ln 2) = \cosh(\ln 2) = \frac{2+\frac{1}{2}}{2} = \frac{5}{4}$$

$$x_0\ddot{x}_0 - \dot{x}_0^2 - 1 = \cosh t \cosh t - \sin h^2 t - 1 = 0.$$

Of course, by satisfying the E-L equation (i.e., the 1st order necessary conditions), the given solution is only a candidate extremal at this point, and may not constitute a minimizing solution to J, since we have not checked the 2nd order conditions.

62. Let x_0 be an extemal. Then, $x_0 + h \in \Omega$ iff

$$h \in \underset{\uparrow}{H} = \{h \in D^{(2)}[0,b] : \quad h(0) + \beta h(b) = 0\}$$

$\uparrow$ space of admisible variations.

The first Gateaux differential of J at $x = x_0$ is

$$\begin{aligned} \delta J(x_0; h) &= \int_0^1 [f_x h(t) + f_{\dot{x}}\dot{h}(t)]dt \\ &= \int_0^1 \left[f_x - \frac{d}{dt}f_{\dot{x}}\right] h(t)dt + f_{\dot{x}}(t, x_0(t), \dot{x}_0(t))h(t)\Big|_0 \end{aligned}$$

$\uparrow$ integration by parts

$$= \int_0^1 \left[f_x - \frac{d}{dt}f_{\dot{x}}\right] h(t)dt + [m(b) + \beta m(0)]h(b)$$

$\uparrow$ for $h \in H$

where

$$m(t) \quad \triangleq \quad f_{\dot{x}}(t, x_0(t), \dot{x}_0(t)).$$

Necessary condition:

$$\begin{aligned}
& \delta J(x_0; h) = 0 && \forall h \in H \\
\Rightarrow \quad & \delta J(x_0; h) = 0 && \forall h \in H, \quad h(b) = 0 \\
\Leftrightarrow \quad & \int_0^1 \left[f_x - \frac{d}{dt} f_{\dot{x}} \right] h(t) dt = 0 && \forall h \in H, \quad h(0) = h(b) = 0
\end{aligned}$$

$$\Rightarrow \boxed{f_x(t, x_0(t), \dot{x}_0(t)) = \frac{d}{dt} f_{\dot{x}}(t, x_0(t), \dot{x}_0(t))} \qquad (1)$$

Using this in $\delta J(x_0; h)$, we obtain

$$[m(b) + \beta m(0)] h(b) = 0 \qquad \forall h(b) \in \mathbf{R}$$

$$\Rightarrow \boxed{\begin{array}{rcl} m(b) + \beta m(0) & = & 0 \\ m(t) & \triangleq & f_{\dot{x}}(t, x_0(t), \dot{x}_0(t)). \end{array}} \qquad (2)$$

(1) and (2) will have to be solved together with

$$\boxed{x_0(0) + \beta x_0(b) = \alpha.}$$

63. Let $\dot{x}(t) = y(t)$, $\Rightarrow$ $x(t) = \int_o^t y(s)\, ds$, where we take y as the variable (of optimization), belonging to $L_2[0, b]$. The constraint equation is

$$H(y) = x(b) - c \equiv \int_0^b y(s)\, ds - c = 0 \,, \quad H : L_2[0, b] \to Z = \mathbf{R} \,.$$

The functional to be minimized is:

$$f(y) = \int_0^b \sqrt{1 + y(t)^2}\, dt \,, \quad f : L_2[0, b] \to \mathbf{R} \,.$$

The Fréchet derivative of H is (as a linear functional on $L_2[0, b]$): $H'(y)h = \int_0^b h(s)$, $\Rightarrow$ $H'(y) = 1$, where 1 is the unit function in $L_2[0, b]$ (that is, the function that is 1 almost everywhere). H' is clearly a **nonzero** linear functional on $L_2[0, b]$, and hence every $y \in L_2[0, b]$ that satisfies the constraint $H(y) = 0$ is a **regular** point. This implies that, given an extremal $y_o \in L_2[0, b]$, there exists a $z^* \in Z^* = \mathbf{R}$ such that

$$(f'(y_o) + z^* H'(y_o))h = 0 \,, \quad \forall h \in L_2[0, b], \; h(0) = h(b) = 0$$

Denoting this scalar z^* by the familiar Lagrange multiplier notation, λ, we have

$$\Rightarrow \qquad \int_0^b \frac{y_o(t)}{\sqrt{1 + y_o(t)^2}} h(t)\, dt + \lambda \int_0^b h(t)\, dt = 0 \,, \quad \forall h \in L_2[0, b], \; h(0) = h(b) = 0$$

$$\Rightarrow \qquad \frac{y_o(t)}{\sqrt{1+y_o(t)^2}} = -\lambda \quad \text{a constant.}$$

This implies that y_o has to be a constant $\quad \Rightarrow$

$$y_o(t) = c/b\,, \quad \lambda = -c/\sqrt{b^2+c^2}\,.$$

64.

i) The solution of $\dot{x} = -x + u(t)^2$, with $x(0) = 1$ is

$$x(t) \;=\; e^{-t} + \int_0^t e^{-(t-\tau)} u(\tau)^2 d\tau.$$

Our second side-constraint is $\quad x(1) = e^{-1}$

$$\Rightarrow \qquad e^{-1} \;=\; e^{-1} + e^{-1}\int_0^1 e^{\tau} u(\tau)^2 d\tau.$$

The only way this can hold is if $u(\tau) = 0 \;\; \forall \tau \in [0,1] \Rightarrow$ constraint set is a singleton $\Rightarrow$ regularity condition does not hold. Note that one can also show directly that the Fréchet derivative H' of the constraints $H(x,u) = \theta$, where

$$H(x,u) := \begin{pmatrix} x(t) - e^{-t} - \int_0^t e^{-(t-\tau)} u(\tau)^2\, d\tau \\ \int_0^1 e^{\tau} u(\tau)^2\, d\tau \end{pmatrix} \quad H : C[0,1] \times L_2[0,1] \to C[0,1] \times \mathbf{R}\,,$$

is not onto at $u = \theta$, but it is not necessary to take this route since the constraint set is a singleton (as mentioned above), and hence there is no direction along which one can move starting at $u = \theta$ and still stay in the constraint set.

ii) If instead of $u^2(t)$ we have $u(t)$ in the state equation: $\dot{x} = -x + u$, then the minimization problem is feasible because $\exists$ a rich class of functions $u(\tau)$, $\tau \in [0,1]$ which satisfy

$$\int_0^1 e^{\tau} u(\tau) d\tau \;=\; 0.$$

The constraints (and there are two of them) can be written as

$$H(x,u) = \begin{pmatrix} x(t) - e^{-t} - \int_0^t e^{-(t-\tau)} u(\tau)\, d\tau \\ \int_0^1 e^{\tau} u(\tau)\, d\tau \end{pmatrix} \quad H : C[0,1] \times L_2[0,1] \to C[0,1] \times \mathbf{R}$$

whose Fréchet derivative is:

$$H'(x,u)h = \begin{pmatrix} h_1(t) - \int_0^t e^{-(t-\tau)} h_2(\tau)\, d\tau \\ \int_0^1 e^{\tau} h_2(\tau)\, d\tau \end{pmatrix} \quad H' \in B(C[0,1] \times L_2[0,1] \to C[0,1] \times \mathbf{R})$$

Note that in this case h has two components $(h_1 \in C[0,1]\,, h_2 \in L_2[0,1])$. Clearly, H' is an **onto** map, since the second component is a nonzero linear functional, and the first component is linear in $h_1(t)$. Hence, every $x \in C[0,1]\,, u \in L_2[0,1]$ satisfying $H(x,u) = \theta$ is a **regular** point.

END !!