Solution Set 6

46. The problem is:

$$\min_{x\in X} \int_0^2 |x(t)|^3 \,\mathrm{d}t \qquad \text{such that} \qquad \int_0^2 t^2\, x(t)\,\mathrm{d}t = 2$$

where $X = L_3[0,2] \equiv Z^*$, $Z = L_{3/2}[0,2]$. Hence we have a minimum norm problem in a dual space (Z^*), subject to an equality constraint : $\min_{<t^2,z^*>=2} \|z^*\|$. By the theory developed in class (Correspondence 14),

$$\min_{<t^2,z^*>=2} \|z^*\| = 2 \max_{\|at^2\|_{3/2}\le 1} a = 2 \max_{|2^{4/3}a|\le 1} a = 2^{-\frac{1}{3}}\,.$$

The minimizing solution is obtained from the condition that z_o^* is aligned with at^2:

$$z_o^* = \alpha\,\mathrm{sgn}(t)\,((t^2)^{\frac{3}{2}})^{\frac{1}{3}} = \alpha t \qquad \text{where} \quad \alpha \int_0^2 t^3\,\mathrm{d}t = 2 \quad \Rightarrow \quad \alpha = \frac{1}{2}.$$

Hence, $x_o(t) = (1/2)t$. The solution is unique, because the alignment condition in Z^* is both necesary and sufficient.

47. Let $D_T = \{x \in X_T : \int_0^T x(t)dt = -3; \int_0^T tx(t)dt = 0\}$, where $T > 0$ is fixed and X_T is a real linear normed space to be delineated later.

Let $I = \{T > 0 : \exists\, x \in D_T, |x(t)| \le 1\; \forall\, 0 \le t \le T\}$.

Then, the problem is to find the infimum of I, to show that it exists, and to find the corresponding solution x. I now show that this problem is equivalent to the following 2-stage minimum norm problem.

Problem A: For each fixed $T > 0$, let $Z_T = L_1[0,T]$ and $X_T = Z_T^* = L_\infty[0,T]$, and consider the two problems given below:

1. $d_T = \min_{x\in D_T} \|x\| = \min_{z^*\in D_T} \|z^*\| = \|x_T^o\|$ (if it exists)
2. $\inf\{T > 0 : \|x_T^o\| \le 1\} = T_o$ (if it exists).

(a) **Proposition:** In Problem A, there exists a solution to both 1 and 2, and $(T_o, x_{T_o}^o)$ also constitutes a solution to the original problem.

Proof:

i) Solution to the first norm minimization problem will exist by Corollary 1 (p. 123) since X_T is dual to a normed linear space and the constraints are linear. I now claim that the distance d_T is a nonincreasing function of T. To show this, assume that, to the contrary, $\exists\, T_1$, T_2, $0 < T_1 < T_2$ for which $d_{T_1} < d_{T_2}$. Let the optimal solution corresponding to T_1 be $x_1(t)$, $0 \le t \le T_1$.

Let

$$\bar{x}_2(t) = \begin{pmatrix} x_1(t) & 0 \le t \le T_1 \\ 0 & T_1 < t \le T_2 \end{pmatrix}$$

which clearly belongs to $L_\infty[0, T_2]$ whenever $x_1 \in L_\infty[0, T_1]$. Furthermore, $\bar{x}_2 \in D_{T_2}$, i.e., it satisfies the constraints (because $x_1 \in D_{T_1}$). However, the strict inequality

$$\begin{aligned} \|\bar{x}_2\|_{[0,T_2]} &= d_{T_1} < d_{T_2} = \min_{x \in D_{T_2}} \|x\|_{[0,T_2]} \\ &\uparrow \text{ by construction} \end{aligned}$$

contradicts with the minimality of d_{T_2}.

ii) Existence of an infimizing T follows because, as we will show later when we compute d_T, d_T is continuous in T and the set $\{T > 0, d_T \le 1\}$ is closed and bounded. [We will in fact see that the infimizing T, T_o, is unique]. To prove that this T_o is indeed a solution to the original problem, assume that, to the contrary, $\exists\ \underline{T}$, $0 < \underline{T} < T_o$, such that $\exists\ \underline{x}(t), t \in [0, \underline{T}]$, $\underline{x} \in D_{\underline{T}}$, $\|\underline{x}\|_{[0,\underline{T}]} \le 1$. But this implies that

$$d_{\underline{T}} \le 1$$

and since d_T is continuous, this contradicts with the minimality of T_o.

(b) Now let us solve Problem A. By Corollary 1, p. 123 of the Text, for each $T > 0$, a solution exists in the dual space $L_\infty(0, T)$, and it is characterized by

$$d_T = \min_{x \in D_T} \|x\|_{[0,T]} = \max_{\substack{a_1, a_2 \\ \int_0^T |a_1 + a_2 t| dt \le 1}} (-3a_1).$$

Clearly, the solution exists, since the cost functional is continuous over a closed and bounded subset of $\mathbf{R}^2$. Furthermore, since the cost functional is linear, and the constraint set is convex, the solution has to be on the boundary. Let $-a_1 = \alpha$. Then, we seek to solve

$$\max_{\alpha, a_2} \alpha \qquad \text{s.t.} \int_0^T |-\alpha + a_2 t| dt = 1.$$

If $(\alpha^\circ, a_2^\circ)$ denotes a solution, clearly $\alpha^\circ > 0$. Furthermore, $-\alpha + a_2 t$ switches sign at most once (say at t_o) with the first sign being negative since $\alpha > 0$. Hence,

$$\begin{aligned} \int_0^T |-\alpha + a_2 t| dt &= \int_0^{t_0} (\alpha - a_2 t) dt + \int_{t_0}^T (a_2 t - \alpha) dt \\ &= 2\alpha t_0 - a_2 t_0^2 - \alpha T + \frac{a_2}{2} T^2 = 1. \end{aligned}$$

Furthermore, $t_o = \alpha / a_2$, and substituting this in the above equation we obtain:

$$f(\alpha, a_2) := \frac{\alpha^2}{a_2} - \alpha T + \frac{a_2}{2} T^2 - 1 = 0$$

as the constraint equation. Now, we have to maximize α, with α and a_2 satisfying $f(\alpha, a_2) = 0$. The unique solution to this optimization problem is

$$\alpha = \frac{1+\sqrt{2}}{T}, \quad a_2 = \frac{2+\sqrt{2}}{T^2}.$$

Since $d_T = 3\alpha = \dfrac{3+3\sqrt{2}}{T}$ (note that d_T is a nonincreasing–in fact decreasing–function of T), the condition that $d_T \leq 1$ leads to

$$\boxed{T_o = 3(1+\sqrt{2})}$$

as the smallest T that satisfies this constraint.

Hence, the optimal solution is (since $t_0 = \dfrac{\alpha}{a_2} = \dfrac{1+\sqrt{2}}{2+\sqrt{2}}(3+3\sqrt{2}) = 3 + \dfrac{3\sqrt{2}}{2}$)

$$\boxed{x_o(t) = \begin{cases} -1, & 0 \leq t \leq 3 + \dfrac{3\sqrt{2}}{2} \\ 1, & 3 + \dfrac{3\sqrt{2}}{2} < t \leq 3 + 3\sqrt{2} \end{cases}}$$

48. The constraints are

$$\int_0^1 (1-t)x(t)\,dt = 1 \qquad (1)$$

$$\int_0^1 (1-t)y(t)\,dt = \frac{3}{2}, \qquad (2)$$

and the optimization problem is:

$$\text{minimize} \int_0^1 \sqrt{x^2(t)+y^2(t)}\,dt \quad \text{subject to (1) and (2).}$$

Choose $Z = C_1^2[0,1]$: space of all continuous functions taking values in $\mathbf{R}^2$. Pick as a norm on this space:

$$\|z\|_\infty = \max_t \ |z(t)|_2 \ = \max_t \sqrt{z_1^2(t) + z_2^2(t)}$$

$\uparrow$ Euclidean norm for a vector in $\mathbf{R}^2$.

The dual space is (see Problem 7, p. 138).

$Z^* = NBV^2[0,1]$: space of all normalized functions with bounded variations, taking values in R^2.

Norm in this case is $\|\nu\|_* = \int_0^1 \sqrt{(d\nu_1)^2 + (d\nu_2)^2}$. By Corollary 1, p. 123, we have

$$\alpha = \min_{z^* \in M^\perp} \|z^*\|_* = \max_{\substack{a_1, a_1 \\ \|a_1 q_1 + a_2 q_2\| \leq 1}} a_1 c_1 + a_1 c_2 = a_1^\circ c_1 + a_2^\circ c_2$$

where

$$q_1 = \begin{pmatrix} 1-t \\ 0 \end{pmatrix}, \; q_2 = \begin{pmatrix} 0 \\ 1-t \end{pmatrix}; \; c_1 = 1, c_2 = \frac{3}{2}.$$

Furthermore, $\|a_1 q_1 + a_2 q_2\| = \max_t |a_1 q_1(t) + a_2 q_2(t)|_2 = \left(\sqrt{a_1^2 + a_2^2}\right) \max_t (1-t) = \sqrt{a_1^2 + a_2^2}$. Hence, the related finite-dimensional optimization problem is

$$\text{maximize} \quad \left(a_1 + \frac{3}{2} a_2\right) \quad \text{s.t.} \quad a_1^2 + a_2^2 \leq 1.$$

The solution is unique (we have a tangent line to a circle of radius 1) and is at the intersection of $a_1 = \frac{2}{3} a_2$ and $a_1^2 + a_2^2 = 1$.

$$\Rightarrow \boxed{a_1^\circ = \frac{2}{\sqrt{13}}, \; a_2^\circ = \frac{3}{\sqrt{13}}} \Rightarrow d = a_1^\circ + a_2^\circ \frac{3}{2} = \boxed{\frac{\sqrt{13}}{2} = d}$$

The desired solution can be obtained from the alignment condition:

$$\begin{aligned} z_0^*(z_0) &= z_0^*(a_1^\circ q_1 + a_2^\circ q_2) = \int_0^1 \left\{ \frac{2}{\sqrt{13}} (1-t) d\nu_1^\circ(t) + \frac{3}{\sqrt{13}} (1-t) d\nu_2^\circ(t) \right\} \\ &= \underbrace{\|a_1^\circ q_1 + a_2^\circ q_2\|}_{1} \underbrace{\|d\nu^\circ\|}_{\sqrt{13}/2} \Rightarrow \begin{array}{lcl} \nu_1^\circ(t) &=& \begin{cases} 0 & t = 0 \\ 1 & 0 < t \leq 1 \end{cases} \\ \nu_2^\circ(t) &=& \begin{cases} 0 & t = 0 \\ \frac{3}{2} & 0 < t \leq 1 \end{cases} \end{array} \end{aligned}$$

Hence

$$\boxed{\begin{aligned} x^\circ(t) &= \delta(t) \\ y^\circ(t) &= \frac{3}{2}\delta(t) \end{aligned}} - \text{ both are impulses at } t = 0.$$

49. **(i)** We are given that $(x_n - x^o, x) \to 0$ (and equivalently that $(x, x_n - x^o) \to 0$) for all $x \in X$, and that $\|x_n\| \to \|x^\circ\|$. Then,

$$\begin{aligned} \|x_n - x^o\|^2 &= \|x_n\|^2 + -(x_n, x^o) - (x^o, x_n - x^o) \\ &= \|x_n\|^2 - \|x^o\|^2 - (x_n - x^o, x^o) - (x^o, x_n - x^o) \to 0 \end{aligned}$$

(ii) Take $x^o = 0$, without any loss of generality. By weak convergence, for every $y \in X$, $(x_n, y) \to 0$ as $n \to \infty$, which means that given $y \in X$, we can find $N > 0$ such that $|(x_n, y)| < \delta$ for all $n > N$. Clearly, also, given $y_1, y_2, \ldots, y_m \in X$ and $\delta > 0$ there exists $N > 0$ such that $|(x_n, y_i)| < \delta$ for all $n > N$, $i = 1, \ldots, m$. Now, given the sequence $\{x_n\}$, choose a subsequence $\{x_{n_k}\}$ as follows:
Choose $\quad x_{n_1} = x_1$.

Choose x_{n_2} such that $|(x_{n_1}, x_{n_2})| < 1$ (such an x_{n_2} exists from the weak convergence property above).
Choose x_{n_3} such that $|(x_{n_1}, x_{n_3})| < \frac{1}{2}, |(x_{n_2}, x_{n_3})| < \frac{1}{2}$ (again use the weak convergence property above, with $y_1 = x_{n_1}$, $y_2 = x_{n_2}$, and $\delta = \frac{1}{2}$).
Iteratively pick $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$, and choose $x_{n_{k+1}}$ such that

$$|(x_{n_i}, x_{n_{k+1}})| < \frac{1}{k}, \quad i = 1, 2, \ldots, k$$

Also, since $(x_n, x) \to 0$ for all $x \in X$, $(x_n, x_n) = \|x_n\|^2$ can be uniformly bounded, say by M^2. Then,

$$\begin{aligned}
\| y_m \|^2 &= \| \frac{1}{m} \sum_{k=1}^{m} x_{n_k} \|^2 \\
&\leq \left(\frac{1}{m}\right)^2 \left(mM^2 + 2 \sum_{i=2}^{m} \sum_{j=1}^{i-1} \left|(x_{n_j}, x_{n_i})\right| \right) \\
&\leq \left(\frac{1}{m}\right)^2 \left(mM^2 + 2(m-1) \right) \to 0 \text{ as } m \to \infty
\end{aligned}$$

Hence, y_m converges strongly to 0.

50. **(i)** We want to show that a linear functional f on a normed space X can be expressed in the form $f(x) =< x, x^* >$, with $x^* \in X^*$, if and only if it is weakly continuous.

First let $f(x) =< x, x^* >$, and in the definition of weak continuity given $\epsilon > 0$ choose $\delta = \epsilon$ and $x_1^* = x^*$. Then,

$$| < x, x_1^* > | < \delta \quad \Rightarrow \quad |f(x)| < \epsilon$$

and hence f is weakly continuous at $x = \theta$ and thereby everywhere (since f is linear). *Note that weak continuity is a stronger notion of continuity than regular continuity, in the sense that weak continuity implies continuity in norm, but not vice versa.*

We now prove the converse. We are given that f is weakly continuous, say at $x_o = \theta$ without any loss of generality. This implies that given $\epsilon > 0$, there exits $\delta > 0$ and a finite collection $\{x_1^*, x_2^*, \ldots, x_n^*\}$ from X^* such that $|f(x)| < \epsilon$ for all x satisfying $< x, x_i^* > < \delta$, $i = 1, \ldots, n$. In particular, $|f(x)| < \epsilon$ for all x such that $< x, x_i^* >= 0$ for all i. We now claim that for all such x it is necessary that $f(x) = 0$. Assume the contrary. Then, there exists $r > 0$ such that $f(x) = r$ for some x with the property $< x, x_i^* >= 0$ for all i, say x^o. Let $y := \frac{1+\epsilon}{r} x^o$, which clearly satisfies $< y, x_i^* >= 0$ for all i. But, using the linearity property of f, $f(y) = 1 + \epsilon > \epsilon$, thus contradicting the initial hypothesis that $|f(x)| < \epsilon$ for all such x. Therefore, we have necessarily: $f(x) = 0$ for all x such that $< x, x_i^* >= 0$ for all i.
Now, using the result of Problem 44, there exist scalars $\lambda_1, \ldots, \lambda_n$, such that

$$f = \sum_{i=1}^{n} \lambda_i x_i^* \quad \Rightarrow \quad f(x) =< x, x^* > \;\text{ with }\; x^* := \sum_{i=1}^{n} \lambda_i x_i^*$$

(ii) Here we want to show that a linear functional g on the dual space X^* can be expressed in the form $g(x^*) =< x, x^* >$, with $x \in X$, if and only if it is weak* continuous. Again, it will be sufficient to consider weak* continuity only at the origin, $x^* = \theta$; that is given $\epsilon > 0$, there exist a finite collection $\{x_1, x_2, \ldots, x_n\}$ and a $\delta > 0$, such that $g(x^*) < \epsilon$ for all x^* satisfying $< x_i, x^* >< \delta,\ i = 1, \ldots, n$. This leads to, as in part (i), $|g(x^*)| < \epsilon$ for all x^* such that $< x_i, x^* >= 0$ for all i. Picking $\delta = \epsilon$ and $x^* = x_1^*$ immediately leads to weak* continuity of $g(x^*)$.

Proof of the converse is also very similar to that in part (i) above. Similar reasoning leads to: $g(x^*) = 0$ for all x^* such that $< x_i, x^* >= 0$ for all i. Again using the result of Problem 44, this time to linear functionals defined on X^*, leads to

$$g \sum_{i=1}^{n} \lambda_i x_i \quad \Rightarrow \quad g(x^*) =< x, x^* > \quad \text{with} \quad x := \sum_{i=1}^{n} \lambda_i x_i$$

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