Solution Set 3

23. **(a)** $F(x,y) = -2x^2 + y^2 + 3xy - x - 2y$ is **jointly continuous** and twice continuously differentiable in (x,y). Hence, it is strictly concave-convex in (x,y) iff $\partial^2 F/\partial x^2 < 0$ and $\partial^2 F/\partial y^2 > 0$. Since,

$$\frac{\partial^2 F}{\partial x^2} = -4 < 0; \qquad \frac{\partial^2 F}{\partial y^2} = 2 > 0,$$

strict concavity-convexity readily follows. Then, in view of the *saddle-point theorem* proved in class, the game admits a unique saddle point solution, since the interval $[0,1]$ is **compact**.

(b) The solution exists and is unique, as justified above.

Let φ and ψ be defined by [both map $[0,1]$ onto itself]

$$\begin{aligned} \max_x F(x,y) &= F(\varphi(y), y) \qquad \forall y \in [0,1] \\ \min_y F(x,y) &= F(x, \psi(x)) \qquad \forall x \in [0,1]. \end{aligned}$$

It readily follows that

$$\varphi(y) = \begin{cases} (3y-1)/4 & \text{if} \quad y \geq \frac{1}{3} \\ 0 & \text{if} \quad 0 \leq y \leq \frac{1}{3} \end{cases}$$

$$\psi(x) = \begin{cases} (2-3x)/2 & \text{if} \quad 0 \leq x \leq \frac{2}{3} \\ 0 & \text{if} \quad x > \frac{2}{3} \end{cases} .$$

Then, solving for x and y from $\left.\begin{aligned} x &= \varphi(y) \\ y &= \psi(x) \end{aligned}\right\}$, we obtain the **unique** solution

$$\boxed{x^* = \frac{4}{17} \cong 0.235; \qquad y^* = \frac{11}{17} \cong 0.647.}$$

The corresponding value of F, which is the *saddle-point value*, is $F(x^*, y^*) \cong -0.7647$.

(c) The sequence generated is

$$\left.\begin{aligned} x_{k+1} &= \varphi(y_k) \\ y_{k+1} &= \psi(x_{k+1}) \end{aligned}\right\}$$

$\Rightarrow$

$$y_{k+1} = \psi(\varphi(y_k)) = \begin{cases} \frac{11}{8} - \frac{9}{8} y_k & y_k \geq \frac{1}{3} \\ 1 & y_k < \frac{1}{3} \end{cases}$$

Since the absolute value of the coefficient of y_k above is larger than 1, the sequence **does not converge**, regardless of what initial condition we choose for y_0 (other than the equilibrium value: 11/17).

24. **(a)** The proof here parallels that of the saddle-point equilibrium, given in class. As proven in class, the conditions given on X, K_X, Y, K_Y, and F_1 lead to the existence of an upper semi-continuous, closed, convex map $g_1 : K_Y \to 2^{K_X}$ such that any $x \in g_1(y)$ minimizes $F_1(x, y)$ over $x \in K_X$, for each fixed $y \in K_Y$. Similarly, there exists an upper semi-continuous map $g_2 : K_X \to 2^{K_Y}$, which is also closed and convex, such that any $y \in g_2(x)$ minimizes $F_2(x, y)$ over $y \in K_Y$, for each fixed $x \in K_X$. Every Nash equilibrium is a common fixed point of these two point-to-set maps, and every such fixed point can be obtained from

$$x \in g_1(g_2(x)); \quad y \in g_2(x)$$

As shown in class, the composite map $(g_1 \circ g_2)(\cdot) := g_1(g_2(\cdot))$ is a multifunction from K_X into 2^{K_X}, which is also upper semi-continuous, closed and convex. Furthermore, since K_X is a compact subset of the metric space X, $(g_1 \circ g_2)$ has a fixed point, say x^*, by Fan's Fixed-Point Theorem (*Correspondence #3, Theorem 9*). Then, it readily follows that $(x^*, y^* \in g_2(x^*))$ is a Nash equilibrium point.

(b) All conditions above hold here; in particular F_1 is strictly convex in x and F_2 is convex in y; furthermore, K_X and K_Y are convex and compact. Hence, we know from the result above that there exists a Nash equilibrium. To compute it (and to determine whether it is unique or not), note that

$$g_1(y) = \frac{1}{y+1}, \quad g_2(x) = \begin{cases} 0 & \text{if } x > 1/2 \\ [0,1] & \text{if } x = 1/2 \\ 1 & \text{if } x < 1/2 \end{cases} \quad \Rightarrow \quad (g_1 \circ g_2)(x) = \begin{cases} 1 & \text{if } x > 1/2 \\ [1/2, 1] & \text{if } x = 1/2 \\ 1/2 & \text{if } x < 1/2 \end{cases}$$

Hence, g_1 is single-valued, but g_2 as well as $g_1 \circ g_2$ are multi-valued. If follows by inspection that $g_1 \circ g_2$ has two fixed points: $x = 1/2$ and $x = 1$. These correspond to $y = 1$, and $y = 0$, respectively. (Note that even though $g_2(1/2) = [0, 1]$, only the point $y = 1$ in the interval $[0, 1]$ leads to $g_1(1) = 1/2$.) Hence, the game admits two Nash equilibria:

$$x^* = \frac{1}{2}, y^* = 1 \quad \text{and} \quad x^\circ = 1, y^\circ = 0$$

25. Let us check the four axioms:

(i) $\overline{(B, A)} = \overline{Tr[B^T Q \bar{A}]} = Tr[\bar{B}^T \bar{Q} A] = Tr[A^T \bar{Q}^T \bar{B}] = (A, B)$, where the third equality holds because $Tr[C] = Tr[C^T]$, and the last one holds because Q is Hermitian.

(ii) $(A + B, C) = Tr[A^T Q \overline{C}] + Tr[B^T Q \overline{C}] = (A, C) + (B, C)$ – holds for all A, B and C.

(iii) $(\lambda A, B) = \lambda Tr[A^T Q \bar{B}] = \lambda (A, B)$ – holds for all A and B.

(iv) $(A, A) = Tr[A^T Q \bar{A}] \to$ this is positive for all $A \neq 0$ because the matrix Q is Hermitian and positive definite, i.e., all its eigenvalues are positive.

Hence, $(A, B) = Tr[A^T Q B]$ is indeed an inner product.

26. **(a)** It is <u>not</u> an inner product on X, because

$$(x, x) \quad = \quad \left| \int_1^4 s^2 x(s) ds \right|^2$$

can be made zero without x being the zero function.

(b) **Yes**, it is an inner product on X, as all four axioms of an inner product are satisfied:

(i) $(x, y) = (y, x)$ for all $x, y \in X$.

(ii) $(x + y, z) = (x, z) + (x, z)$ for all $x, y, z \in X$.

(iii) $(\lambda x, y) = \lambda(x, y)$ for all real numbers λ and all $x, y \in X$.

(iv) $(x, x) = \int_1^4 t^3 x^2(t)\, dt \geq 0$ and is equal to zero iff $x(t) = 0$ for all $t \in [1, 4]$, which is the zero element in X.

27. **(a)** Here the Hilbert space is $H = L_2[-1, 2]$, and $M = \{m \in H : \; m(t) = a + bt, \; a, b \in \mathbf{R}\}$. Note that M is a 2-dimensional subspace of H, which is also closed (it is in fact isomorphic to $\mathbf{R}^2$). Hence the problem can be viewed as the optimization problem of minimizing $\|x - m\|$ over $m \in M$, where $x(t) = t^3$ is an element of H. The Projection Theorem directly applies here, leading to the conclusion that there exists a unique $m_o \in M$ that solves this minimization problem, and that $x - m_o \perp M$.

(b) We have to solve for a and b from the two relationships:

$$x - m_o \perp 1 \;\Rightarrow (t^3 - m_o, 1) = 0 \qquad \text{and} \qquad x - m_o \perp t \;\Rightarrow (t^3 - m_o, t) = 0$$

Using $m_o(t) = a + bt$, we have $\quad \frac{5}{4} = a + \frac{1}{2}b\,, \;\; \frac{21}{5} = \frac{1}{2}a + b \quad \Rightarrow \quad a = 0.2, \; b = 2.1$

(c) The minimum value of F is

$$\|x - m_o\|^2 = (x - m_o, x - m_o) = (x - m_o, x) = \|x\|^2 - (m_o, x) = \|x\|^2 - \|m_o\|^2$$

where $\quad \|x\|^2 = \frac{129}{7} \quad$ and $\quad \|m_o\|^2 = 14.61 \quad \Longrightarrow \quad \min_{m \in M} F(m) = F(m_o) = 3.8187$

28. Here the space $L_2[-1, 2]$ is replaced by the same with only the inner product different:

$$(x, y)_2 = \int_{-1}^{2} t^2 x(t) y(t)\, dt$$

This is also a Hilbert space (say, H), and M as defined above is a closed subspace of H. Projection Theorem again applies, leading to existence of a unique solution. We again have to solve for a and b from the two relationships:

$$x - m_o \perp 1 \;\Rightarrow (t^3 - m_o, 1)_2 = 0 \qquad \text{and} \qquad x - m_o \perp t \;\Rightarrow (t^3 - m_o, t)_2 = 0$$

Substituting for $m_o(t) = a + bt$, and solving for a and b: $\quad a = 0.03361\,, \; b = 2.773.$

We have $\quad \|x\|^2 = \|t^3\|^2 = 56.778 \quad$ and $\quad \|m_o\|^2 = 51.547$

and hence the minimum value of F is: $\quad F(m_o) = \|x\|^2 - \|m_o\|^2 = 5.543.$

29. Let $y(t) = x(t)\sqrt{t}$. Then, the problem is equivalent to one of minimizing $\|y\|^2$ over $H = L_2[1, 2]$, subject to: $(y, t^{-\frac{1}{6}}) = 1$ and $(y, t^{\frac{1}{6}}) = -1$, where the inner product $(\cdot, \cdot)$ is the standard one on H.

Let Z be the closed subspace of H generated by $\{t^{-\frac{1}{6}}, t^{\frac{1}{6}}\}$. Then, $H = Z \oplus Z^{\perp}$, and hence any $y \in H$ admits the unique additive decomposition $y = z + m$, where $z \in Z$ and $m \in M := Z^{\perp}$. The constraints can then be written as $(z, t^{-\frac{1}{6}}) = 1$ and $(z, t^{\frac{1}{6}}) = -1.$, and the functional to be minized is $\|z\|^2 + \|m\|^2$ (since $m \perp z$). But, since Z is two-dimensional (because $t^{-\frac{1}{6}}$ and $t^{\frac{1}{6}}$ are linearly independent), and we have two linearly independent constraints, the choice of an element out of Z that satisfies both constraints is unique. Letting $y(t) = at^{-\frac{1}{6}} + bt^{\frac{1}{6}}$, and solving for a and b, we have $a = 486.3295$, $b = -427.5050$. Hence, the unique solution to the optimization problem is (in view of the relationship $x(t) = y(t)/\sqrt{t}$:

$$x^o(t) = 486.3295\, t^{-\frac{2}{3}} - 427.5050\, t^{-\frac{1}{3}}$$

30. Choose the Hilbert space $H = \mathbf{R}^n$, with the inner product $(x, y) = x^T Q y$, where $Q > 0$ (positive definite).

Let $A^T = [a_1, a_2, \ldots, a_m]$, $b^T = [b_1, b_2, \ldots, b_m]$, where a_i's are vectors and b_j's are scalars. Further assume that $a_1, \ldots, a_m$ are linearly independent in $\mathbf{R}^n$, which is equivalent to saying that $\underline{\text{rank } (A) = m}$.

Then, the constraint equation $Ax = b$ can be written as

$$\begin{aligned} a_i^T x = b_i, &\qquad i = 1, \ldots, m \\ \Leftrightarrow (x, \tilde{a}_i) = b_i, &\qquad i = 1, \ldots, m \end{aligned}$$

where $\tilde{a}_i = Q^{-1} a_i$.

Note that since $\det Q^{-1} \neq 0$, $\{\tilde{a}_i\}_{i=1}^m$ is also a linearly independent set.

Hence, our optimization problem is

$$\text{minimize } \|x\| \quad \text{subject to} \quad (x, \tilde{a}_i) = b_i, i = 1, \ldots, m.$$

Applying Thm 2 (p. 65, Luenberger), which was discussed in class, we know that the solution is $\underline{\text{unique}}$, and is given by

$$x_0 \;=\; \sum_{i=1}^{m} \lambda_i \tilde{a}_i = Q^{-1} A^T \lambda$$

where the $\tilde{a}_i' s$ satisfy

$$\underbrace{\begin{bmatrix} (\tilde{a}_1, \tilde{a}_1) & (\tilde{a}_2, \tilde{a}_1) & \cdots & (\tilde{a}_m, \tilde{a}_1) \\ \vdots & & & \\ \vdots & & & \\ (\tilde{a}_1, \tilde{a}_m) & \cdots & \cdots & (\tilde{a}_m, \tilde{a}_m) \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \vdots \\ \lambda_m \end{bmatrix}}_{\lambda} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}}_{b}.$$

The matrix G can easily be shown to be $G = AQ^{-1}A^T$, which is nonsingular because of the linear independence of columns of A^T. Then, λ is uniquely solved to yield

$$\lambda = G^{-1} b \;=\; (AQ^{-1}A^T)^{-1} b$$

and substituting this into the expression for x_0, we obtain:

$$\boxed{x_0 = Q^{-1}A^T(AQ^{-1}A^T)^{-1}b} \Rightarrow \text{ unique solution.}$$

For those of you who have taken ECE 490, or are familiar with the contents of any nonlinear programming course, the parameter vector λ obtained above is the Lagrange multiplier, and the problem solved is the "quadratic programming problem with linear equality constraints".

If A is not a full rank matrix, but the linear constraints are still consistent (compatible), then one reduces the number of equations by eliminating the redundant ones and arrive at a new matrix A which is now of full rank. Then, this new A would be used in the solution given above.

31. Let Q be the space of polynomials on $[a, b]$, of degree n or less, and P be a subset of Q, consisting of elements, p, of Q, which further satisfy the constraint

$$(p, 1) = \int_a^b p(t)\, dt = 0\,,$$

that is, are orthogonal to the constant 1. Clearly, both Q and P are closed subspaces of $L_2[a, b]$, and P is further a subspace of Q. By the Projection Theorem, both minimization problems

$$\min_{p \in P} \|x - p\| \qquad \text{and} \qquad \min_{q \in Q} \|x - q\|$$

admit unique solutions, say p^o and q^o, respectively. Furthermore, $x - p^o \perp P$ and $x - q^o \perp Q$, and p^o and q^o are the unique elements of P and Q, respectively, that satisfy these orthogonality relationships. By the same reasoning, the minimization problem $\min\limits_{p \in P} \|q^o - p\|$ also admits a unique solution; let us denote it by $\hat{p}$, and note that $q^o - \hat{p} \perp P$, with this relationship again satisfied uniquely by $\hat{p}$. Now, for any $p \in P$:

$$(x - \hat{p}, p) = (x - q^o + q^o - \hat{p}, p) = (x - q^o, p) + (q^o - \hat{p}, p) = 0$$

where the last result follows because $q^o - \hat{p} \perp P$, and $x - q^o \perp Q \;\Rightarrow\; x - q^o \perp P \subset Q$. But since p^o was the unique vector with the property $x - p^o \perp P$, it follows that $\hat{p} = p^o$.

32. *i)* Let $Y = \overline{[y_1, \ldots, y_n]}$. Write $x \in X$ as $x = x_y + x^\perp$ where $x_y \in Y$, $x^\perp \in Y^\perp$. Then, because of orthogonality, $\|x\|^2 = \|x_y + x^\perp\|^2 = \|x_y\|^2 + \|x^\perp\|^2$. Let $K = \{k \in Y : (k, y_i) \geq c_i\}$. Then, since $(x_y + x^\perp, y_i) = (x_y, y_i)$,

$$\inf_{\substack{x \in X \\ \ni (x, y_i) \geq c_i \\ i=1,\ldots,n}} \|x\|^2 = \inf_{k \in K} \|k\|^2$$

K is closed (because of $\geq$) and convex (because (k, y_i) is a linear functional). Hence, what we have is a problem of *minimum distance to a convex set in a Hilbert space.* Thm 1 (p. 69 of *Luenberger*) applies with $x = \theta$, to ensure that $\exists$ a unique $k_0 \in K$ satisfying $\inf\limits_{k \in K} \|k\| = \|k_0\|$ and the solution is characterized by

$$(k_0, k_0 - k) \leq 0 \qquad \forall k \in K \tag{1}$$

where k_0 can be written as

$$k_0 = \sum_{i=1}^{n} a_i \, y_i \text{ for some } a_1, \ldots, a_n. \tag{2}$$

ii) First note that, for $k_0 = \sum_{j=1}^{n} a_j y_j \in K$, we have $(k_0, y_i) \geq c_i \Rightarrow \sum_{j=1}^{n} a_j (y_j, y_i) \geq c_i \Rightarrow$

$$\boxed{G^T a \geq c}$$

where G is the Gram matrix, $c := (c_1, c_2, \ldots, c_n)^T$ and $a := (a_1, \ldots, a_n)^T$. Now we have to show that $a \geq \theta$. Toward this end, first note that for any fixed j, we can find a vector z_j such that

$$(z_j, y_j) > 0 \quad \text{and} \quad z_j \perp \overline{[y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n]}.$$

Choosing $k = k_0 + z_j$, clearly $(k, y_i) = (k_0, y_i) + \underbrace{(z_j, y_i)}_{\geq 0} \geq \underbrace{(k_0, y_i)}_{\geq c_i} \Rightarrow k \in K$.

Using this k in (1), we obtain $-(k_0, z_j) \leq 0 \Rightarrow \sum_{i=1}^{n} a_i \, (y_i, z_j) \geq 0 \Rightarrow a_j \, (y_j, z_j) \geq 0$. Since $(y_j, z_j) > 0$ by construction, this says that a_j <u>cannot</u> be negative, and since j was arbitrary,

$$\boxed{a \geq \theta}$$

Now we prove the last part, i.e., $(k_0, y_i) > c_i \Rightarrow a_i = 0$. Since $\{y_j\}_{j=1}^{n}$ is a linearly independent set,

$$\exists z_i \text{ such that } z_i \perp \overline{[y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n]} \text{ and } (z_i, y_i) = -\epsilon < 0.$$

Let $k = k_0 + z_i$, which belongs to K if ϵ is sufficiently small, since $(k_0, y_i) > c_i$.

Now, using this k in (1), we obtain

$$(k_0, -z_i) \leq 0 \quad \Rightarrow \quad \sum_{j=1}^{n} a_j (y_j, z_i) \geq 0 \quad \Leftrightarrow \quad a_i \underbrace{(y_i, z_i)}_{-\epsilon} \geq 0\,.$$

Since $a_i \geq 0$, this is possible only if $a_i = 0$. Hence,

$$\boxed{a_i = 0 \text{ if } (k_0, y_i) > c_i}$$

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