Notes On

HILBERT SPACES

SEPARABILITY AND EXISTENCE OF BASIS

As a topic that will be covered in future lectures following the basic material on Hilbert spaces, I discuss in these notes the separability of Hilbert spaces, and in particular prove the important result that separability of a Hilbert space implies and is implied by the existence of a complete countable orthonormal sequence (that is, a basis). This result is given in Theorems 1 and 2 below, which are followed by some discussion on the separability of $L_2(-\infty, \infty)$.

Let us first recall the notions of *denseness* and *separability*, which were introduced in class while discussing normed linear spaces.

Definition 1. *Given a normed linear space X, a subset $D \subset X$ is dense in X if for each $x \in X$ and each $\epsilon > 0$, there exists $d \in D$ such that $\|x - d\| < \epsilon$. X is separable, if it contains a countable dense set.* $\diamond$

We now state and prove the two main theorems.

Theorem 1. *Let H be a* **separable** *Hilbert space. Then, every orthonormal system of vectors in H consists of a finite or a countable number of elements.*

Proof : Let $\{x_1, x_2, \ldots\}$ be a sequence of vectors dense in H, and let M be an orthonormal family in H. We need to show that M is countable.

Let e_1 and e_2 be two distinct vectors in M. Choose x_{k_1} and x_{k_2} such that

$$\|e_1 - x_{k_1}\| < \frac{1}{2}\sqrt{2} \qquad \text{and} \qquad \|e_2 - x_{k_2}\| < \frac{1}{2}\sqrt{2}\,.$$

By orthonormality,

$$\|e_1 - e_2\|^2 = \|e_1\|^2 + \|e_2\|^2 = 2\,,$$

and hence

$$\sqrt{2} = \|e_1 - e_2\| \le \|e_1 - x_{k_1}\| + \|e_2 - x_{k_1}\| < \frac{1}{2}\sqrt{2} + \|e_2 - x_{k_1}\| \quad \Rightarrow \quad \|e_2 - x_{k_1}\| > \frac{1}{2}\sqrt{2}$$

Hence $x_{k_1} \neq x_{k_2}$ and $k_1 \neq k_2$. Thus, we can associate with each element of M a different integer k, which shows that M is enumerable (that is, countable). $\diamond$

The next result says that separability is actually equivalent to existence of a complete orthonormal sequence.

Theorem 2. *A Hilbert space H contains a complete orthonormal sequence if and only if it is separable.*

Proof : Let us first assume that H is separable, and let D be a countable set of vectors dense in H (which exists by separability). This set could contain linearly dependent vectors; hence first eliminate in some arbitrary but fixed order linearly dependent ones, and then orthonormalize the remaining countable set (using Gram-Schmidt procedure). Denote the resulting set of orthonormal vectors by M. We first show that M is complete, that is the only vector in H that is orthogonal to M is the null vector. Toward this end, let $h \in H$ be orthogonal to M. Then, by the construction of M, h is also orthogonal to D. By denseness, for each $\epsilon > 0$ there exists an $f \in D$ such that $\|h - f\| < \epsilon$, which implies (by orthogonality) that

$$\|h\|^2 = (h, h) = (h - f, h) \le \|h - f\| \cdot \|h\| < \epsilon\|h\| \quad \Rightarrow \quad \|h\| < \epsilon$$

But ϵ was arbitrary. Therefore $h = \theta$, and hence M is complete.

We next prove the reverse implication. The main idea to be used in this proof is similar to that used in the proof of separability of ℓ_p, $1 \le p < \infty$, in Example 2, page 43 of the text. Now, let $\{e_1, e_2, \ldots\}$ be a complete orthonormal sequence in H. Let D be the set of all linear combinations of the form

$$\alpha_1^{(n)} e_1 + \alpha_2^{(n)} e_2 + \ldots + \alpha_n^{(n)} e_n \,, \quad n = 1, 2, 3, \ldots \,,$$

where $\alpha_i^{(n)}$'s are rational numbers. The set D is countable. Since the sequence $\{e_1, e_2, \ldots\}$ is complete, the closure of $[e_1, e_2, \ldots]$ is H, which implies that given any $h \in H$ and any $\epsilon > 0$, there exists an integer N such that

$$\|h - \sum_{i=1}^{N} (h, e_i)\, e_i\| < \frac{\epsilon}{2} \,.$$

Now, the Fourier coefficients (h, e_i) need not be rational, but they can be approximated by rational numbers to any degree, that is given $\epsilon > 0$ (the same ϵ as above), we can find rational numbers $\alpha_i^{(N)}$'s such that

$$\left\| \sum_{i=1}^{N} \left((h, e_i) - \alpha_i^{(N)} \right) e_i \right\| < \frac{\epsilon}{2} \,.$$

Thus there exists $d \in D$, given by

$$d = \sum_{i=1}^{N} \alpha_i^{(N)} e_i \,,$$

for which

$$\|h - d\| < \epsilon \,.$$

Since ϵ was arbitrary, this implies that the countable set D is dense in H, and hence H is separable. $\diamond$

Remark 1. The proof given above has assumed that the Hilbert space H is defined over the reals, that is its elements are real-valued functions. This was explicitly used at the point where the Fourier coefficients (h, e_i) were approximated by rational numbers. Now, if (h, e_i) is complex valued, one can separately approximate its real and imaginary parts again by rational numbers, and hence the proof goes through almost verbatim in this more general case also. Hence, the bottom line is that the result of Theorem 2 is valid for real-valued as well as complex-valued Hilbert spaces. $\diamond$

SEPARABILITY (OR INSEPARABILITY) OF $L_2(-\infty, \infty)$

When discussing separability of Banach spaces, I had mentioned in class that the space $L_p(a, b)$ with $1 \le p < \infty$, and a and b finite, and under the standard p-norm

$$\|x\|_p = \left(\int_a^b |x(t)|^p \, dt \right)^{\frac{1}{p}}$$

is separable, whereas $L_\infty(a, b)$ is not. On the other hand, if we have the doubly infinite interval, $(-\infty, \infty)$, and take the norm (in the limit as $a \to -\infty$ and $b \to \infty$) as

$$\|x\|_p = \left(\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |x(t)|^p \, dt \right)^{\frac{1}{p}} \,,$$

then the resulting space, denoted by $L_p(-\infty, \infty)$, is not separable. I will now prove this result for $p = 2$. Consider a continuum of elements $\{\sin \alpha t\}$, where α is an arbitrary real number. For $\alpha \ne \beta$, let us compute the distance between $\sin \alpha t$ and $\sin \beta t$:

$$\| \sin \alpha t - \sin \beta t\|^2 = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \left[\sin \alpha t - \sin \beta t \right]^2 dt = 2$$

Consequently, there exists an uncountable set of elements such that the distance between any two of them is $\sqrt{2}$. This readily implies that there cannot exist a countable everywhere dense set in this space, as the following argument shows. Suppose there is such a set, with x_o an arbitrarily picked element. Consider an ϵ-neighborhood of x_o, with say $\epsilon = \frac{1}{2}$. Since, by hypothesis, there is only a countable number of such ϵ-neighborhoods, there is at least one such ϵ-neighborhood which contains two distinct elements, $\sin \alpha t$ and $\sin \beta t$, with $\alpha \neq \beta$. Then,

$$\sqrt{2} = \| \sin \alpha t - \sin \beta t \| \leq \| \sin \alpha t - x_o \| + \| x_o - \sin \beta t \| \leq \frac{1}{2} + \frac{1}{2} = 1$$

which, however, is impossible – leading to a contradiction.

This result says in particular that $L_2(-\infty, \infty)$ is not separable if we use as the norm

$$\|x\|_2 = \left(\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |x(t)|^2 \, dt \right)^{\frac{1}{2}} . \tag{1}$$

This, however, is not the only possible norm for $L_2(-\infty, \infty)$, as yet another one would be

$$\|x\| = \left(\lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt \right)^{\frac{1}{2}} . \tag{2}$$

The L_2 space endowed with this norm is also denoted by $L_2(-\infty, \infty)$, and the two are distinguished only by the norm that they are endowed with. To make the distinction here, let us denote the first one, with norm (1), by H_1, and the second one, with norm (2), by H_2. Clearly, $H_2 \subset H_1$. Now, as the following result shows, even though H_1 is not separable, H_2 is.

Theorem 3. *The space H_2, of square-integrable Lebesgue integrable functions on $(-\infty, \infty)$, with the inner product*

$$(h_1, h_2) = \int_{-\infty}^{\infty} h_1(t) h_2(t) \, dt$$

is a separable Hilbert space.

Proof : I prove here only separability. In view of Theorem 2, all we need to do is find a complete orthonormal sequence in H_2. The sequence that does the job (which is not yet orthogonal) is

$$e^{-\frac{t^2}{2}}, \; te^{-\frac{t^2}{2}}, \; t^2 e^{-\frac{t^2}{2}}, \ldots, \; t^n e^{-\frac{t^2}{2}}, \ldots$$

Orthonormalization of this system yields the orthonormal family of *Hermite functions*

$$g_k(t) = \frac{(-1)^k \pi^{-\frac{1}{4}}}{2^{\frac{k}{2}} \sqrt{k!}} e^{\frac{t^2}{2}} \frac{d^k e^{-t^2}}{dt^k} \, , \quad k = 0, 1, 2, \ldots$$

To prove that this countable family is complete, suppose that to the contrary there exists a nonzero $f \in H_2$ which is orthogonal to every one of the g_k's. This is equivalent to saying

$$\int_{-\infty}^{\infty} f(t) e^{-\frac{t^2}{2}} t^k \, dt = 0 \, , \quad k = 0, 1, 2, \ldots \tag{$*$}$$

Introduce the function

$$F(z) = \int_{-\infty}^{\infty} f(t) e^{-\frac{t^2}{2}} e^{\imath t z} \, dt \, ,$$

which exists for every complex number z (here $\imath := \sqrt{-1}$). The derivative of $F(z)$ is

$$F^{(1)}(z) = \int_{-\infty}^{\infty} f(t) e^{-\frac{t^2}{2}} e^{\imath t z} \imath t \, dt \, ,$$

which holds everywhere in the complex plane, which makes $F(z)$ an entire function. But, by $(*)$, the k-th order derivative of $F(z)$ at $z = 0$ is

$$F^{(k)}(0) = \int_{-\infty}^{\infty} f(t) e^{-\frac{t^2}{2}} (\imath t)^k \, dt = 0 \,, \quad k = 0, 1, 2, \dots \,,$$

and thus $F(z)$ is identically zero. Therefore,

$$\int_{-\infty}^{\infty} f(t) e^{-\frac{t^2}{2}} e^{\imath t x} \, dt = 0 \,, \quad \forall \, x \in (-\infty, \infty)$$

Multiply this equality by $e^{-\imath x y}$, where y is real, and integrate it with respect to x from $-\omega$ to ω, to get:

$$\int_{-\infty}^{\infty} f(t) e^{-\frac{t^2}{2}} \left[\frac{\sin \omega (t - y)}{t - y} \right] dt = 0 \,,$$

which is valid for every real y and ω. This then implies that $f(t) = 0$ almost everywhere, which contradicts the initial assumption. $\diamond$

Remark 2. Yet another basis for H_2 is the Haar basis, which is useful particularly in the context of *wavelets*. $\diamond$

Remark 3. We have the counterpart of Theorem 3 for the interval $(0, \infty)$ as well, where a complete orthonormal sequence is

$$r_k(t) = \frac{e^{-\frac{t}{2}}}{k!} L_k(t) \,, \quad k = 0, 1, 2, \dots \,,$$

where $L_k(t)$'s are the *Laguerre* functions, defined through

$$L_k(t) = e^t \frac{d^k}{dt^k} \left(t^k e^{-t} \right) \,, \quad k = 0, 1, 2, \dots \,.$$

Hence, the Hilbert space $L_2(0, \infty)$ with the inner product

$$(h_1, h_2) = \int_0^{\infty} h_1(t) h_2(t) \, dt$$

is separable. $\diamond$

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