*Notes On*

# FIXED POINTS

Let $X$ be some set (in a vector space), and $T : X \to X$ a mapping. A point $x^*$ is said to be a *fixed point of $T$* if it solves the (*fixed-point*) equation :

$$T(x) = x .$$

Such equations can have nonunique solutions (e.g. $T(x) = x^2, X = [0, 1]$), unique solutions (e.g. $T(x) = 2x - 1, X = (-\infty, \infty)$), or no solution at all (e.g. $T(x) = \begin{cases} x + \frac{1}{2} & \text{for } 0 \le x \le \frac{1}{2} \\ \frac{1}{3} & \text{for } \frac{1}{2} < x \le 1 \end{cases}, X = [0, 1]$). We study here conditions under which :

1. A given mapping T has a fixed point,
2. The fixed point is unique, and
3. It can be obtained through an iterative process.

**Note** that every fixed point of a mapping $T : X \to X$ is also a fixed point of $T^r$ (*$r$-th power of $T$*) for every positive integer $r$. (**SHOW !**)

**Definition 1.** *Let $(X, \rho)$ be a metric space, and $T : X \to X$ a mapping. $T$ is said to be Lipschitz if there exists a real number $\alpha \ge 0$ such that for all $x, y \in X$, we have*

$$\rho(T(x), T(y)) \le \alpha \, \rho(x, y) .$$

*$T$ is said to be a contraction mapping if, in the above inequality, $\alpha < 1$, and it is nonexpansive if $\alpha = 1$. Finally, $T$ is said to be a contractive mapping if, for all $x, y \in X$ and $x \ne y$,*

$$\rho(T(x), T(y)) < \rho(x, y) .$$

**Note** that contraction $\Rightarrow$ contractive $\Rightarrow$ nonexpansive $\Rightarrow$ Lipschitz, and all such mappings are *continuous*. (**SHOW !**)

The next result is known as *Banach's contraction principle*, being due to *Banach (1922)*.

**Theorem 1.** *Let $(X, \rho)$ be a **complete** metric space (e.g. a Banach space, if normed), and $T : X \to X$ a **contraction** mapping. Then, $T$ has a unique fixed point, $x^* \in X$, and this fixed point can be obtained as the limit of every sequence generated by the iteration:*

$$x_{n+1} = T(x_n) , \quad x_0 \text{ an arbitrary element of } X .$$

*Proof :* Let $\{x_n\}$ be the sequence generated by the iteration above. Then, using the contraction property of $T$, we obtain:

$\rho(x_1, x_2) = \rho(T(x_0), T(x_1)) \le \alpha \, \rho(x_0, x_1)$

$\rho(x_2, x_3) = \rho(T(x_1), T(x_2)) \le \alpha \, \rho(x_1, x_2) \le \alpha^2 \, \rho(x_0, x_1)$ where $x_1 = T(x_0)$;

and likewise for general $n$: $\rho(x_n, x_{n+1}) \le \alpha^n \, \rho(x_0, x_1)$.

Furthermore, repeated applications of the triangle inequality yields, for arbitrary positive integers $n$ and $p$:

$$\rho(x_n, x_{n+p}) \le \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p})$$

Combining this with the previous set of inequalities leads to:

$$\rho(x_n, x_{n+p}) \le \frac{\alpha^n (1 - \alpha^p)}{1 - \alpha} \, \rho(x_0, x_1) < \frac{\alpha^n}{1 - \alpha} \, \rho(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty$$

where the strict inequality and convergence to *zero* above have followed because $\alpha \in [0, 1)$. Hence $\{x_n\}$ is a Cauchy sequence, and since $X$ is complete it has a limit, say $x^*$ in $X$. Since $T$ is a continuous mapping, we clearly have $x^*$ a fixed point of $T$.

To prove uniqueness, suppose that to the contrary $T$ has two fixed points: $x^*$ and $y^*$, with $\rho(x^*, y^*) > 0$. But since

$$\rho(x^*, y^*) = \rho(T(x^*), T(y^*)) \le \alpha\, \rho(x^*, y^*),$$

and $\alpha \in [0, 1)$, we have a contradiction. ◇

**Corollary 1.1.** *Under the hypotheses of the theorem above, if $x^* \in X$ is the unique fixed point of $T$,*

$$\rho(x_n, x^*) \le \frac{\alpha^n}{1 - \alpha}\, \rho(x_0, x_1)\,.$$

[This provides a measure of how close we can get to $x^*$ as $n$ increases (as a function of the distance between the starting point and the first iterate).]

**Corollary 1.2.** *If $S$ is a closed subset of a complete metric space $X$, and $T : S \to S$ is a contraction on $S$, then $T$ has a unique fixed point in S.*

The following two examples now illustrate the result of *Theorem 1*.

**Example 1.1.** *Let $X = L_2[a, b]$, and consider the following integral equation defined on this space:*

$$x(t) = f(t) + \lambda \int_a^b K(t, s)\, x(s)\, ds =: T(x)(t)$$

*where $f \in X$, $\lambda$ is a given parameter, and $K$ is a square-integrable function on $[a, b] \times [a, b]$. Let*

$$M := \int_a^b \int_a^b |K(t, s)|^2\, ds\, dt\,, \qquad |\lambda|^2\, M < 1\,.$$

**Show** *(using the Cauchy-Schwartz inequality) that if $\lambda$ is chosen as above, then $T$ is a contraction, and hence the integral equation admits a unique solution. This further implies that the unique solution can be obtained as the limit of the following iteration with an arbitrary starting point $x_{(0)} \in X$:*

$$x_{(n+1)}(t) = T(x_{(n)})(t)\,, \quad n = 0, 1, \dots$$

**Example 1.2.** *Let $X = \mathbf{R}^n$, and consider the linear equation $\quad Ax = b\,, \quad$ where $A = \{a_{ij}\}$ is a given $n \times n$ matrix, with $a_{ii} \ne 0$ for $i = 1, \dots, n$, and $b \in \mathbf{R}^n$ is also given. Further, by appropriate scaling, we can assume $a_{ii} = 1$ for all $i = 1, \dots, n$. As a metric on $\mathbf{R}^n$ pick*

$$\rho(x, y) = \max_i |x_i - y_i|$$

*where the index $i$ on a vector designates the $i$'th component of the vector (only here). To obtain the conditions under which the solution $x$ of this equation exists and can be obtained iteratively, we first re-write the equation as*

$$x = (I - A)x + b =: T(x)$$

*where $I$ is the $n \times n$ identity matrix. Then, one can show that $T$ is a contraction (under the metric defined above) if, for all $i$,*

$$\sum_{j \ne i} |a_{ij}| < 1$$

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where $a_{ij}$ denotes the $ij$'th entry of the matrix $A$. A matrix with such a property (with also diagonal elements all 1) is said to be *diagonally dominant*. If $a_{ii}$'s are not normalized to 1, then a matrix $A$ is diagonally dominant if, for all $i$,

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|$$

Hence, if $A$ is diagonally dominant with all diagonal elements equal to 1, by the contraction theorem the iteration

$$x_{(n+1)} = (I - A)x_{(n)} + b, \quad x_{(0)} \text{ arbitrary}, \qquad n = 0, 1, \ldots$$

converges to the unique solution of the linear algebraic equation above. Note that the condition of diagonal dominance is only sufficient for the iteration to converge. Consider, for example, the case where $A = \begin{pmatrix} 1 & -2 \\ -\frac{1}{3} & 1 \end{pmatrix}$. The corresponding iteration converges here, even though $T$ is not a contraction. The reason why we have convergence here is that even though $T$ is not a contraction, $T^2$ is.

In the spirit of the very last observation made above, the following result is now a useful generalization of *Theorem 1*, which says that even if $T$ is not a contraction, provided that $T^r$ is contraction for some positive integer $r$, then $T$ has a unique fixed point.

**Theorem 2.** *If $(X, \rho)$ is a complete metric space, and $T : X \to X$ is a mapping such that $T^r$ is a contraction for some positive integer $r$, then $T$ has a unique fixed point.*

*Proof.* Let $u \in X$ be the unique fixed point of $T^r$. Then,

$$T(u) = T(T^r(u)) = T^{r+1}(u) = T^r(T(u))$$

which implies that $T(u)$ is a fixed point of $T^r$. By uniqueness, we have $u = T(u)$, that is $u$ is also a fixed point of $T$. To prove the uniqueness of the fixed point of $T$, let $v$ be some other fixed point of $T$. Then,

$$v = T(v) = T(T(v)) = \ldots = T^r(v) ,$$

that is, $v$ is also a fixed point of $T^r$. Hence $v = u$. $\diamond$

The three examples below further show that there indeed exist cases when a mapping may not be contraction, while some positive power of it is:

**Example 2.1.** *Consider $T : C[0, 2] \to C[0, 2]$, with the standard maximum norm, defined as*

$$[T(x)](t) = \int_0^t x(s)\, ds$$

**Show** *that its $r$-th power is given by*

$$[T^r(x)](t) = \frac{1}{(r-1)!} \int_0^t (t-s)^{r-1} x(s)\, ds ,$$

*and hence*

$$\|T^r(x) - T^r(y)\|_\infty \leq \frac{2^r}{r!} \|x - y\|_\infty$$

*and is contraction for sufficiently large $r$, even though $T$ is not.*

**Example 2.2.** *This is a slight generalization of Example 2.1. Let $f(x, t)$ be continuous in $t \in [0, t_f]$ where $t_f > 0$ is finite, and be Lipschitz in $x \in \mathbf{R}$ with a finite Lipschitz constant $K$, i.e.,*

$$|f(x, t) - f(y, t)| \leq K|x - y| , \text{ for all } t \in [0, t_f], \text{and all } x, y \in \mathbf{R}$$

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*Let* $T : C[0, t_f] \to C[0, t_f]$ *be defined by*

$$[T(x)](t) = \int_0^t f(x(s), s) \, ds + x_0 \,,$$

*where* $x_0 \in \mathbf{R}$ *is given, and the underlying norm is the standard maximum norm on* $C[0, t_f]$. **Show** *again that* $T^r$ *is a contraction mapping for sufficiently large* $r$, *and in particular that*

$$\|T^r(x) - T^r(y)\| \leq \frac{K^r |t_f|^r}{r!} \, \|x - y\| =: \alpha_r \|x - y\| \,,$$

*where* $\alpha_r < 1$ *for* $r$ *sufficiently large.*

(A consequence of this result is the existence of a unique continuously differentiable solution to the differential equation $\quad \dot{x} = f(x, t) \,, \; x(0) = x_0 \quad$ over the finite interval $[0, t_f]$.)

**Example 2.3.** *Let* $X = \mathbf{R}$, *and* $T(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$. *Clearly,* $T$ *is not continuous, and hence cannot be contraction, but* $T^2(x) \equiv 1$, *and is trivially a contraction mapping. This example also shows that* $T$ *does not have to be continuous for Theorem 2 to apply.*

We have introduced earlier the notion of a *contractive mapping*, which is weaker than a *contraction mapping*. One may wonder at this point as to whether *Theorem 1* covers also contractive mappings. The answer is "no," as the following example demonstrates.

**Example 2.4.** *Let* $X = \{x \in \mathbf{R} : x \geq 1\}$, *with the natural metric* $\rho(x, y) = |x - y|$. *Let* $T : X \to X$ *be given by* $\quad T(x) = x + \frac{1}{x}$. **Show** *that* $T$ *is contractive, but has no fixed point. Note that* $T$ *is not a contraction.*

In order to obtain a result like *Theorem 1* for contractive mappings, it is necessary to bring in some additional restrictions on $T$. The theorem given below is one such result, which is due to *Edelstein (1962)*.

**Theorem 3.** *Let* $(X, \rho)$ *be a metric space, and* $T : X \to X$ *a contractive mapping. Let* $x_0$ *be a point in* $X$ *such that the sequence* $\{T^n(x_0)\}$ *has a convergent subsequence which converges to a point* $x^* \in X$. *Then,* $x^*$ *is a fixed point of* $T$, *which is also unique.*

A corollary follows from this result:

**Corollary 3.1.** *Let* $(X, \rho)$ *be a metric space, and* $T : X \to S$ *a contractive mapping, where* $S$ *is a* **compact** *subset of* $X$. *Then,* $T$ *has a unique fixed point* $x^*$ *in* $S$, *and* $x^* = \lim_{n \to \infty} T^n(x_0)$ *where* $x_0$ *is an arbitrary point of* $X$.

The following result even does away with "contractiveness," at the expense of some additional conditions:

**Theorem 4.** *Let* $(X, \rho)$ *be a* **complete** *metric space (e.g. a Banach space, if normed), and* $T : X \to X$ *a continuous mapping. Suppose that there exists a nonnegative real-valued function* $\phi$ *on* $X$ *such that, for all* $x \in X$,

$$\rho(x, T(x)) \leq \phi(x) - \phi(T(x)) \,.$$

*Then* $T$ *has a fixed point in* $X$ *(but it is not necessarily unique).*

*Proof :* For any $x_0 \in X$, let $x_n := T^n(x_0)$. Then

$$\rho(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1}) \tag{$\circ$}$$

This implies that $\{\phi(x_n)\}$ is a decreasing sequence. Summing $(\circ)$ from 0 to $N$ yields

$$\sum_{n=0}^{N} \rho(x_n, x_{n+1}) \leq \phi(x_0) - \phi(x_{N+1}) \leq \phi(x_0) \,.$$

Since this holds for arbitrary $N$,

$$\sum_{n=0}^{\infty} \rho(x_n, x_{n+1}) \leq \phi(x_0) \,.$$

This implies that $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete there exists an $x^* \in X$ such that $x_n \to x^*$. By continuity of $T$, it follows that $T(x^*) = x^*$.                    ⋄

**Remark.** Note that the theorem above does not claim that the fixed point is unique. Furthermore, the theorem is valid even if $T$ is not continuous (see, *Caristi (1976)*), in which case we require that the functional $\phi$ be lower semicontinuous. The proof in this more general case is much more involved, requiring background beyond the scope of this course.                    ⋄

Not all mappings that have fixed points are contraction, or contractive, or nonexpansive. We now discuss some important theorems on fixed points, which utilize the topological structure of the underlying vector space rather than any (contraction) property of the mapping $T$. These theorems, however, are not "constructive" in the sense that they do not suggest any schemes for the computation of fixed points (as in the case of *Theorem 1*).

First a definition:

**Definition 2.** *A metric (or normed linear) space $(X, \rho)$ possesses the* **fixed point property** *if every continuous mapping of $X$ into itself has a fixed point.*

It is a simple exercise in geometry to see that the unit interval $[0, 1]$ has the fixed point property. Generalization of this to $n$-dimensions ($n > 1$) is not so easy, and was first studied by *Brouwer (1910)* (with a slightly less general version proven by *Poincare* 24 years earlier) who proved the following:

**Theorem 5.** *Let $(X, \|\cdot\|)$ be the $n$-dimensional Euclidean space $\mathbf{R}^n$, with the natural Euclidean norm. Let $B_1$ be the unit ball in $\mathbf{R}^n$, i.e., $B_1 := \{x \in \mathbf{R}^n : \|x\| \leq 1\}$. $B_1$ has the fixed point property.*

**Remark.** One of the simplest proofs for this theorem can be found in *Kuga (1974)*.                    ⋄

A generalization of *Theorem 5* from $B_1$ to more "irregular" sets in $\mathbf{R}^n$ (which can also be viewed as a corollary to *Theorem 5*, and hence also called *Brouwer's theorem* by some authors), is the following:

**Theorem 6.** *Every nonempty compact convex subset $S$ of $\mathbf{R}^n$ (or of any finite-dimensional normed linear space) has the fixed point property.*

We now explore the possibility of obtaining a similar result in infinite-dimensional spaces. First two useful definitions:

**Definition 3.** *Two subsets, $W$ and $V$, of a metric space $(X, \rho)$ are homeomorphic if there exists a continuous map $T : W \to V$ which is one-to-one and onto, and has a continuous inverse.*

**Definition 4.** *The Hilbert cube $\mathcal{H}_0$ is the subset of $\ell_2$ consisting of points $a = (a_1, a_2, \ldots)$ such that $|a_r| \leq 1/r$ for all $r = 1, 2, \ldots$.*

**Theorem 7.** *If $W$ and $V$ are homeomorphic, and $W$ has the fixed point property, then so does $V$.*

**Theorem 8.**
 (i) *$\mathcal{H}_0$ has the fixed point property.*
 (ii) *Any nonempty compact convex subset $K$ of $\mathcal{H}_0$ has the fixed point property.*
(iii) *Every nonempty compact convex subset $K$ of a Banach space $X$ is homeomorphic under a linear mapping to a nonempty compact convex subset of $\mathcal{H}_0$, and hence has the fixed point property.*

The result of part $(iii)$ of the theorem was first proven by *Schauder (1930)*, and is therefore referred to as *Schauder's theorem*.

Finally, we discuss below fixed points for *multi-valued functions*, or equivalently *multifunctions*. A *multifunction* maps points in a vector space, say $X$, to subsets of another vector space, $Y$. More formally, a multifunction $T$ maps $X$ into $2^Y$. We say that $x^* \in X$ is a fixed point of $T : X \to 2^X$ if $x^* \in T(x^*)$. Existence of fixed points for multifunctions was first studied by *Kakutani (1941)* in finite dimensions, whose results were then extended to infinite-dimensional spaces by several authors with different degrees of generality. We state below the one due to *Fan (1960)*. First a useful definition:

**Definition 5.** *Let $X$ and $Y$ be normed linear spaces. A multifunction $T : X \to 2^Y$ is upper semicontinuous (usc) at a point $x_0 \in X$ if, for any sequence $\{x_i\}$ converging to $x_0$ and any sequence $\{y_i \in T(x_i)\}$ converging to $y_0 \in Y$, we have $y_0 \in T(x_0)$. The multifunction $T$ is usc on $X$ if it is usc at every point of $X$.*

**Theorem 9.** *Let $X$ be a normed linear space, $K$ a nonempty compact convex subset of $X$, and $T$ a mapping that assigns to each $x \in K$ a nonempty closed convex subset $T(x)$ of $K$. If $T$ is upper semicontinuous on $K$, then it has a fixed point in $K$, that is an $x^*$ with the property that $x^* \in T(x^*)$.*

**Remark.** As mentioned earlier, this result was first proved by *Fan (1960)*; its finite-dimensional version (obtained by simply taking $X = \mathbf{R}^n$) is due to *Kakutani (1941)*, and is known as the *Kakutani fixed-point theorem*. It has found important applications in control theory, game theory, and mathematical economics (see *Franklin (1980)* for a nice exposition). A comprehensive treatment of fixed-point theory can be found in the book by *Smart (1980)*. ◇

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