TWO MAIN THEOREMS OF OPTIMIZATION INVOLVING DUAL SPACES

The two results below (Theorems 1 and 2) will be discussed in class on Tuesday (March 12). I provide here a proof for Theorem 2; proof of Theorem 1 follows similar arguments, with careful attention paid to the spaces involved. But first a few definitions and concepts, which parallel those we had seen for Hilbert spaces, but in this case for dual spaces.

Let X be a real normed space, X^* be the corresponding dual space, and $< x, x^* >$ be the value of $x^* \in X^*$ evaluated at $x \in X$. As discussed earlier in class, $| < x, x^* > | \leq \|x\|\|x^*\|$, which resembles the property of inner product in Hilbert spaces.

Definition 1. *A vector $x^* \in X^*$ is aligned with a vector $x \in X$ if $| < x, x^* > | = \|x\|\|x^*\|$.*

Definition 2. *Two vectors $x^* \in X^*$ and $x \in X$ are orthogonal if $< x, x^* >= 0$.*

Definition 3. *Let $S \subset X$. The orthogonal complement of S in X^* is $S^\perp := \{x^* \in X^* :< s, x^* >= 0, \forall s \in S\}$.*

And the counterpart of this notion in the dual space is:

Definition 4. *Let $U \subset X^*$. The orthogonal complement of U in X is ${}^\perp U := \{x \in X :< x, u >= 0, \forall u \in U\}$.*

A result (fact) in this context is the following:

Fact. *Let M be a closed subspace of X. Then, ${}^\perp[M^\perp] = M$.*

And now the two main theorems:

Theorem 1. *Let X be a real normed linear space, and $M \subset X$ be a subspace. Let $x \in X$ be a fixed element of X, at a distance d from M. Then:*

(i) There exists $x_o^* \in M^\perp$, with $\|x_o^*\| = 1$, such that

$$d = \inf_{m \in M} \|x - m\| = \max_{x^* \in M^\perp, \|x^*\| \leq 1} < x, x^* >=< x, x_o^* >$$

(ii) If the infimum is achieved above, say by $m_o \in M$, then x_o^* is aligned with $x - m_o$.

(iii) If there exist $x_o^* \in M^\perp, m_o \in M$ such that x_o^* is aligned with $x - m_o$, then the infimum in (i) above is achieved by m_o.

Theorem 2. *Let X be a real normed linear space, and $M \subset X$ be a subspace. Let $x^* \in X^*$ be a fixed element of X^*, at a distance d from $M^\perp$. Then:*

(i) There exists $m_o^* \in M^\perp$, such that

$$\|x^* - m_o^*\| = d = \min_{m^* \in M^\perp} \|x^* - m^*\| = \sup_{x \in M, \|x\| \leq 1} < x, x^* >=: \|x^*\|_M$$

(ii) *If the supremum is achieved above, say by $x_o \in M$, then $x^* - m_o^*$ is aligned with x_o.*

(iii) *If there exist $m_o^* \in M^\perp, x_o \in M$ such that $x^* - m_o^*$ is aligned with x_o, then the supremum in (i) above is achieved by x_o.*

Proof. (i) For any $m^* \in M^\perp$,

$$\|x^* - m^*\| = \sup_{x \in X, \|x\| \leq 1} [< x, x^* > - < x, m^* >] \geq \sup_{x \in M, \|x\| \leq 1} [< x, x^* > - < x, m^* >] = \sup_{x \in M, \|x\| \leq 1} < x, x^* >$$

which implies that

$$\|x^*\|_M \leq \|x^* - m^*\| \quad \forall m^* \in M^\perp$$

To complete the proof of (i), we have to show that $\exists\, m^*$ which makes this an equality. Let $x^* \in X^*$ be restricted to M. Then, its norm on this subspace is $\|x^*\|_M$. Let y^* be an extension of x^* to X^* (by the Hahn-Banach Theorem, such an extension exists, which preserves the norm). Then, $\|y*\| = \|x^*\|_M$ and $x^* - y^* = 0$ on M. Let $m_o^* = x^* - y^*$ on X. Note that $m_o^* \in M^\perp$, and hence $\|x^* - m_o^*\| = \|x^*\|_M$.

(ii) Let the sup be achieved by $x_o \in M$. Clearly, $\|x_o\| = 1$ and $\|x^* - m_o^*\| =< x_o, x^* >=< x_o, x^* - m_o^* >$. Hence, x_o is aligned with $x^* - m_o^*$ (and *vice versa*).

(iii) Consider the following sequence of equalities and inequalities (where m_o^* is not necessarily the one in (i)). Take x_o to have unit norm, without any loss of generality. Then,

$$\|x^* - m_o^*\| =< x_o, x^* - m_o^* >=< x_o, x^* >\leq \sup_{x \in M, \|x\| \leq 1} < x, x^* >\leq \|x^* - m_o^*\|$$

where the 1st equality is due to the alignment condition with $\|x_o\| = 1$, the 2nd equality is due to $x_o \in M$ and $m_o^* \in M^\perp$, the next inequality is from the definition of supremum, and the last inequality follows because m_o^* may not be the minimizing one in (i). Now, since the extreme right and extreme left quantities are the same, supremum exists as a maximum, and is equal to $< x_o, x^* >$. ◇

♢ ♢ ♢